# MATHEMATICAL EXPECTATION OF PRODUCT MOMENTS OF SAMPLES DRAWN FROM A SET OF INFINITE POPULATIONS

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## Introduction

In the second part of his investigations, "On the Mathematical Expectation of Moments of Frequency Distributions,"[2] Tchouproff presented a method which may be interpreted as sampling from a set of infinite univariate populations. In the present paper this method is extended to the study of moments of product moments of samples drawn from a set of infinite bivariate populations. It is also shown how this method may be extended to populations of higher order by deriving some of the simpler formulae for populations of three and four variables.

Tchouproff's method has been criticised[3] because of the complicated algebra. On close examination it is found, however, that it is not the algebra which is complicated but rather the symbolism. Tchouproff introduced a great variety of symbols both in his derivations and in his results. As a consequence his work seems very intricate. If, however, the number of symbols is reduced, and the symbols themselves are simplified, which can be easily accomplished, the underlying idea of Tchouproff's method is found to be very simple.

Quite a complete study of product moments of any bivariate population has been made by Joseph Pepper in his "Studies in the Theory of Sampling."[4] His method is essentially an extension of Church's[5] method, in his studies of univariate populations, to bivariate populations. He does not, however, derive any generalized formulae. In the present study generalized formulae for both the first moment and the variance of product moments of any order are obtained.

It may be noted here, that all of Pepper's formulae for any infinite population can be obtained from those of the present study as special cases, by assuming that all the populations in the set are identical.

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[1] A dissertation presented to the Board of Graduate Studies of Washington University in partial fulfilment of the requirements for the degree of Doctor of Philosophy, June 1933.

[2] *Biometrika*, Vol. XXI, Dec. 1929, pp. 231–258.

[3] Church, A. E. R. "On the Means and Squared Standard Deviations of Small Samples from any Population," *Biometrika*, Vol. XVIII, Nov., 1926, pp. 321–394.

[4] *Biometrika*, Vol. XXI, Dec. 1929, pp. 231–258.

[5] Church, A. E. R., "On the Means and Squared Standard Deviations of Small Samples rom any Population," *Biometrika*, Vol. XVIII, Nov., 1926, pp. 321–394.

## CHAPTER I. Notations and Definitions

Let $(X_1, Y_1), (X_2, Y_2), \cdots (X_n, Y_n)$ be $n$ bivariate populations each following any law of distribution whatever. The product moment of order $a$ in $X$ and $b$ in $Y$ of the $k^{th}$ population will be denoted by $P^k_{a\,b}$. It is defined as

$$P^k_{a\,b} = E(X_k - a_k)^a \, (Y_k - b_k)^b \tag{1.11}$$

where
$$a_k = E(X_k), \qquad b_k = E(Y_k), \tag{1.12}$$

and where the symbol $E$ signifies the expected value or the mathematical expectation of a quantity.

Regarding each of the $n$ populations of the set as infinite,[6] samples of $n$ are drawn, each member of a sample from one of the $n$ populations.[7] The individual which is drawn from the $k^{th}$ population will be denoted by $(x_k, y_k)$; and the product moment of order $a$ in $x$ and $b$ in $y$, of such a sample will be denoted by $p_{ab}$. This product moment may then be defined as

$$p_{ab} = n^{-1} \, S \, (x_k - x)^a \, (y_k - y)^b \tag{1.13}$$

where
$$x = n^{-1} \, Sx_k, \qquad y = n^{-1} \, Sy_k \, . \tag{1.14}$$

The symbols $a$ and $b$ will now be defined by the equations

$$a = n^{-1} \, Sa_k, \qquad b = n^{-1} \, Sb_k \, . \tag{1.15}$$

Obviously
$$E(x) = E(n^{-1} \, Sx_k) = n^{-1} \, SE(X_k) = n^{-1} \, Sa_k = a. \tag{1.16}$$

Similarly $E(y) = b$. That is, the mathematical expectation of the mean, of such a sample as was described above, is equal to the average of the means of all the populations.[8]

In order to make the equations as compact as possible the following additional symbols will be employed:

$$\left.\begin{array}{lll} x_k - a_k = u_k, & x - a = u, & \text{and } u_k - u = U_k \\ y_k - b_k = v_k, & y - b = v, & \text{and } v_k - v = V_k \end{array}\right\} \tag{1.17}$$

also $a_k - a = A_k, b_k - b = B_k$.

From the above definitions it easily follows that

$$E(u_k) = E(v_k) = E(U_k) = E(V_k) = E(u) = E(v) = 0\, . \tag{1.18}$$

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[6] The term infinite is used here in the probability sense. It is defined very clearly by Church in his "Means and Squared Standard Deviations of Small Samples," *Biometrika*, Vol. XVIII, Nov., 1926, p. 322.

[7] It may be easily shown that this is equivalent to drawing a sample of $n$ from a set of any finite number of populations. The number drawn from each population, however, must be specified. See *Biometrika*, Vol. XIII, 1920–21, p. 295, footnote.

[8] This, of course, is a result of the Lexis Theory, for Poisson and Lexis Series.

The notation is now completed with the definition of the symbol $Q_{ij}$ by the equation:

$$Q_{ij} = S(a_k - a)^i (b_k - b)^j = SA_k^i B_k^j . \tag{1.19}$$

## Chapter II. The Mathematical Expectation of $p_{ab}$

The mathematical expectation of $p_{ab}$ will be denoted by $\bar{p}_{ab}$. In the terminology of moments this would be called the mean or first moment of the distribution of $p_{ab}$.

**1. The Mathematical Expectation of $p_{11}$.** According to the above notation the expected value of $p_{11}$ is $\bar{p}_{11}$. By definition

$$\bar{p}_{11} = E(p_{11}) = En^{-1}S(x_i - x)(y_i - y), \tag{2.11}$$

and obviously $En^{-1}S(x_i - x)(y_i - y) = n^{-1}SE(x_i - x)(y_i - y)$.
Writing

$$x_i - x = [(x_i - a_i) - (x - a)] + [a_i - a] = U_i + A_i$$

$$y_i - y = [(y_i - b_i) - (y - b)] + [b_i - b] = V_i + B_i,$$

equation (2.11) may be written as

$$\bar{p}_{11} = n^{-1}SE(U_i + A_i)(V_i + B_i)$$

$$= n^{-1}SE(U_iV_i) + n^{-1}SA_iE(V_i) + n^{-1}SB_iE(U_i) + n^{-1}SE(A_iB_i).$$

Since for any given population $A_i$ and $B_i$ are constants, it follows that $E(A_iB_i) = A_iB_i$. Hence

$$n^{-1}SE(A_iB_i) = n^{-1}SA_iB_i = n^{-1}Q_{11} .$$

Making use of (1.18), it is seen that the terms $n^{-1}SA_iE(V_i)$ and $n^{-1}SB_iE(U_i)$ are zero. The only term left to evaluate is therefore $n^{-1}SE(U_iV_i)$. Since $U_i$ and $V_i$ are symmetric functions of the corresponding small letters, their product is symmetric in $u_iv_i$. There is therefore no loss in generality if attention is concentrated on a single subscript, say 1.

We may therefore write

$$n^{-1}SE(U_iV_i) = n^{-1}E(U_1V_1) + n^{-1}\underset{*2}{S}E(U_iV_i).$$

Remembering that $U_i = u_i - u = u_i - n^{-1}Su_i$, we may write,

$$U_i = u_i - u = u_i - n^{-1}(u_1 + u_2 + \cdots + u_n)$$

$$= n^{-1}[n_1u_i - (u_1 + u_2 + \cdots + u_{i-1} + u_{i+1} + \cdots + u_n)]$$

* The 2 at the bottom of the $S$ simply indicates that the summation begins with $i = 2$.

where $n_1 = n - 1$. In general, $n_i$ will denote the number $n - i$. Similarly

$$V_i = n^{-1}[n_1 v_1 - (v_1 + v_2 + \cdots + v_{i-1} + v_{i+1} + \cdots + v_n)].$$

Thus

$$n^{-1}SE(U_i V_i) = n^{-3}E(n_1 u_1 - u_2 - \cdots - u_n)(n_1 v_1 - v_2 - \cdots - v_n)$$
$$+ n^{-3}\underset{2}{S}E(n_1 u_i - u_1 - \cdots - u_{i-1} - u_{i+1} - \cdots u_n)$$
$$(n_1 v_i - v_1 - \cdots - v_{i-1} - v_{i+1} - \cdots v_n).$$

When the right hand side of the last equation is expanded the only terms which appear are of the form $E(u_i v_i)$ and $E(u_i v_j)$. The last one must vanish for $u_i$ and $v_j$ are independent and hence $E(u_i v_j) = E(u_i)E(v_j) = 0$. From the last equation above it is easily seen that the coefficient of $E(u_1 v_1)$ is

$$n^{-3}(n_1^2 + n_1) = n^{-3}\, n_1(n_1 + 1) = n^{-2}\, n_1;$$

and because of the symmetry this is obviously the coefficient of any term of that form. Hence

$$n^{-1}SE(U_i V_i) = n^{-2} n_1 SE(u_i v_i).$$

Since $u_i = x_i - a_i$, $v_i = y_i - b_i$, then

$$E(u_i v_i) = E(x_i - a_i)(y_i - b_i) = E(X_i - a_i)(Y_i - b_i) = P^i_{1\,1}$$

and in general,

$$E(u_k^i v_k^j) = P^k_{i\,j}. \tag{2.12}$$

We thus get the formula

$$\bar{p}_{11} = n^{-2} n_1 S P^i_{1\,1} + n^{-1} Q_{11}. \tag{1}$$

Now suppose all the $n$ populations are identical. Then all the $A$'s and also all the $B$'s vanish and therefore, $Q_{11} = 0$. The formula (1) thus becomes

$$\bar{p}_{11} = \frac{n-1}{n} P_{11}. \tag{1'}$$

This is exactly Pepper's formula for $\bar{p}_{11}$ for an infinite population.[9]

2. **The Mathematical Expectation of $p_{21}$.** By definition

$$\bar{p}_{21} = En^{-1}S(x_i - x)^2(y_i - y). \tag{2.21}$$

[9] *Biometrika*, Vol. XXI, p. 233, Eq. A, N = ∞. As was already stated in the introduction, all the formulae of the present study reduce to Pepper's when the above assumption is made.

