SAMPLES FROM TWO BIVARIATE NORMAL POPULATIONS[1]

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1. Introduction. In multivariate analysis involving p variates, or in analysis of variance of m samples from univariate populations, we are often interested in the hypothesis of the equality of variances; viz., that

$$\sigma_1 = \sigma_2 = \cdots = \sigma_p, \qquad \text{in the case of } p \text{ variates;}$$

or

$$\sigma_1 = \sigma_2 = \cdots = \sigma_m, \qquad \text{in the case of } m \text{ samples.}$$

As a matter of fact, it seldom occurs that these hypotheses are true, but the ratio between the variances might be known.

Hotelling [5] has suggested that if

$$\sigma_1^2/k_1 = \sigma_2^2/k_2 = \cdots = \sigma_m^2/k_m = \sigma^2,$$

where the k's are known constants, we can apply the transformation

$$\begin{aligned} x_1' &= w_1x_1, \\ x_2' &= w_2x_2, \\ &\cdots\cdots\cdots\cdots \\ x_m' &= w_mx_m, \end{aligned}$$

where

$$w\sqrt{k_1} = w_2\sqrt{k_2} = \cdots = w_m\sqrt{k_m} = 1,$$

so that after transformation the variances become equal, i.e.,

$$\sigma_1' = \sigma_2' = \cdots = \sigma_m',$$

and the required analysis can be carried out. This method is similarly applicable in the multivariate case.

In a previous paper [7], I developed a series of hypotheses concerning samples from a bivariate normal population under the assumption that

$$\sigma_1 = \sigma_2.$$

In case $\sigma_1^2/k_1 = \sigma_2^2/k_2$, where k_1 and k_2 are two distinct known constants, similar results may be obtained by the use of the transformation $x_1' = w_1x_1$; $x_2' = w_2x_2$; where $w_1\sqrt{k_1} = w_2\sqrt{k_2} = 1$.

[1] Presented to the American Mathematical Society at Washington, D. C., May 3, 1941.

In multivariate analysis, the hypotheses usually of interest concerning correlation coefficients may be classified in two categories, viz.,

(i) that the correlation coefficient is equal to a specified value, e.g., in simple correlation $\rho_{12} = \rho_0$, in partial correlation, $\rho_{12\cdot3} = \rho_0$, in multiple correlation, $\rho_{1\cdot23} = \rho_0$, or in correlation between two sets of variates [4][2], $Q = Q_0$; of special interest is the hypothesis of the vanishing of such correlation coefficients.

(ii) that two given correlation coefficients are equal, e.g., (1) correlation coefficients ρ_1 and ρ_2 in the correlation matrix of a multivariate distribution are equal (Hotelling [6]), or (2) the correlation coefficients ρ_{12} and ρ'_{12} in two bivariate populations are equal.

R. A. Fisher in his earlier paper [3] introduced the transformation $z = \frac{1}{2}\log\frac{1+r}{1-r}$ which provides a very satisfactory, though approximate, method for the comparison of two correlation coefficients. Brander [1] treated the same problem by the method of the likelihood ratio criterion.

The present paper is an attempt to obtain different criteria by the likelihood ratio method (Neyman and Pearson [9], [10], [11]) for testing, by means of samples, the equality of correlation coefficients in two bivariate normal populations under the following sets of conditions: (1) $\sigma_1 = \sigma_2$ and $\sigma'_1 = \sigma'_2$; (2) $\sigma_1 = \sigma_2$, $\xi_1 = \xi_2$ and $\sigma'_1 = \sigma'_2$, $\xi'_1 = \xi'_2$. The results may be extended to the cases (3) $\sigma_1^2/k_1 = \sigma_2^2/k_2$ and $\sigma_1'^2/k'_1 = \sigma_2'^2/k'_2$; (4) $\sigma_1^2/k_1 = \sigma_2^2/k_2$, $\xi_1^2/k_1 = \xi_2^2/k_2$ and $\sigma_1'^2/k'_1 = \sigma_2'^2/k'_2$, $\xi_1'^2/k'_1 = \xi_2'^2/k'_2$, where the k's are known constants.

