REPLY TO MR. WERTHEIMER'S PAPER

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The attainment of rigor both in applied as well as pure mathematics is a slow process, and for this reason criticism of my paper, if constructive, is welcomed.

Properties like continuity, differentiability, and dimensionality are *local* properties, that is to say a function may be continuous or differentiable over a certain range but not outside this range, or otherwise a function may be continuous or differentiable over a given range except for singular points.

The presence of singularities in functions does not necessarily cancel their utility. Thus the function $y = \tan x$ contains points where it is discontinuous, but ordinarily it is regarded as a continuous function and the presence of these singular points seldom handicaps one when working with this function. Similarly, the function $f = \bar{x} - \frac{1}{2}\frac{\mu_3}{\mu_2}$ is a function which satisfies all four Axioms as stated in Whittaker and Robinson's book and expresses the mode of Pearson's Type III curve as a symmetric function of the measures. The fact that this function is not differentiable along the line $x_1 = x_2 = x_3 = \cdots = x_n$ will never handicap the investigator for unless the frequency distribution is clearly skew the Type III curve would not be used to represent it.

It seems that Mr. Wertheimer bases nearly all his criticisms on the tacit addition of the word "*everywhere*" to Axiom IV as stated in Whittaker and Robinson's book. The word "everywhere" is not in the statement of Axiom IV and I assumed nothing else than stated in the axiom.

If one deliberately adds the word "everywhere" to Axiom IV then nearly all my criticisms of previous writers are incorrect, unfair, and unjust. However, it does not seem that clearness and rigor in mathematics are increased by reading into an axiom a word that is not there.

Consider first the criticism in my paper which remains valid even when the word "everywhere" is added. (Schimmack uses the word "everywhere" on page 127 although Whittaker and Robinson do not.) Both Schimmack and Whittaker and Robinson proceed as at the top of page 217 of the book by the latter authors with the statement: "In this equation make $k \to 0$ then each of the quantities $\left[\frac{\partial f}{\partial x_n}\right]$ tends to a value which is independent of the x's $\cdots$." This statement rests on the tacit assumption that the quantities $\left[\frac{\partial f}{\partial x_n}\right]$ are functions of k. Even if such were true the use of tacit assumptions in a rigorous proof is objectionable, but as a matter of fact these quantities are not functions of k. Thus the particular proof given in Whittaker and Robinson's book as

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well as in Schimmack's paper is altogether lacking in rigor even when the word "everywhere" is added to Axiom IV. Both Schiaparelli's and Broggi's proofs appear to be entirely rigorous if the word "everywhere" is added to Axiom IV.

In preparing my paper I assumed that no prohibition on functions which had singular points was contained in Axiom IV. In other words, I assumed since the word "everywhere" did not appear there was no valid objection to introduce and discuss functions with singularities. The functions I introduced are everywhere *continuous* but they are not *differentiable* along the line in Euclidian n-space defined by $x_1 = x_2 = x_3 = \cdots = x_n$. They are differentiable at every other point in the space.

It seems to me since Axiom IV as stated in Whittaker and Robinson's book does not exclude functions which are not everywhere differentiable that all my criticism is fair and just, and moreover nearly all my statements are correct. Mr. Wertheimer is entirely correct in pointing out that the words "everywhere" on page 181 of my paper are contradictory. As a matter of fact the whole paragraph beginning with line 7 on page 181 appears to me, on reexamining it, to be unsatisfactory. Except for this single paragraph I believe my paper to be rigorous, but I welcome further criticism.

Mr. Wertheimer's conclusions in his paragraph number 4 are clearly erroneous. To show this, consider a function of k. As $k \to 0$ any one of three situations may arise, namely: (1) The function may become infinite, (2) the function may become indeterminate, that is it may take on any value whatever, (3) the function may approach a unique finite value independent of k. Neither Schimmack nor Whittaker and Robinson nor Mr. Wertheimer has established as a definite fact that the particular type of function here in question approaches a unique finite value independent of k as $k \to 0$. The truth of the matter is that this conclusion cannot be established because the function in question does not involve k either explicitly or implicitly.

In conclusion there are two things I wish to emphasize. First, even when the word "everywhere" is added to Axiom IV, the proof given in Whittaker and Robinson's book is faulty, but if one consults the references given there in the footnotes he will find two other proofs which are rigorous with this addition to Axiom IV. Second, the mode of a skew bell shaped Pearson Frequency Curve satisfies all four axioms as stated in Whittaker and Robinson's book, and the fact that these expressions for the mode are not differentiable along a certain line is never a handicap to the statistician.

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