Proceeding as above it is seen that

$$En^{-1}S(x_i - x)^2(y_i - y) = n^{-1}SE(x_i - x)^2(y_i - y)$$
$$= n^{-1}SE(U_i + A_i)^2(V_i + B_i) = n^{-1}SE(U_i^2 V_i) + 2n^{-1}SE(U_i V_i A_i)$$
$$+ n^{-1}SE(U_i^2 B_i) + n^{-1}SE(V_i A_i^2) + 2n^{-1}SE(A_i B_i U_i) + n^{-1}SE(A_i^2 B_i) \cdots \cdot (2.22)$$

It is quite evident that the two terms before the last vanish. To evaluate the remaining terms, we employ the reasoning of section 1 of this chapter and write:

$$SE(U_i^2 V_i) = E(U_1^2 V_1) + \underset{2}{S}E(U_i^2 V_i)$$

$$= n^{-3}E(n_1^* u_1 - u_2 - \cdots)(n_1 v_1 - v_2 - \cdots) + n^{-3}\underset{2}{S}E(n_1 u_i - u_1 - \cdots)$$
$$(n_1 v_i - v_1 - \cdots).$$

Since terms of the form $E(u_i^2 v_j)$ vanish, only the coefficient of the term $E(u_i^2 v_i)$ must be found. Again considering the subscript 1, the coefficient of $E(u_1^2 v_1)$ is easily found from the last equation to be

$$n^{-3}(n_1^3 - n_1) = n^{-3}n_1(n_1 + 1)(n_1 - 1) = n^{-2}n_1 n_2 .$$

Thus

$$n^{-1}SE(U_i^2 V_i) = n^{-2}n_1 n_2 SE(u_i^2 v_i) = n^{-2}n_1 n_2 SP_{21}^i . \tag{2.23}$$

For the second term of (2.22) we have

$$SE(U_i V_i A_i) = E(U_1 V_1 A_1) + \underset{2}{S}E(U_i V_i A_i)$$

$$= n^{-2}E(n_1 u_1 - u_2 - \cdots)(n_1 v_1 - v_2 - \cdots)A_1 + n^{-2}\underset{2}{S}E(n_1 u_i - u_1 - \cdots)$$
$$(u_1 v_i - v_1 - \cdots)A_i.$$

The coefficient of $E(u_1 v_1)$ in the first term of the right hand side of the last equation is $n^{-2}n_1^2 A_1$. In the second term it is $n^{-2}\underset{2}{S}A_i = -n^{-2}A_1$, since $SA_i = 0$. It therefore follows that

$$2n^{-1}SE(U_i V_i A_i) = 2n^{-2}n_2 SP_{11}^i A_i . \tag{2.24}$$

Quite similarly

$$n^{-1}SE(U_i^2 B_i) = n^{-2}n_2 SP_{20}^i B_i, \tag{2.25}$$

and it is obvious that

$$n^{-1}SE(A_i^2 B_i) = n^{-1}Q_{21}. \tag{2.26}$$

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* Note that the $u$ which has the coefficient $n_1$ does not occur among the u's which have the negative sign.

We thus get the formula

$$\bar{p}_{21} = n^{-3}n_1n_2SP^i_{21} + n^{-2}n_2S(2P^i_{11}A_i + P^i_{20}B_i) + n^{-1}Q_{21}. \quad (2)$$

3. **The Mathematical Expectation of $p_{31}$ and $p_{22}$.**

$$\bar{p}_{31} = En^{-1}S(x_i - x)^3(y_i - y) = n^{-1}SE(x_i - x)^3(y_i - y)$$
$$= n^{-1}SE(U_i + A_i)^3(V_i + B_i) = n^{-1}S\{E(U^3_iV_i + U^3_iB_i + 3U^2_iV_iA_i$$
$$+ 3U^2_iA_iB_i + 3U_iV_iA^2_i + 3U_iA^2_iB_i + V_iA^3_i + A^3_iB_i)\}. \quad (2.31)$$

The two terms before the last are zero. The last term is

$$n^{-1}SE(A^3_iB_i) = n^{-1}Q_{31}. \quad (2.32)$$

By (2.23) and (2.24) and some slight manipulation

$$3n^{-1}SE(U^2_iA_iB_i + U_iV_iA^2_i)$$
$$= 3n^{-3}n_2S(P^i_{20}A_iB_i + P^i_{11}A^2_i) + 3n^{-3}(Q_{11}SP^i_{20} + Q^i_{20}SP^i_{11}), \quad (2.33)$$

and by (2.22)

$$n^{-1}SE(U^3_iB_i + 3U^2_iV_iA_i) = n^{-4}(n^3_1 + 1)S(P^i_{30}B_i + 3P^i_{21}A_i). \quad (2.34)$$

The only new term which is to be evaluated is $SE(U^3_iV_i)$. This may be written as follows:

$$SE(U^3_iV_i) = n^{-4}SE(n_1u_i - u_1 - \cdots)^3(n_1v_i - v_1 - \cdots).$$

When the right hand side is expanded it is found that the only non-vanishing terms are of the form $E(U^3_iV_i)$ and $E(u^2_iu_jv_j)$. Only two subscripts, therefore, have to be considered. Without any loss in generality these may be taken as 1 and 2, and the right hand side of the last equation may then be written as follows:

$$SE(n_1u_i - u_1 - \cdots)^3(n_1v_i - v_1 - \cdots) = E(n_1u_1 - u_2 - \cdots)^3(n_1v_1 - v_2 - \cdots)$$
$$+ E(n_1u_2 - u_1 - \cdots)^3(n_1v_1 - v_2 - \cdots) + \underset{3}{S}E(n_1u_i - u_1 - u_2 - \cdots)^3$$
$$(n_1v_i - v_1 - v_2 - \cdots).$$

From this last expansion it is easily seen that the coefficient of $E(u^3_iv_i)$ is $(n^4_1 + n_1)$ and that of $E(u^2_iu_jv_j)$, $(6n^2_1 + 3n_2) = 3(2n^2_1 + n_2)$. We thus finally obtain

$$SE(U^3_iV_i) = n^{-4}\{(n^4_1 + n_1)SE(u^3_iv_i) + 3(2n^2_1 + n_2)SE(u^2_iu_jv_j)\}.$$

But by (2.12) $E(u^3_iv_i) = P^i_{31}$, and since $u_i$ and $u_j$ and $u_i$ and $v_j$ are independent $E(u^2_iu_jv_j) = E(u^2_i)E(u_jv_j) = P^i_{20}P^j_{11}$. Whence

$$E(U^3_iV_i) = n^{-4}\{(n^4_1 + n_1)SP^i_{31} + 3(2n^2_1 + n_2)SP^i_{20}P^j_{11}\}. \quad (2.35)$$

From (2.31) and the succeeding equations we finally get

$$\bar{p}_{31} = n^{-5}\{(n_1^4 + n_1)SP^i_{31} + 3(2n_1^2 + n_2)SP^{*i}_{20}P^j_{11}\}$$
$$+ n^{-4}\{(n_1^3 + 1)S(P^i_{30}B_i + 3P^i_{21}A_i)\} + 3n^{-3}\{(n_1^2 - 1)S(P^i_{20}A_iB_i + P^i_{11}A_i^2)$$
$$+ Q_{11}SP^i_{20} + Q_{20}SP^i_{11}\} + n^{-1}Q_{31}. \quad (3)$$

The derivation of $\bar{p}_{22}$ is so similar to that of $\bar{p}_{31}$, that it would be mere repetition to go through the details again. We shall therefore merely write down the formula for $\bar{p}_{22}$ which is

$$\bar{p}_{22} = n^{-5}\{(n_1^4 + n_1)SP^i_{22} + (2n_1^2 + n_2)S(P^i_{20}P^j_{02} + 4P^i_{11}P^j_{11})\}$$
$$+ 2n^{-4}\{(n_1^3 + 1)S(P^i_{21}B_i + P^i_{12}A_i)\} + n^{-3}\{(n_1^2 - 1)S(P^i_{20}B_i^2 + 4P^i_{11}A_iB_i$$
$$+ P^i_{02}A_i^2) + Q_{20}SP^i_{02} + Q_{02}SP^i_{20} + 4Q_{11}SP^i_{11}\} + n^{-1}Q_{22}. \quad (4)$$

4. **The Mathematical Expectation of the General Product Moment** $p_{ab}$. So far, formulae for the mathematical expectation of $p_{ab}$, for particular values of $a$ and $b$, have been derived. The method used in deriving these is, however, perfectly general, and now, that it has been sufficiently illustrated, it can be easily generalized.

By definition we have

$$\bar{p}_{ab} = E[n^{-1}S(x_i - x)^a(y_i - y)^b].$$

Making use of the notation of Chapter I this may be written as

$$n\bar{p}_{ab} = ES(U_i + A_i)^a(V_i + B_i)^b = \mathop{S}_{q,r=0}^{a,b} C_q^a C_r^b \mathop{S}_{1}^{n} E(U_i^{a-q} V_i^{b-r} A_i^q B_i^r) \quad (2.41)$$

where

$$C_q^a = \frac{a!}{q!(a-q)!}, \qquad C_r^b = \frac{b!}{r!(b-r)!}.$$

Expressing the $U$'s and $V$'s in terms of the $u$'s and $v$'s and setting $a - q = l$, $b - r = m$; we may write for a particular pair of values $q$ and $r$:

$$n^{l+m}SE(U_i^l V_i^m A_i^q B_i^r) = SE(n_1u_i - u_1 - \cdots)^l(n_1v_i - v_1 - \cdots)^m A_i^q B_i^r. \quad (2.42)$$

Consider, now, the general term in the expansion of the right hand side of (2.42). It is of the form:

$$\frac{l!m!}{\Pi\alpha_h!\Pi\beta_h!}(-1)^{l+m}(-n_1)^{\alpha_h+\beta_h}E(n_1u_{j_1}^{\alpha_1}\cdots u_{j_k}^{\alpha_k}v_{j_1}^{\beta_1}\cdots v_{j_k}^{\beta_k}A_i^qB_i^r), \quad (2.43)$$

where $\Pi\alpha_h! = \alpha_1!\alpha_2!\cdots\alpha_k!$

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* In this case, and also in the formulae that follow, whenever two or more indices appear in a summation, it will be understood that no two of them can have the same value simultaneously.

For particular sets of values $j_1, j_2, \cdots j_k, \alpha_1, \alpha_2, \cdots \alpha_k$, and $\beta_1, \beta_2, \cdots \beta_k$, this term will appear in every member of the summation of the right hand side of (2.42), and its coefficient will differ only in the exponent of $(-n_1)$ and in the subscript $i$ of $A^q B^r$. Because of the symmetry there is no loss in generality if we take for $j_1, j_2, \cdots j_k$, the first $k$ integers. We now break up the summation of the right hand side of (2.42) as follows:

$$\begin{aligned} \overset{n}{\underset{1}{S}} E(n_1 u_i - u_1 - \cdots)^l (n_1 v_i - v_1 - \cdots)^m A_i^q B_i^r \\ = E(n_1 u_1 - u_2 - \cdots)^l (n_1 v_1 - v_2 - \cdots)^m A_1^q B_1^r \\ + E(n_1 u_2 - u_1 - \cdots)^l (n_1 v_2 - v_1 - \cdots)^m A_2^q B_2^r + \cdots + E(n_1 u_k - u_1 - \cdots)^l \\ (n_1 v_k - v_1 - \cdots)^m A_k^q B_k^r + \underset{i=k+1}{\overset{n}{S}} E(n_1 u_i - u_1 - \cdots)^l \\ (n_1 v_i - v_1 - \cdots)^m A_i^q B_i^r . \end{aligned} \tag{2.44}$$

From (2.44) we easily get for the total coefficient (excluding the numerical factor) the expression

$$\underset{h=1}{\overset{k}{S}} (-n_1)^{\alpha_h + \beta_h} A_h^q B_h^r + \underset{h=k+1}{\overset{n}{S}} A_h^q B_h^r .$$

Writing

$$\underset{k+1}{\overset{n}{S}} A_h^q B_h^r = \overset{n}{\underset{1}{S}} A_h^q B_h^r - \overset{k}{\underset{1}{S}} A_h^q B_h^r = Q_{qr} - \overset{k}{\underset{1}{S}} A_h^q B_h^r ,$$

the general term, (2.43), together with the total coefficient, may then be written as

$$(-1)^{l+m} \frac{l\,!\; m\,!}{\Pi \alpha_h !\; \Pi \beta_h !} \left\{ \underset{h=1}{\overset{k}{S}} [(-n_1)^{\alpha_h + \beta_h} - 1]\, A_h^q B_h^r + Q_{qr} \right\} \underset{h=1}{\overset{k}{E}}\; u_h^{\alpha_h} v_h^{\beta_h} .$$

Since $u_i$ and $u_j$, $v_i$ and $v_j$, and $u_i$ and $v_j$ are independent while $u_i$ and $v_i$ are not, we have:

I. $E \Pi u_h v_h = \Pi E u_h v_h = \Pi P^h_{\alpha_h \beta_h}$

II. Any term in which $\alpha_h + \beta_h = 1$ must vanish.

From II it follows that the maximum number of subscripts which can appear in any term in the expansion of (2.42), i.e. the upper limit of $k$, which will be denoted by $t$, cannot exceed $(l + m)/2$. In fact when $l + m$ is even, $t = (1 + m)/2$, while when $l + m$ is odd, $t$ is the largest integer less than $(l + m)/2$.

Making use of (2.41), the equations following it, and the reasoning of the last paragraph, we finally get the formula:

$$\begin{aligned} n(-n)^{a+b} \bar{p}_{ab} = (a\,!)\,(b\,!) \underset{j_h=1}{\overset{n}{S}} \underset{q,r=0}{\overset{a,b}{S}} \frac{(-n)^{q+r}}{q\,!\; r\,!} \underset{\alpha_h=0,\beta_h=0}{\overset{a-q,b-r}{S}} \underset{k=1}{\overset{t}{S}} \\ \left\{ \underset{h=1}{\overset{k}{S}} [(-n_1)^{\alpha_h+\beta_h} - 1]\, A_{j_h}^q B_{j_h}^r + Q_{qr} \right\} \Pi \frac{P^{j_h}_{\alpha_h \beta_h}}{\alpha_h !\; \beta_h !} . \end{aligned} \tag{5}$$

The following restrictions on the $\alpha$'s and $\beta$'s must be observed

$$\text{(a) } \alpha_1 + \alpha_2 + \cdots + \alpha_k = a - q$$
$$\text{(b) } \beta_1 + \beta_2 + \cdots + \beta_k = b - r$$
$$\text{(c) } \alpha_h + \beta_h \neq 1 .$$

In case the $n$ populations are identical (5) reduces as follows: For $q = 0$, $r = 0$, $A_i^0 = 1$, $B_i^0 = 1$, and $Q_{00} = n$; while in every other case $A_i^q B_i^r = 0$, $Q_{qr} = 0$. The summations with respect to $q$ and $r$, therefore disappear.

Consider now the summations

$$\underset{j_1=1}{\overset{n}{S}} \; \underset{j_2=1}{\overset{n}{S}} \cdots \underset{j_k=1}{\overset{n}{S}} \; P^{j_1} P^{j_2} \ldots P^{j_k} .$$

Since all the populations are the same we may drop the $j$ by actually carrying out the indicated summations. If, then, there are $c$ repetitions among the $k$ pairs of integers $\alpha_h\beta_h$, in which $\alpha_1\beta_1$, $\alpha_2\beta_2$, $\cdots$ $\alpha_c\beta_c$, are repeated $l_1$, $l_2$, $\cdots$ $l_c$ times respectively, then we have;

$$\underset{j_1=1}{\overset{n}{S}} \cdots \underset{j_k=1}{\overset{n}{S}} \quad P^{j_h}_{\alpha_h\beta_h} = \frac{k\,!\,C^n_k}{l_1\,!\,l_2\,!\cdots l_c\,!}\,\Pi\,P_{\alpha_h\beta_h} .$$*

We thus arrive at the following corollary: The mathematical expectation, $\bar{p}_{ab}$, of the product moment, $p_{ab}$, in samples of $n$ from a single infinite population having any law of distribution is given by

$$n(-n)^{a+b}\bar{p}_{ab} = \underset{\alpha_h,\beta_h=0}{\overset{a,b}{S}} \frac{(a\,!)\,(b\,!)}{\Pi\alpha_h\,!\,\Pi\beta_h\,!} \underset{k=1}{\overset{t}{S}} \left[ \underset{h=1}{\overset{n}{S}} (-n_1)^{\alpha_h+\beta_h} + n_k \right] \frac{k\,!\,C^n_k}{l_1\,!\cdots l_c\,!}\,\Pi P^*_{\alpha_h\beta_h}. \quad (5')$$

*Note*: In deriving these general formulae it was assumed that $n > t$. There is however, no loss in generality in this assumption. For, if $t > n$, we may suppose that, $x_{n+1} = x_{n+2} = \cdots = x_c = 0$, and hence $P^{n+1}_{\alpha\beta} = \cdots = P^t_{\mu\nu} = 0$, and thus the above reasoning is still valid.

5. **Formulae for $\bar{p}_{41}, \bar{p}_{32}, \bar{p}_{51}, \bar{p}_{42}, \bar{p}_{33}$.** Formulae for $\bar{p}_{ab}$ in which $a + b = 5, 6, 7, 8$ have been obtained. But for $(a + b) > 6$ these formulae become very long, and since these will be of no use in the subsequent work, only those of order 5 and 6 are given below.

$$\bar{p}_{41} = n^{-6}\,\{(n_1^5 - n_1)SP^i_{41} + 2nn_2^2S(2P^i_{30}P^i_{11} + 3P^i_{21}P^i_{20})\}$$
$$+ n^{-5}\,\{(n_1^4 + n_1)S(P^i_{40}B_i + 4P^i_{31}A_i) + 6nn_2S(P^i_{20}B_iP^j_{20} + 2P^{i}_{11}{}^{\dagger}A_iP^j_{20})\}$$

* This is a generalization of Pepper's results for $N = \infty$. See *Biometrika* Vol. XXI, pp. 231–240.

† The symbol $P^i_{11}A_iP^j_{20}$ is an abbreviation of the full term $(A_i + A_j)\,(P^i_{11}P^j_{20} + P^j_{11}P^i_{20})$. Similar abbreviations will be used in the other formulae.

$$+ 2n^{-4} \{(n_1^3 + 1) S(2P^i_{30}A_iB_i + 3P^i_{21}A_i^2) - 2Q_{11}SP^i_{30} - 3Q_{20}SP^i_{21}\}$$
$$+ 2n^{-3} \{(n_1^2 - 1)S(2P^i_{11}A_i^3 3P^i_{20}A_i^2B_i) + 2Q_{30}SP^i_{11} + 3Q_{21}SP^i_{20}\} + n^{-1}Q_{41}. \quad (6)$$

$$\bar{p}_{32} = n^{-6} \{(n_1^5 - n_1)SP^i_{32} + nn_2^2S(P^i_{30}P^j_{02} + 6P^i_{21}P^j_{11} + 3P^i_{20}P^j_{12})\}$$
$$+ n^{-5} \{(n_1^4 - 1)S(2P^i_{31}B_i + 3P^i_{22}A_i) + 3nn_2S(P^i_{20}P^j_{11}B_i + [P^i_{20}P^j_{02}$$
$$+ 4P^i_{11}P^j_{11}] A_i)\} + n^{-4} \{(n_1^3 + 1)S(P^i_{30}B_i^2 + 6P^i_{21}A_iB_i$$
$$+ 3P^i_{12}A_i^2) - Q_{02}SP^i_{30} - 6Q_{11}SP^i_{21} - 3Q_{20}SP^i_{12}\} + n^{-3} \{(n_1^2 - 1)S(3P^i_{20}A_iB_i^2$$
$$+ 6P_{11}A_iB_i + P^i_{02}A_i^3) + 3Q_{12}SP^i_{20} + 6Q_{21}SP^i_{11} + Q_{30}SP^i_{02}\} + n^{-1}Q_{32}. \quad (7)$$

$$\bar{p}_{51} = n^{-7} \{(n_1^6 + n_1)SP^i_{51} + 5(n_1^4 + n_1^2 + n_2)S(P^i_{40}P^j_{11}$$
$$+ 2P^i_{31}P^j_{20}) - 10(2n_1^3 - n_2)SP^i_{21}P^j_{30} + 30(3n_1^2 + n_3)SP^i_{20}P^j_{20}P^k_{11}\} + n^{-6}\{(n_1^5$$
$$+ 1)S(P^i_{50}B_i + 5P^i_{41}A_i) + 10(n_1^3 + 1)S^*[2P^i_{30}P^j_{20}B_i + (2P^i_{30}P^j_{11}$$
$$+ 3P^i_{21}P^j_{20})A_i] - 10nn_2S^*[2P^i_{30}P^j_{20}B_j + (2P^i_{30}P^j_{11} + 3P^i_{21}P^j_{20})A_j]\}$$
$$+ 5n^{-5} \{(n_1^4 - 1)S(P^i_{40}A_iB_i + 2P^i_{31}A_i^2) + 6nn_2S(P^i_{20}P^j_{20}A_iB_i + 2P^i_{20}P^j_{11}A_i^2)$$
$$+ Q_{11}S(P^i_{40} + 6P^i_{20}P^j_{20}) + 2Q_{20}S(P^i_{31} + 6P^i_{20}P^j_{11})\} + 10n^{-4}\{(n_1^3$$
$$+ 1)S(P^i_{30}A_i^2B_i + P^i_{21}A_i^3) - Q_{21}SP^i_{30} - Q_{30}SP^i_{21}\} + 5n^{-3}\{(n_1^3 - 1)S(2P^i_{20}A_i^3B_i$$
$$+ P^i_{11}A_i^4) + 2Q_{31}SP^i_{20} + Q_{40}SP^i_{11}\} + n^{-1}Q_{51}. \quad (8)$$

$$\bar{p}_{42} = n^{-7}\{(n_1^6 + n_1)SP^i_{42} + (n_1^4 + n_1^2 + n_2)S(P^i_{40}P^j_{02} + 8P^i_{31}P^j_{11}$$
$$+ 6P^i_{22}P^j_{20}) + 4(2n_1^3 - n_2)S(P^i_{30}P^j_{12} + 3P^i_{21}P^j_{21}) + 6(3n_1^2 + n_3)S(P^i_{20}P^j_{20}P^k_{02}$$
$$+ 4P^i_{20}P^j_{11}P^k_{11})\} + 2n^{-6}\{(n_1^5 + 1)S(P^i_{41}B_i + 2P^i_{32}A_i) + 2(n_1^3 + 1)S[(2P^i_{30}P^j_{11}$$
$$+ 3P^i_{21}P^j_{20})B_i + (P^i_{30}P^j_{02} + 6P^i_{21}P^j_{11} + 3P^i_{12}P^j_{20})A_i] - 2nn_2S[(2P^i_{30}P^j_{11}$$
$$+ 3P^i_{21}P^j_{20})B_j + (P^i_{30}P^j_{12} + 6P^i_{21}P^j_{11} + 3P^i_{12}P^j_{20})A_j]\} + n^{-5}\{(n_1^4 - 1)S(P^i_{40}B_i^2$$
$$+ 8P^i_{31}A_iB_i + 6P^i_{22}A_i^2) + 6nn_2S[P^i_{20}P^j_{21}B_i^2 + 4P^i_{20}P^j_{11}A_iB_i + (P^i_{20}P^j_{02}$$
$$+ 4P^i_{11}P^j_{11})A_i^2] + Q_{02}S(P^i_{40} + 6P^i_{20}P^j_{20}) + 8Q_{11}S(P^i_{31} + 3P^i_{20}P^j_{11})$$
$$+ 6P_{20}S(P^i_{22} + P^i_{20}P^j_{02} + 4P^i_{11}P^j_{11})\} + 4n^{-4}\{(n_1^3 + 1)S(P^i_{30}A_iB_i^2$$
$$+ 3P^i_{21}A_i^2B_i + P^i_{12}A_i^3) - Q_{12}SP^i_{30} - 3Q_{21}SP^i_{21} + Q^i_{30}SP^i_{12}\}$$
$$+ n^{-3}\{S(6P^i_{20}A_i^2B_i^2 + 8P^i_{11}A_i^3B_i + P^i_{02}A_i^4) + Q_{40}SP^i_{02} + 8Q_{31}SP^i_{11}$$
$$+ 6Q_{22}SP^i_{20}\} + n^{-1}Q_{42}. \quad (9)$$

$$\bar{p}_{33} = n^{-7}\{(n_1^6 + n_1)SP^i_{33} + 3(n_1^4 + n_1^2 + n_2)S(P^i_{31}P^j_{p2} + 3P^i_{22}P^j_{11}$$
$$+ P^i_{13}P^j_{20}) - (2n_1^3 - n_2) S(P^i_{30}P^j_{03} + 9P^i_{21}P^j_{12}) + 9(3n_1^3 + n_3)S(P^i_{20}P^j_{11}P^k_{02}$$

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* The repetition of this expression signifies that $A$ and $B$ factors are coupled only with those $P$ factors which have corresponding indices.

$$+ 4P^i_{11}P^j_{11}P^k_{11})\} + 3n^{-6}\{(n_1^5 + 1)S(P^i_{32}B_i + P^i_{23}A_i) + (n_1^3 + 1)\, S[(P^i_{30}P^j_{02} + 6P^i_{21}P^j_{11} + 3P^i_{12}P^j_{20})B_i + (P^i_{03}P^j_{20} + 6P^i_{12}P^j_{11} + 3P^i_{21}P^j_{02})A_i] - nn_2S[(P^i_{30}P^j_{02} + 6P^i_{21}P^j_{11} + 3P^i_{12}P^j_{20})B_j + (P^i_{03}P^j_{20} + 6P^i_{12}P^j_{11} + 3P^i_{21}P^j_{02})A_j]\} + 3n^{-5}\{(n_1^4 - 1)S(P^i_{31}B_i^2 + 3P^i_{22}A_iB_i + P^i_{13}A_i^2) + 3n_1n_2S[P^i_{20}P^j_{11}B_i^2 + (P^i_{20}P^j_{02} + 4P^i_{11}P^j_{11})A_iB_i + P^i_{11}P^j_{02}A_i^2)] + S[Q_{02}(P^i_{31} + 3P^i_{20}P^i_{11}) + 3Q_{11}(P^i_{22} + P^i_{20}P^j_{02} + 4P^i_{11}P^j_{11}) + Q_{20}(P^i_{13} + 3P^i_{02}P^i_{11})]\} + n^{-4}\{(n_1^3 + 1)S(P^i_{30}B_i^3 + 9P^i_{21}A_iB_i^2 + 9P^i_{12}A_i^2A_i + P^i_{03}A_i^3) - S(Q_{03}P^i_{30} + 9Q_{12}P^i_{21} + 9Q_{21}P^i_{12} + Q_{30}P^i_{03})\} + 3n^{-3}\{(n_1^2 - 1)S(P^i_{20}A_iB_i^3 + 3P^i_{11}A_i^2B_i^2 + P^i_{02}A_i^3B_i) + S(Q_{13}P^i_{20} + 3Q_{22}P^i_{11} + Q_{31}P^i_{02})\} + n^{-1}Q_{33}. \quad (10)$$

## CHAPTER III. The Mathematical Expectation of the Variance of $p_{ab}$

**1. The Symbols $_2m_{p_{ab}}$ and $_2M_{p_{ab}}$.** Denoting the variance of $p_{ab}$ by $m$ and the mathematical expectation of $_2m_{p_{ab}}$ by $_2M_{p_{ab}}$, we have the definition,

$$_2m_{p_{ab}} = \{n^{-1}S(x_i - x)^a(y_i - y)^b - \bar{p}_{ab}\}^2$$
$$= n^{-2}S^2(x_i - x)^a(y_i - y)^b - 2n^{-1}\bar{p}_{ab}S(x_i - x)^a(y_i - y)^b + \bar{p}^2_{ab}, \text{ and}$$
$$_2M_{p_{ab}} = E(_2m_{p_{ab}}) = E\{n^{-2}S^2(x_i - x)^a(y_i - y)^b - 2n^{-1}\bar{p}_{ab}S(x_i - x)^a(y_i - y)^b + \bar{p}^2_{ab}\}$$
$$= n^{-2}E[S(x_i - x)^{2a}(y_i - y)^{2b}] + 2n^{-2}E[S(x_i - x)^a(x_j - x)^a(y_i - y)^b(y_j - y)^b] - 2n^{-1}\bar{p}_{ab}E[S(x_i - x)^a(y_i - y)^b] + \bar{p}^2_{ab} = n^{-1}\bar{p}_{2a2b} + 2n^{-2}E[S(x_i - x)^a(y_i - y)^b(x_j - x)^a(y_j - y)^b] - \bar{p}^2_{ab}. \quad (3.11)$$

Before attempting to expand the right hand side of (3.11) for any values $a$, $b$ we shall derive the formula for $_2M_{p_{11}}$ to illustrate the procedure.

**2. The Mathematical Expectation of $_2m_{p_{11}}$.** By (3.11) we have

$$_2M_{p_{11}} = n^{-1}\bar{p}_{22} + 2n^{-2}E[S(x_i - x)(y_i - y)(x_j - x)(y_j - y)] - \bar{p}^2_{11}. \quad (3.21)$$

The first term is given by (4) and the last by (1). The only new term is the middle one. To expand it let us write it in terms of $U$ and $V$. We then have:

$$n^{-2}SE[(x_i - x)(y_i - y)(x_j - x)(y_j - y)] = n^{-2}SE[(U_i + A_i)(V_i + B_i)(U_j + A_j)(V_j + B_j)] = n^{-2}\{SE[U_iV_iU_jV_j + (U_iV_iU_jB_j + U_jV_jU_iB_i) + (U_iV_iV_jA_j + U_jV_jV_iA_i) + (U_iV_iA_jB_j + U_jV_jA_iB_i) + (U_iV_jA_jB_i + U_jV_iA_iB_j) + U_iU_jB_iB_j + V_iV_jA_iA_j + 4 \text{ vanishing terms} + A_iB_iA_jB_j]\}. \quad (3.22)$$

The evaluation of the last term is very simple. For

$$SE(A_iB_iA_jB_j) = S(A_iB_iA_jB_j),$$

and from the elementary theory of symmetric functions we have:

$$S(A_iB_iA_jB_j) = \frac{S^2(A_iB_i) - S(A_i^2B_i^2)}{2}.$$

Hence

$$SE(A_iB_iA_jB_j) = \frac{S^2(A_iB_i) - S(A_i^2B_i^2)}{2} = \frac{Q_{11}^2 - Q_{22}}{2}. \tag{3.23}$$

To expand the first term and also the remaining ones, we return to the $u$, $v$, notation defined in Chapter I. We then write

$$SE(U_iV_iU_jV_j) = n^{-4}SE[(n_1u_i - u_1 - \cdots)(n_1v_i - v_1 - \cdots)$$
$$(n_1u_j - u_1 - \cdots)(n_1v_j - v_1 - \cdots)].$$

The only terms which can appear in the expansion of the right hand side of the last equation have the following form:

$$E(u_i^2v_i^2), \quad E(u_i^2v_j^2), \quad E(u_iv_iu_jv_j),$$

i.e., exactly those which appear in the evaluation of $\bar{p}_{22}$. Remembering the symmetry, there will be no loss in generality if we take for $i$ and $j$ the integers 1 and 2. To find the coefficients of the three characterstic terms, the above summation may be broken up as follows:

$$n^4SE(U_iV_iU_jV_j) = E[(n_1u_1 - u_2 - \cdots)(n_1v_1 - v_2 - \cdots)(n_1u_2 - u_1 - \cdots)$$
$$(n_1v_2 - v_1 - \cdots)] + E\{[n_1u_1 - u_2 - \cdots)(n_1v_1 - v_2 - \cdots) + (n_1u_2 - u_1$$
$$- \cdots)(n_1v_2 - v_1 - \cdots)]\underset{3}{S}(n_1u_i - u_1 - \cdots)(n_1v_i - v_1 - \cdots)\} + \underset{3}{SE}[(n_1u_i$$
$$- u_1 - \cdots)(n_1v_i - v_1 - \cdots)(n_1u_j - u_1 - \cdots)(n_1v_j - v_1 - \cdots)]. \tag{3.24}$$

Writing the three terms in a row and their coefficients from the three parts of (3.24) in columns below these terms, we get the following scheme:

| | $E(u_1^2v_1^2)$ | $E(u_1^2v_2^2 + u_2^2v_1^2)$ | $E(u_1v_1u_2v_2)$ |
|---|---|---|---|
| | $n_1^2$ | $n_1^2$ | $(n_1^2 + 1)^2$ |
| | $n_2(n_1^2 + 1)$ | $-2n_1n_2$ | $2n_2^3$ |
| | $\frac{n_2n_3}{2}$ | $\frac{n_2n_3}{2}$ | $2n_2n_3$ |
| Total coeff. | $\frac{nn_1(2n_1 - 1)}{2}$ | $\frac{-nn_3}{2}$ | $n(n_1^3 + n_1^2 - 3n_1 + 3).$ |

With the aid of the above equations we finally get:

$$SE(U_iV_iU_jV_j) = n^{-4}\left\{\frac{n_1n(2n_1-1)}{2}\overset{*}{S}P^i_{22} - \frac{nn_3}{2}SP^i_{20}P^j_{02} + n(nn_1^2 - 3n_2)SP^i_{11}P^j_{11}\right\}$$

Proceeding in the same way we find:

$$SE(U_iV_iU_jB_j + U_jV_jU_iB_i) = n^{-3}(2n_1^2 + n_2)SP^i_{21}B_i$$

$$SE(U_iV_iV_jA_j + U_jV_jV_iA_i) = n^{-3}(2n_1^2 + n_2)SP^i_{12}A_i$$

$$SE(U_iV_iA_jB_j + U_jV_jA_iB_i) = -nn_2SP^i_{11}A_iB_i + (n_1^2 + n_2)Q_{11}SP^i_{11}$$

$$SE(U_iV_jA_jB_i + U_jV_iA_iB_j) = 2nSP^i_{11} - Q_{11}SP^i_{11}$$

$$SE(U_iU_jB_iB_j + V_iV_jA_iA_j) = nS(P^i_{20}B_i^2 + P^i_{02}A_i^2) - \tfrac{1}{2}S(Q_{20}P^i_{02} + Q_{02}P^i_{20}).$$

Collecting terms and simplifying we finally get:

$$_2M_{p_{11}} = n^{-4}\{n_1^2SP^i_{22} + S(P^i_{20}P^j_{02} + 2P^i_{11}P^j_{11}) - n^2S(P^i_{11})^2\}$$
$$+ 2n^{-3}n_1\{S(P^i_{21}B_i + P^i_{12}A_i)\} + n^{-2}\{S(P^i_{20}B_i^2 + 2P^i_{11}A_iB_i + P^i_{02}A_i^2)\}. \quad (11)$$

Corollary 1. In case $X_i = Y_i$, i.e., when the set of populations are univariate, (11) becomes

$$_2M_{p_{20}} = n^{-4}\{n_1^2S[P^i_{40} - (P^i_{20})^2] + 4SP^i_{20}P^j_{20}\} + 4n^{-3}n_1SP^i_{30}A_i + 4n^{-2}SP^i_{20}A_i^2. \quad (11')$$

This is Tchouproff's formula for the expected value of the variance of samples of $n$.[10]

Corollary 2. In case the $n$ populations are identical (11) becomes

$$_2M_{p_{11}} = n^{-3}n_1[n_1P_{22} + P_{20}P_{02} - n_2P_{11}^2]. \quad (11'')$$[11]

**3. The Mathematical Expectation of $_2M_{p_{ab}}$.** We now return to the general equation

$$_2M_{p_{ab}} = n^{-1}\bar{p}_{2a2b} - \bar{p}_{ab}^2 + 2n^{-2}\underset{i=1,j=1}{\overset{n}{S}} E(x_i - x)^a(y_i - y)^b(x_j - x)^a(y_j - y)^b. \quad (3.11)$$

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* Since $E(u_i^2v_i^2) = P^i_{22}$, $E(u_i^2v_j^2) = P^i_{20}P^j_{02}$, etc.

[10] See *Biometrika*, Vol. XIII p. 295.

[11] See *Biometrika*, Vol. XXI p. 234, Cor. 1.

The first two terms are given by (5). To evaluate the last term we write:

$$SE[(x_i - x)^a(y_i - y)^b(x_j - x)^a(y_j - y)^b] = SE[(U_i + A_i)^a(V_i + B_i)^b(U_j + A_j)^a$$

$$(V_j + B_j)^b] = SE(U_i^a V_i^b U_j^a V_j^b) + \underset{r_1, r_2, r_3, r_4 = 0}{\overset{a,a,b,b,}{S}} C^a_{r_1} C^a_{r_2} C^b_{r_3} C^b_{r_4}$$

$$\overset{n}{\underset{1}{SE}}(U_i^\alpha V_i^\gamma U_j^\beta V_j^\delta A_i^{r_1} B_i^{r_3} A_j^{r_2} B_j^{r_4}) = n^{-2(a+b)} SE\{(n_1u_i - \cdots)^a(n_1v_i - \cdots)^b$$

$$(n_1u_j - \cdots)^a(n_1v_j - \cdots)^b + \underset{r_1 \cdots r_4}{S} n^{(r_1+r_2+r_3+r_4)} C^a_{r_1} \cdots C^b_{r_4} SE[(n_1u_i - \cdots)^\alpha$$

$$(n_1v_i - \cdots)^\gamma(n_1u_j - \cdots)^\beta(n_1v_j - \cdots)^\delta A_i^{r_1} B_i^{r_3} A_j^{r_2} B_j^{r_4}, \tag{3.31}$$

where $\alpha = a - r_1, \beta = a - r_2, \gamma = b - r_3, \delta = b - r_4$.

The right hand side of (3.31) has been broken up into two parts because the first part is symmetrical, while the second part, in general, is not except when $r_1 = r_2$, and $r_3 = r_4$.

Let us now consider the expression

$$SE[(n_1u_i - \cdots)^a(n_1v_i - \cdots)^b(n_1u_j - \cdots)^a(n_1v_j - \cdots)^b]. \tag{3.32}$$

This is a double summation in which $c_{ij} = c_{ji}$ and in which the diagonal terms, $c_{ii}$, are missing.

Consider next a general term of $k$ factors from the expansion of each bracket of (3.32). As we are dealing with symmetric functions, there will be no loss in generality if we consider the first $k$ subscripts only; and if we let the lower limits of the exponents of the $u$'s and $v$'s begin with zero we may consider that each parenthesis of a given bracket contributes exactly $k$ factors. Such a term, omitting the coefficient, may be written as follows:

$$E(u_1^{\alpha_1} \cdots u_k^{\alpha_k} v_1^{\beta_1} \cdots v_k^{\beta_k} u_1^{\alpha'_1} \cdots u_k^{\alpha'_k} v_1^{\beta'_1} \cdots v_k^{\beta'_k}) = \prod_{h=1}^{k} E(u_h^{\alpha_h+\alpha'_h} v_h^{\beta_h+\beta'_h})$$

$$= \prod_{h=1}^{k} P^h(\alpha_h + \alpha'_h)\,(\beta_h + \beta'_h). \tag{3.33}$$

This term occurs in every one of the $\frac{1}{2}nn_1$ brackets of (3.32), having the same numerical coefficient in every one of them, which is

$$\frac{(a\,!)^2\,(b\,!)^2}{\Pi\alpha_h\,!\,\Pi\alpha'_h\,!\,\Pi\beta_h\,!\,\Pi\beta'_h\,!}. \tag{3.34}$$

To obtain the $n_1$ coefficient of (3.33) we break up (3.32) into the following partial summations:

$$E[(n_1u_i - \cdots)^a(n_1v_i - \cdots)^b(n_1u_j - \cdots)^a(n_1v_j - \cdots)^b] = E[(n_1u_1 - \cdots)^a$$

$$(n_1v_1 - \cdots)^b\,(n_1u_2 - \cdots)^a(n_1v_2 - \cdots)^b] + \cdots + E[(n_1u_{k-1} - \cdots)^a$$

$$(n_1v_{k-1} - \cdots)^b\ (n_1u_k - \cdots)^a(n_1v_k - \cdots)^b] + \overset{k}{\underset{i=1}{S}} E\Big[(n_1u_i - \cdots)^a$$

$$(n_1v_i - \cdots)^b \overset{n}{\underset{j=k+1}{S}} (n_1u_j - \cdots)^a(n_1v_j - \cdots)^b\Big] + \overset{n}{\underset{i,j=k+1}{S}} [E\{(n_1u_i - \cdots)^a$$

$$(n_1v_i - \cdots)^b(n_1u_j - \cdots)^a(n_1v_j - \cdots)^b\}] .$$

From this equation we get for the total coefficient in $n$ of the term (3.33) the following expression:

$$\overset{k}{\underset{h,h'=1}{S}} (-n_1)^{\alpha_h+\alpha'_{h'}+\beta_h+\beta'_{h'}} + n_k \overset{k}{\underset{h=1}{S}} [(-n_1)^{\alpha_h+\beta_h} + (-n_1)^{\alpha'_h+\beta'_h}] + C_2^{n_k} .$$

The following restrictions on the $\alpha$'s and $\beta$'s must be observed.

$$\text{(a)}\ \begin{array}{l}\alpha_1 + \alpha_2 + \cdots + \alpha_k = a\\ \alpha'_1 + \alpha'_2 + \cdots + \alpha'_k = a\end{array} \qquad \text{(b)}\ \begin{array}{l}\beta_1 + \beta_2 + \cdots + \beta_h = b\\ \beta'_1 + \beta'_2 + \cdots + \beta'_k = b\end{array}$$

$$\text{(c)}\ \alpha_h + \alpha'_h + \beta_h + \beta'_h \neq 1 .$$

From (c) we obtain the upper limit of $k$, namely: $t = a + b$.

Combining the various above equations we finally obtain:

$$(n)^{2(a+b)} S(U_i^a V_i^b U_j^a V_j^b) = (a\,!)^2(b\,!)^2 \overset{n}{\underset{j_h=1}{S}} \ \overset{n,b}{\underset{\alpha_h,\alpha'_{h'},\beta_h,\beta'_{h'}=0}{S}}$$

$$\overset{t}{\underset{k=1}{S}} \left\{ \overset{k}{\underset{h,h'=1}{S}} (-n_1)^{\alpha_h+\beta_h+\alpha'_{h'}+\beta'_{h'}} + n_k \overset{k}{\underset{h=1}{S}} [(-n_1)^{\alpha_h+\beta_h} + (-n_1)^{\alpha'_h+\beta'_h}] + C_2^{n_k}\right\}$$

$$\frac{\Pi\, P^{j_h}_{(\alpha_h+\alpha'_h)\,(\beta_h+\beta'_h)}}{\Pi\alpha_h\,!\ \Pi\alpha'_h\,!\ \Pi\beta_h\,!\ \Pi\beta'_h\,!} . \tag{3.35}$$

Turning to the second part of (3.31) let us consider the expression

$$\overset{n}{\underset{i=1,j=1}{S}} E[(n_1u_i - \cdots)\,(n_1v_i - \cdots)\,(n_1u_j - \cdots)\,(n_1v_j - \cdots)\,A_i^{r_1}B_i^{r_3}A_j^{r_2}B_j^{r_4}$$

for a given set of $r$'s. The term (3.33) may also be considered as a general term of this last expression; of course, the exponents of the $u$'s and $v$'s will be different in this case. In order to evaluate the complete coefficient of a term like (3.33) we again write;

$$SE[(n_1u_i - \cdots)^\alpha(n_1v_i - \cdots)^\gamma(n_1u_j - \cdots)^\beta(n_1v_j - \cdots)^\delta\, A_i^{r_1}B_i^{r_3}A_j^{r_2}B_j^{r_4}$$

$$= E[(n_1u_1 - \cdots)^\alpha(n_1v_1 - \cdots)^\gamma(n_1u_2 - \cdots)^\beta(n_1v_2 - \cdots)^\delta A_1^{r_1}B_1^{r_3}A_2^{r_2}B_2^{r_4}]$$

$$+ E[n_1u_2 - \cdots)^\alpha(n_1v_2 - \cdots)^\gamma(n_1u_1 - \cdots)^\beta(n_1v_1 - \cdots)^\delta A_2^{r_1}B_2^{r_3}A_1^{r_2}B_1^{r_4}$$

$$+ \cdots + E[(n_1u_k - \cdots)^\alpha(n_1v_k - \cdots)^\gamma(n_1u_{k-1} - \cdots)^\beta(n_1v_{k-1} - \cdots)^\delta$$

$$A_k^{r1}B_k^{r3}A_{k-1}^{r2}B_{k-1}^{r4}] + \underset{i=1}{\overset{k}{S}} E[(n_1u_i - \cdots)^\alpha(n_1v_i - \cdots)^\gamma A_i^{r1}B_i^{r3} \underset{j=k+1}{\overset{n}{S}}$$

$$(n_1u_j - \cdots)^\beta(n_1v_j - \cdots)^\delta A_j^{r2}B_j^{r4}] + \underset{j=1}{\overset{k}{S}} E[(n_1u_j - \cdots)^\beta(n_1v_j - \cdots)^\delta$$

$$A_j^{r2}B_j^{r4} \underset{i=k+1}{\overset{n}{S}} (n_1u_i - \cdots)^\alpha(n_1v_i - \cdots)^\gamma A_i^{r1}B_i^{r3}] + \underset{i,j=k+1}{\overset{n}{S}} E[(n_1u_i - \cdots)^\alpha$$

$$n_1v_i - \cdots)^\gamma(n_1u_j - \cdots)^\beta(n_1v_j - \cdots)^\delta A_i^{r1}B_i^{r3}A_j^{r2}B_j^{r4}]. \tag{3.36}$$

It is now quite easy to write down the complete coefficient of a term of the form (3.33). The numerical coefficient of this term is the same in every bracket of (3.36), and is

$$\frac{(-1)\overset{4}{\underset{1}{S}}{}_{,i}(a-r_1)!\,(a-r_2)!\,(b-r_3)!\,(b-r_4)!}{\Pi\alpha_h!\,\Pi\alpha_h'!\,\Pi\beta_h!\,\Pi\beta_h'!}. \tag{3.37}$$

The coefficient in $n_1$ and $A_i^{r1}\,B_i^{r3}\,A_j^{r2}\,B_j^{r4}$ is broken up by (3.36) into the following four parts:

I. $\underset{h=1,h'=1}{\overset{k}{S}} (-n_1)^{\alpha_h+\alpha_{h'}'+\beta_h+\beta_{h'}'} A_h^{r1}B_h^{r3}A_{h'}^{r2}B_{h'}^{r4}$, from the first $k(k-1)$ brackets.

II. $$\underset{h=1}{\overset{k}{S}} (-n_1)^{\alpha_h+\beta_h} A_h^{r1}B_h^{r3} \underset{h'=k+1}{\overset{n}{S}} A_{h'}^{r2}B_{h'}^{r4} = \underset{h=1}{\overset{k}{S}} (-n_1)^{\alpha_h+\beta_h} A_h^{r1}B_h^{r3} \cdot \left[Q_{r2r4} - \underset{h'=1}{\overset{k}{S}} A_{h'}^{r2}B_{h'}^{r4}\right],$$

from the next $k(n-k)$ brackets. Similarly

III. $\underset{h'=1}{\overset{k}{S}} (-n_1)^{\alpha_{h'}'+\beta_{h'}'} A_{h'}^{r2}B_{h'}^{r4}\left[Q_{r1r3} - \underset{h=1}{\overset{k}{S}} A_h^{r1}B_h^{r3}\right]$, from the next $k(n-k)$.

And finally:

IV. $$\underset{i,j=k+1}{\overset{n}{S}} A_i^{r1}B_i^{r3}A_j^{r2}B_j^{r4} = \overset{n}{\underset{1}{S}}A_h^{r1}B_h^{r3}\overset{n}{\underset{1}{S}}A_h^{r2}B_h^{r4} - \overset{n}{\underset{1}{S}}A_h^{(r1+r2)}B_h^{(r3+r4)}$$

$$- \underset{h,h'=1}{\overset{k}{S}} A_h^{r1}B_h^{r3}A_{h'}^{r2}B_{h'}^{r4} - \underset{h=1}{\overset{n}{S}} A_h^{r1}B_h^{r3} \underset{h'=1}{\overset{k}{S}} A_{h'}^{r2}B_{h'}^{r4} - \underset{h=1}{\overset{n}{S}} A_{h'}^{r2}B_{h'}^{r4} \underset{h=1}{\overset{k}{S}} A_h^{r1}B_h^{r3}$$

$$+ 2 \underset{h=1}{\overset{k}{S}} A_h^{r1}B_h^{r3} \underset{h'=1}{\overset{k}{S}} A_{h'}^{r2}B_{h'}^{r4} = Q_{r1r3}Q_{r2r4} - Q_{(r1+r2)(r3+r4)} - Q_{r1r3}\overset{k}{\underset{1}{S}}A_h^{r2}B_h^{r4}$$

$$- Q_{r2r4}\overset{k}{\underset{1}{S}}A_h^{r1}B_h^{r3} - \underset{h,h'=1}{\overset{k}{S}} A_h^{r1}B_h^{r3}A_{h'}^{r2}B_{h'}^{r4} + 2 \underset{h=1}{\overset{k}{S}} A_h^{r1}B_h^{r3} \underset{h'=1}{\overset{k}{S}} A_{h'}^{r2}B_{h'}^{r4},$$ from the last $c_2^{n-k}$ brackets.

The restrictions on the $\alpha$'s and $\beta$'s differ from those given above in that $a$ is replaced by $a - r_1$ and $a - r_2$, and $b$ by $b - r_3$ and $b - r_4$; and from the restriction (c) we get for the upper limit of $k$, in this case,

$$t_1 = \frac{\alpha + \beta + \gamma + \delta}{2} = a + b - \frac{r_1 + r_2 + r_3 + r_4}{2}$$

when $\overset{4}{\underset{1}{S}} r_i$ is even, or the greatest integer less then $\frac{S\alpha}{2}$ when $Sr_i$ is odd.

Combining (3.37) with $C^a_{r_1} \cdots C^b_{r_4}$ we get for the general numerical coefficient in the expansion of the second part of (3.31), the expression

$$\frac{(-1)^{Sr_i}(a!)^2(b!)^2}{\Pi r_i!\,\Pi\alpha_h!\,\Pi\alpha'_h!\,\Pi\beta_h!\,\Pi\beta'_h!}.$$

By an obvious manipulation we have

$$\begin{aligned}
\mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} = &\overset{k}{\underset{h,h'=1}{S}} \left[(-n_1)^{\alpha_h+\beta_h+\alpha'_{h'}+\beta'_{h'}} - 1\right] A_h^{r_1} B_h^{r_3} A_{h'}^{r_2} B_{h'}^{r_4} + Q_{r_2r_4} \\
&\overset{k}{\underset{h=1}{S}} \left[(-n_1)^{\alpha_h+\beta_h} - 1\right] A_h^{r_1} B_h^{r_3} + Q_{r_1r_3} \overset{k}{\underset{h=1}{S}} \left[(-n_1)^{\alpha'_h+\beta'_h} - 1\right] A_h^{r_2} B_h^{r_4} \\
&- \overset{k}{\underset{h=1}{S}} A_h^{r_2} B_h^{r_4} \overset{k}{\underset{h=1}{S}} \left[(-n_1)^{\alpha_h+\beta_h} - 1\right] A_h^{r_1} B_h^{r_3} - \overset{k}{\underset{h=1}{S}} A_h^{r_1} B_h^{r_3} \\
&\overset{k}{\underset{h=1}{S}} \left[(-n_1)^{\alpha'_h+\beta'_h} - 1\right] A_h^{r_2} B_h^{r_4} + Q_{r_1r_3} Q_{r_2r_4} - Q_{(r_1+r_2)(r_3+r_4)}.
\end{aligned} \tag{3.38}$$

Finally, combining the various equations we get the formula:

$$\begin{aligned}
{}_2M_{p_{ab}} = &\, n^{-1}\bar{p}_{2a2b} - \bar{p}^2_{a\,b} + 2(n)^{-2(a+b+1)}(a\,!)^2(b\,!)^2 \overset{n}{\underset{j\,h=1}{S}} \overset{a,b}{\underset{\alpha_h,\alpha'_{h'},\beta_h,\beta'_{h'}=0}{S}} \overset{t}{\underset{k=1}{S}} \\
&\left\{ \overset{k}{\underset{h,h'=1}{S}} (-n_1)^{\alpha_h+\beta_h+\alpha'_{h'}+\beta'_{h'}} + n_k \overset{k}{\underset{h=1}{S}} [(-n_1)^{\alpha_h+\beta_h} + (-n_1)^{\alpha'_h+\beta'_h}] + C_2^{n_k} \right\} \\
&\frac{\Pi P^{ih}(\alpha_h + \alpha'_h)(\beta_h + \beta'_h)}{\Pi\alpha_h\,!\,\Pi\beta_h\,!\,\Pi\alpha'_h\,!\,\Pi\beta'_h\,!} + 2(n)^{-2(a+b+1)}(a\,!)^2(b\,!)^2 \overset{n}{\underset{j\,h=1}{S}} \overset{a,b}{\underset{r_1,r_2,r_3,r_4=0}{S}} \frac{(-n)^{Sr_1} Sr_i}{\Pi r_1\,!} \\
&\overset{\alpha,\beta,\gamma,\delta}{\underset{\alpha_h,\alpha'_{h'},\beta_h,\beta'_{h'}=0}{S}} \overset{t}{\underset{k=1}{S}} \left\{ \overset{k}{\underset{h,h'=1}{S}} [(-n_1)^{\alpha_h+\beta_h+\alpha'_{h'}+\beta'_{h'}} - 1]\, A_h^{r_1} B_h^{r_3} A_{h'}^{r_2} B_{h'}^{r_4} \right. \\
&- \overset{k}{\underset{h=1}{S}} [(-n_1)^{\alpha_h+\beta_h} - 1]\, A_h^{r_1} B_h^{r_3} \overset{k}{\underset{h=1}{S}} A_h^{r_2} B_h^{r_4} - \overset{k}{\underset{h=1}{S}} [(-n_1)^{\alpha'_h+\beta'_h} - 1] \\
&A_h^{r_2} B_h^{r_4} \overset{k}{\underset{h=1}{S}} A_h^{r_1} B_h^{r_3} + Q_{r_2r_4} \overset{k}{\underset{h=1}{S}} [(-n_1)^{\alpha_h+\beta_h} - 1]\, A_h^{r_1} B_h^{r_3}
\end{aligned}$$

$$+ Q_{r_1r_3} \mathop{S}_{h=1}^{k} [(-n_1)^{\alpha'_h+\beta'_h} - 1] A_h^{r_2} B_h^{r_4} + Q_{r_1r_3}Q_{r_2r_4} - Q_{(r_1+r_2)(r_3+r_4)}\Big\}$$

$$\frac{\Pi P^{j_h}_{(\alpha+\alpha'_h)(\beta_h+\beta'_h)}}{\Pi\alpha_h!\,\Pi\beta_h!\,\Pi\alpha'_h!\,\Pi\beta'_h!}. \tag{12}$$

In case the $n$ populations are identical the second part of (12) must vanish, and in the first part the summations

$$\mathop{S}_{j_h=1}^{n} \prod_{h=1}^{k} P^{j_h}_{(\alpha_h+\alpha'_h)(\beta_h+\beta'_h)} = \frac{k!\,C_k^n \,\Pi\, P_{(\alpha_h+\alpha'_h)(\beta_h+\beta'_h)}}{l_1!\,l_2!\cdots l_c!},$$

where $l_1, l_2, \cdots l_c$ are the number of repetitions of the pairs of integers $(\alpha_1+\alpha'_1)\,(\beta_1+\beta'_1), \cdots (\alpha_k+\alpha'_k)\,(\beta_k+\beta'_k)$, respectively.
We then have the following
Corollary: The mathematical expectation of the variance, ${}_2m_{p_{ab}}$, of the product moment, $p_{ab}$, in samples of $n$ from a single infinite population is given by

$${}_2M_{p_{ab}} = \bar{p}_{2a2b} - \bar{p}^2_{ab} + 2(n)^{-2(a+b+1)}(a!)^2(b!)^2 \mathop{S}_{\alpha_h,\alpha'_h,\beta_h,\beta'_h=0}^{a,a,b,b}$$

$$\mathop{S}_{k=1}^{t} \frac{k!\,C^{nk}}{l_1!\,l_2!\cdots l_c!}\Big\{\mathop{S}_{h,h'=1}^{k}(-n_1)^{\alpha_h+\beta_h+\alpha'_{h'}+\beta'_{h'}} + n_k \mathop{S}_{h=1}^{k}[(-n_1)^{\alpha_h+\beta_h}$$

$$+ (-n_1)^{\alpha'_h+\beta'_h}] + C_2^{nk}\Big\}\frac{\prod_{1}^{k} P(\alpha_h+\alpha'_h)\,(\beta_h+\beta'_h)}{\Pi\alpha_h!\,\Pi\beta_h!\,\Pi\alpha'_h!\,\Pi\beta'_h!}. \tag{12'}$$

4. **The Formula for ${}_2M_{p_{21}}$.** Formula (12) can by no means be used mechanically. It does, however, summarize to a great extent the details in finding ${}_2M_{p_{ab}}$ for any given values $a$, $b$. Formulae for ${}_2M_{p_{21}}$, ${}_2M_{p_{31}}$ have been obtained, but the one for ${}_2M_{p_{31}}$ is too long to be included in the paper, especially since with a little work it can be easily derived by applying (12). The one for ${}_2M_{p_{21}}$ is given immediately below.

$${}_2M_{p_{21}} = n^{-6}\{n_1^2n_2^2S[P^i_{42} - (P^i_{21})^2] + n_2^2S[P^i_{40}P^i_{02} + 4(P^i_{30}P^j_{12} - n_2P^i_{31}P^j_{11})]$$

$$- 2n_2^2n_3SP^i_{22}P^j_{20} + (n_2^2+2)S(P^i_{20}P^j_{20}P^k_{02} + 8P^i_{20}P^j_{11}P^k_{11}) + 6SP^i_{20}P^j_{20}P^k_{02}\}$$

$$+ 2n^{-5}\{n_1n_2^2S(P^i_{41}B_i + 2P^i_{32}A_i - P^i_{30}P^i_{11}B_i - 2P^i_{21}P^i_{11}A_i)$$

$$- 4n_2n_4S(P^i_{30}B_iP^j_{11} + P^i_{12}A_iP^j_{20}) - 2n_2S[n_5P^i_{21}B_iP^j_{20} + 2(2n_2-3)P^i_{21}A_iP^j_{11}]$$

$$+ 6nSP^i_{21}P^j_{11}A_j + 4n_2S(P^i_{30}P^j_{11}B_j + P^i_{30}P^j_{02}A_j + P^i_{30}A_iP^j_{02} + 2P^i_{21}P^j_{20}B_j$$

$$+ P^i_{12}P^j_{20}A_j)\} + n^{-4}\{n_2^2S[P^i_{40}B_i^2 - (P^i_{20}B_i)^2] + 4SP^i_{20}P^j_{20}(B_i+B_j)^2$$

$$+ 3(n_2^2+n_2)SP^i_{22}A_i^2 + 4SP^i_{20}P^j_{02}(A_i+A_j)^2 - 2n_4S[P^i_{20}P^j_{02}A_j^2$$

$$+ 2P^i_{11}P^j_{11}(A_i + A_j)^2] + 16SP^i_{11}A_iP^j_{11}A_j - 4n_2^2S(P^i_{11}A_i)^2$$
$$+ 4(2n_2^2 + n_2)SP^i_{31}A_iB_i - 4n_4SP^i_{11}A_iB_iP^j_{20} - 8n_3SP_{11}P^j_{20}A_jB_j$$
$$+ 8S(P^i_{11}B_iP^j_{20}A_j + P^i_{11}A_iP^j_{20}B_j) - 4n_2^2SP^i_{11}A_iP^i_{20}B_i$$
$$- 2n_1n_2n^{-1}S(Q_{20}P^i_{22} + 2Q_{11}P^i_{31}) + 2n_2n^{-1}S[6Q_{11}P^i_{11}P^j_{20}$$
$$+ Q_{20}(P^i_{20}P^j_{02} + 4P^i_{11}P^j_{11})]\} + 2n^{-4}\{nn_2S(2P^i_{30}A_iB_i^2 + 2P^i_{12}A_i^3 + 5P^i_{21}A_i^2B_i)$$
$$- n_1S[Q_{20}(P^i_{21}B_i + P^i_{12}A_i) + 2Q_{11}8P^i_{30}B_i + 2P^i_{21}A_i)]\} + n^{-4}\{n^2S[P^i_{02}A_i^4$$
$$+ 4(P^i_{20}A_i^2B_i^2 + P^i_{11}A_i^2B_i)] - 2nS[(Q_{20}A_iB_i + Q_{11}A_i^2)P^i_{11} + Q_{11}P^i_{20}A_iB_i$$
$$+ Q_{20}P^i_{02}A_i^2] + S[Q^2_{20}P^i_{02} + 4Q_{20}(Q_{11}P^i_{11} + Q^2_{11}P^i_{20})]\}. \qquad (13)[12]$$

## Chapter IV. The Mathematical Expectation of the Third Moment of $p_{11}$

**1. The Mathematical Expectation of ${}_3m_{p_{11}}$.** Following the notation of the last chapter we shall denote the third moment of $p_{11}$ about its mean by ${}_3m_{p_{11}}$ and the mathematical expectation of ${}_3m_{p_{11}}$ by ${}_3M_{p_{11}}$. We have then by definition.

$${}_3m_{p_{11}} = \{n^{-1}S(x_i - x)(y_i - y) - \bar{p}_{11}\}^3,$$

and by a well known formula we have:

$${}_3M_{p_{11}} = \overline{p^3_{11}} - 3{}_2M_{p_{11}}\bar{p}_{11} - \bar{p}^3_{11}. \qquad (4.11)$$

The last two terms of (4.11) are given by (1) and (11). To evaluate $\overline{p^3_{11}}$ we write:

$$\overline{p^3_{11}} = E\{n^{-1}S(x_i - x)(y_i - y)\}^3 = n^{-3}SE(x_i - x)^3(y_i - y)^3$$
$$+ 3n^{-3}SE(x_i - x)^2(y_i - y)^2(x_j - x)(y_j - y)$$
$$+ 6n^{-3}SE(x_i - x)(y_i - y)(x_j - x)(y_j - y)(x_k - x)(y_k - y).$$

The first term is simply $n^{-2}\bar{p}_{33}$ which is given by (10). The evaluation of the second term is not essentially different from the evaluation of the left hand side of (3.22), and since all details have been given there we shall omit them here.

To evaluate the last expression let us write:

$$SE(x_i - x)(y_i - y)(x_j - x)(y_j - y)(x_k - x)(y_k - y)$$
$$= SE[(U_i + A_i)(V_i + B_i)(U_j + A_j)(V_j + B_j)(U_k + A_k)(V_k + B_k)]$$
$$= SE(U_iV_iU_jV_jU_kV_k) + SE(U_iV_iU_jV_jU_kB_k) + \cdots + SE(A_iB_iA_jB_jA_kB_k). \qquad (4.12)$$

[12] In case the $n$ populations are identical this reduces to one of Pepper's formulae, *Biometrika*, Vol. XXI, p. 238, Cor. 1.

As there is a great deal of similarity among the various terms of the right hand side of (4.12), it will not be necessary to go into the details of the expansion of every one of them. We shall, therefore, indicate the details for the expansion of only two of them—one symmetrical and one non-symmetrical; and as the first two terms are of that type we shall use these for the purpose of illustration.

Using the $u$, $v$ notation we have

$$SE(U_iV_iU_jV_jU_kV_k) = n^{-6}SE[(n_1u_i - \cdots)(n_1v_i - \cdots)(n_1u_j - \cdots) (n_1v_j - \cdots)(n_1u_k - \cdots)(n_1u_k - \cdots)].$$

The maximum number of subscripts appearing in any term evidently being 3, we can write without any loss in generality:

$$\begin{aligned} SE[(n_1u_i - \cdots) \cdots (n_1v_k - \cdots)] &= E[(n_1u_1 - \cdots)(n_1v_1 - \cdots)(n_1u_2 - \cdots) \\ & n_1v_2 - \cdots)(n_1u_3 - \cdots)(n_1v_3 - \cdots)] + E\{(n_1u_1 - \cdots)(n_1v_1 - \cdots)[(n_1u_2 - \cdots) \\ & (n_1v_2 - \cdots) + (n_1u_3 - \cdots)(n_1v_3 - \cdots)] + (n_1u_2 - \cdots)(n_1v_2 - \cdots) \\ & (n_1u_3 - \cdots)(n_1v_3 - \cdots)\}\underset{4}{S}(n_1u_i - \cdots)(n_1v_i - \cdots) + E\{(n_1u_1 - \cdots) \\ & (n_1v_1 - \cdots) + (n_1u_2 - \cdots)(n_1v_2 - \cdots) + n_1u_3 - \cdots)(n_1v_3 - \cdots)\} \\ & \underset{4}{S}(n_1u_i - \cdots)(n_1v_i - \cdots)(n_1u_j - \cdots)(n_1v_j - \cdots) + \underset{4}{SE}\{(n_1u_i - \cdots) \cdots \\ & (n_1v_k - \cdots)\}. \end{aligned} \tag{4.13}$$

The coefficients of the various terms arising in this expansion can now be found quite easily. For example, the coefficient of $P^1_{33}$, which is, of course, the same as the coefficient of $P^i_{33}$, is easily found to be

$$n_1^2 + n_3(2n_1^2 + 1) + \frac{n_3n_4(n_1^2 + 2)}{2} + \frac{n_3n_4n_5}{6} = \frac{nn_1n_2(3n_1 - 2)}{6}.$$

To evaluate the summation $SE(U_iV_iU_jV_jU_kB_k) = n^{-5}SE[(n_1u_i - \cdots)(n_1v_i - \cdots)(n_1u_j - \cdots)(n_1v_j - \cdots)(n_1v_k - \cdots)B_k]$, we break it up into partial summations as follows:

$$\begin{aligned} & SE[(n_1u_i - \cdots)(n_1v_i - \cdots)(n_1u_j - \cdots)(n_1v_j - \cdots)(n_1u_k - \cdots)B_k] \\ &= E\{(n_1u_1 - \cdots)(n_1v_1 - \cdots)[(n_1u_2 - \cdots)(n_1v_2 - \cdots)(n_1u_3 - \cdots)B_3 \\ &+ (n_1u_2 - \cdots)B_2(n_1u_3 - \cdots)(n_1v_3 - \cdots)] + (n_1u_1 - \cdots)B_1(n_1u_2 - \cdots) \\ & (n_1v_2 - \cdots)(n_1u_3 - \cdots)(n_1v_3 - \cdots)\} + E\{(n_1u_1 - \cdots)(n_1v_1 - \cdots) \\ & [(n_1u_2 - \cdots)(n_1v_2 - \cdots) + (n_1u_3 - \cdots)(n_1v_3 - \cdots)] + (n_1u_2 - \cdots) \\ & (n_1v_2 - \cdots)(n_1u_3 - \cdots)(n_1v_3 - \cdots)\}\underset{4}{S}(n_1u_j - \cdots)B_j + E\{(n_1u_1 - \cdots) \\ & (n_1v_1 - \cdots)[(n_1u_2 - \cdots)B_2 + (n_1u_3 - \cdots)B_3] + (n_1u_2 - \cdots)(n_1v_2 - \cdots) \end{aligned}$$

$$[(n_1u_1 - \cdots)B_1 + (n_1u_3 - \cdots)B_3] + (n_1u_3 - \cdots)(n_1v_3 - \cdots)$$

$$[(n_1u_1 - \cdots)B_1 + (n_1v_2 - \cdots)B_2]\}\underset{4}{S}(n_1u_j - \cdots)(n_1v_j - \cdots)$$

$$+ E\{(n_1u_1 - \cdots)(n_1v_1 - \cdots) + (n_1u_2 - \cdots)(n_1v_2 - \cdots) + (n_1u_3 - \cdots)$$

$$(n_1v_3 - \cdots)\}\underset{4}{S}(n_1u_i - \cdots)(n_1v_i - \cdots)(n_1u_j - \cdots)B_j + E\{(n_1u_1 - \cdots)B_1$$

$$+ (n_2u_2 - \cdots)B_2 + (n_1u_3 - \cdots)B_3\}\underset{4}{S}(n_1u_i - \cdots)(n_1v_i - \cdots)$$

$$(n_1u_j - \cdots)(n_1v_j - \cdots) + E\underset{4}{S}(n_1u_i - \cdots)(n_1v_i - \cdots)(n_1u_j - \cdots)$$

$$(n_1v_j - \cdots)(n_1u_k - \cdots)B_k. \tag{4.14}$$

The expansion of (4.14) is not as difficult as it appears for only two subscripts can appear in any term: the explicit appearance of the subscript 3 is due to the fact that we are dealing with a triple summation. We, consequently, do not need to expand those parentheses in which $B$ appears.

We shall now, without any further details, state the final result, which is:

$$
\begin{aligned}
{}_3M_{p_{11}} = n^{-6}\{&S[n_1^3P^i_{33} - P^i_{30}P^j_{03} + 3n_1(P^i_{31}P^j_{02} + P^i_{20}P^j_{13}) + 3n_1(n_1^2 + 2)P^i_{22}P^j_{11} \\
&- 3(2n_1^2 + 1)P^i_{21}P^j_{12} + 3n_3P^i_{11}P^j_{02}P^k_{20} + 6(n_1^3 + 3n_1 - 2)P^i_{11}P^j_{11}P^k_{11}] \\
&- 3n_1SP^i_{11}[S(n_1^2P^i_{22} + P^i_{20}P^j_{02} - n_1^2(P^i_{11})^2 + 2P^i_{11}P^j_{11})] - n_1^3(SP^i_{11})^3]\} \\
&+ 3n^{-5}\{S[n_1^2(P^i_{32}B_i + P^i_{23}A_i) + 2\alpha(P^i_{21}P^j_{11}B_i + P^i_{12}P^j_{11}A_i) \\
&- 2n_1(P^i_{21}P^i_{11}B_i + P^i_{12}P^i_{11}A_i) - 2n_1(P^i_{11}P^j_{21}B_i + P^i_{11}P^j_{12}A_i) \\
&+ (P^i_{12}P^j_{20}B_i + P^i_{21}P^j_{02}A_i) - 2n_1(P^i_{12}P^j_{20}B_j + P^i_{21}P^j_{02}A_j) \\
&+ (P^i_{30}P^j_{02}B_j + P^i_{03}P^j_{20}A_j)]\} + 3n^{-4}\{S[n_1(P^i_{31}B_i^2 + P^i_{13}A_i^2) \\
&+ n_2(P^i_{20}P^j_{11}B_i^2 + P^i_{02}P^j_{11}A_i^2) - (P^i_{11}P^i_{20}B_i^2 + P^i_{02}P^i_{11}A_i^2) \\
&- 2(P^i_{20}B_iP^j_{11}B_j + P^i_{02}A_iP^j_{11}A_j) + 2n_1P^i_{22}A_iB_i - 2P^i_{20}B_iP^j_{02}A_j \\
&- 2P^i_{11}A_iP^j_{11}B_j + 2n_2P^i_{11}P^j_{11}A_iB_i - 2(P^i_{11})^2A_iB_i]\} + n^{-3}\{S[(P^i_{30}B_i^3 + P^i_{03}A_i^3) \\
&+ 3(P^i_{21}A_iB_i^2 + P^i_{12}A_i^2B_i)]\}.
\end{aligned} \tag{14}
$$

[13]

Where $\alpha = n_1^2 + n_1 + 1$.

This formula is shorter and simpler than the formula for ${}_2M_{p_{21}}$, although they are of the same order. This is due to the symmetry of ${}_3M_{p_{11}}$.

## Chapter V. Product Moments of Trivariate and Quadrivariate Populations

**1. Some additional definitions and notation.** In this chapter we shall indicate briefly how the method of the previous chapters may be extended to populations

[13] Cf. *Biometrika*, Vol. XXI, p. 253, formula (19).

of more than two variables. We shall do this by deriving some of the simpler formulae, corresponding to those of Chapter II, for trivariate and quadrivariate populations.

The notation will be slightly changed in that we shall symbolize the new variables by priming the symbols for the variables used in the previous chapters. Thus, we shall indicate the $k^{\text{th}}$ trivariate population by $(X_k, Y_k, X'_k)$ and the $k^{\text{th}}$ quadrivariate population by $(X_k, Y_k, X'_k, Y'_k)$, and samples from such populations by $(x_k, y_k, x'_k)$ and $(x_k, y_k, x'_k, y'_k)$ respectively.

We shall denote by $P^m_{ijk}$ the product moment of the $m^{\text{th}}$ population of order $i$ in $X$, $j$ in $Y$, and $k$ in $X'$, and by $P^m_{ijkl}$ the similar product moment for a quadrivariate population. These are defined by the following equations:

$$P^m_{ijk} = E(X_m - a_m)^i(Y_m - b_m)^j(X'_m - c_m)^k, \tag{5.11}$$

$$P^m_{ijkl} = E(X_m - a_m)^i(Y_m - b_m)^j(X'_m - c_m)^k(Y_m - d_m)^l \tag{5.12}$$

where $a_m$, $b_m$, etc. are defined as in Chapter I part 2.

The sample product moments corresponding to $P^m_{ijk}$, $P^m_{ijkl}$ will be denoted by $p_{ijk}$ and $p_{ijkl}$ respectively. They are defined by:

$$p_{ijk} = n^{-1} \overset{n}{\underset{m=1}{S}} (x_m - x)^i(y_m - y)^j(x'_m - x')^k, \tag{5.13}$$

$$p_{ijkl} = n^{-1} \overset{n}{\underset{m=1}{S}} (x_m - x)^i(y_m - y)^j(x'_m - x')^k(y'_m - y')^l. \tag{5.14}$$

Finally we shall designate $E(p_{ijk})$ and $E(p_{ijkl})$ by $\bar{p}_{ijk}$ and $\bar{p}_{ijkl}$ respectively.

2. **The Mathematical Expectation of $p_{111}$ and $p_{211}$.** By definition we have

$$\bar{p}_{111} = E[n^{-1}S(x_i - x)(y_i - y)(x'_i - x')]. \tag{5.21}$$

Applying the transformations (1.17) this equation becomes

$$np_{111} = E[S(U_i + A_i)(V_i + B_i)(U'_i + C_i)] = SE(U_iV_iU'_i) + SE(U_iV_iC_i)$$
$$+ SE(U_iU'_iB_i) + SE(V_iU'_iA_i) + \text{vanishing terms} + SE(A_iB_iC_i). \tag{5.22}$$

Since $EA_iB_iC_i = A_iB_iC_i$, $SE(A_iB_iC_i) = SA_iB_iC_i$. Following the previous notation we shall put $SA_iB_iC_i = Q_{111}$.

When the expression $SE(U_iV_iU'_i)$ is expanded, no other non-vanishing terms except those of the form $E(u_iv_iu'_i) = P^i_{111}$ can appear. The coefficient of this term will evidently be the same as that of $P^i_{21}$ in (2.23), namely: $n^{-2}n_1n_2$. Whence:

$$SE(U_iV_iU'_i) = n^{-2}n_1n_2SP^i_{11}.$$

The three terms following the first of (5.22) are by (2.24) equal to

$$n^{-1}n_2S(P^i_{110}C_i + P^i_{101}B_i + P^i_{011}A_i).$$

We thus get:

$$\bar{p}_{111} = n^{-3}n_1n_2SP^i_{111} + n^{-2}n_2S(P^i_{110}C_i + P^i_{101}B_i + P^i_{011}A_i) + n^{-1}Q_{111}. \quad (15)$$

With the aid of the formulae of II, 3 we easily find the formula

$$\bar{p}_{212} = n^{-5}\{(n_1^4 - 1)SP^i_{211} + (2n_1^2 + n_2)S(P^i_{200}SP^i_{011} + 2P^i_{110}SP^i_{101} - P^i_{200}P^i_{011} - 2P^i_{110}P^i_{101})\} + n^{-4}\{(n_1^3 + 1)S(P^i_{201}B_i + P^i_{210}C_i + 2P^i_{111}A_i)\} + n^{-3}\{(n_1^2 - 1)S(P^i_{011}A_i^2 + 2P^i_{101}A_iB_i + 2P^i_{110}A_iC_i + P^i_{200}B_iC_i) + Q_{200}SP^i_{011} + 2Q_{110}SP^i_{101} + 2Q_{101}SP^i_{110} + Q_{011}SP^i_{200})\} + n^{-1}Q_{211}. \quad (16)$$

**3. The Mathematical Expectation of $p_{1111}$.** The procedure for finding the formula for $p_{1111}$ is very similar to the above. We shall therefore merely state the result.

$$\bar{p}_{1111} = n^{-5}\{(n_1^4 - 1)SP^i_{1111} + (2n_1^2 + n_2)S(P^i_{1100}P^j_{0011} + P^i_{1001}P^j_{0110} + P^i_{1010}P^j_{0101})\} + n^{-4}\{(n_1^3 + 1)S(P^i_{1110}D_i + P^i_{1101}C_i + P^i_{1011}B_i + P^i_{0111}A_i)\} + n^{-3}\{(n_1^2 + 1)S(P^i_{1100}C_iD_i + P^i_{1010}B_iD_i + P^i_{0110}A_iD_i + P^i_{0101}A_iC_i + P^i_{0011}A_iB_i + P^i_{1001}B_iC_i + S(Q_{0011}P^i_{1100} + Q_{0101}P^i_{1010} + Q_{1001}P^i_{0110} + Q_{1010}P^i_{0101} + Q_{1100}P^i_{0011} + Q_{0110}P^i_{1001})\} + n^{-1}Q_{1111}. \quad (17)$$

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