2. The hypotheses. Two samples, each being of two variates (x_1, x_2) and (x'_1, x'_2), of size N and N', are supposed to be drawn at random, respectively, from two independent normal bivariate populations, with the following distributions:

$$(1)\quad \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_1-\xi_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\xi_1}{\sigma_1}\right)\left(\frac{x_2-\xi_2}{\sigma_2}\right) + \left(\frac{x_2-\xi_2}{\sigma_2}\right)^2\right]\right\},$$

$$(2)\quad \frac{1}{2\pi\sigma'_1\sigma'_2\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho'^2)}\left[\left(\frac{x'_1-\xi'_1}{\sigma'_1}\right)^2 - 2\rho'\left(\frac{x'_1-\xi'_1}{\sigma'_1}\right)\left(\frac{x'_2-\xi'_2}{\sigma'_2}\right) + \left(\frac{x'_2-\xi'_2}{\sigma'_2}\right)^2\right]\right\},$$

where $\xi_1, \xi_2, \sigma_1, \sigma_2, \rho; \xi'_1, \xi'_2, \sigma'_1, \sigma'_2, \rho'$ are the unknown parameters of the populations.

The hypotheses to be considered in the present paper are:

H_1: Assuming $\sigma_1 = \sigma_2$ and $\sigma'_1 = \sigma'_2$, to test $\rho = \rho'$.

H_2: Assuming $\sigma_1 = \sigma_2$, $\xi_1 = \xi_2$, and $\sigma'_1 = \sigma'_2$, $\xi'_1 = \xi'_2$, to test $\rho = \rho'$.

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[2] See bibliography at the end of the paper.

The derivation and the distribution of the criteria for testing these hypotheses may be simplified by the following simultaneous transformations:

$$X = \frac{1}{\sqrt{2}}(x_1 - x_2) \qquad Y = \frac{1}{\sqrt{2}}(x_1 + x_2) \tag{3}$$

$$X' = \frac{1}{\sqrt{2}}(x_1' - x_2') \qquad Y' = \frac{1}{\sqrt{2}}(x_1' + x_2') \tag{4}$$

The corresponding normal bivariate distributions in the transformed variables (X, Y) and (X', Y') are obtained, viz.

$$\frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho_{XY}^2}} \exp\left\{-\frac{1}{2(1-\rho_{XY}^2)}\left[\left(\frac{X-\xi}{\sigma_X}\right)^2 - 2\rho_{XY}\left(\frac{X-\xi}{\sigma_X}\right)\left(\frac{Y-\eta}{\sigma_Y}\right) + \left(\frac{Y-\eta}{\sigma_Y}\right)^2\right]\right\} dX\,dY, \tag{5}$$

$$\frac{1}{2\pi\sigma_X'\sigma_Y'\sqrt{1-\rho_{XY}'^2}} \exp\left\{-\frac{1}{2(1-\rho_{XY}'^2)}\left[\left(\frac{X'-\xi'}{\sigma_X'}\right)^2 - 2\rho_{XY}'\left(\frac{X'-\xi'}{\sigma_X'}\right)\left(\frac{Y'-\eta'}{\sigma_Y'}\right) + \left(\frac{Y'-\eta'}{\sigma_Y''}\right)^2\right]\right\} dX'\,dY'. \tag{6}$$

The conditions corresponding to

$$\sigma_1 = \sigma_2 \quad \text{and} \quad \sigma_1' = \sigma_2', \tag{7}$$

are that

$$\rho_{XY} = 0 \quad \text{and} \quad \rho_{XY}' = 0. \tag{8}$$

Also, for a given ρ and ρ', we have from (7)

$$\sigma_Y^2 = \gamma\sigma_X^2 \quad \text{and} \quad \sigma_Y'^2 = \gamma'\sigma_X'^2, \tag{9}$$

where

$$\gamma = \frac{1+\rho}{1-\rho} \quad \text{and} \quad \gamma' = \frac{1+\rho'}{1-\rho'}. \tag{10}$$

Following the notation of (9) and (10), the hypotheses H_1' and H_2' corresponding to H_1 and H_2 are:

H_1' : Assuming $\rho_{XY} = 0$, and $\rho_{XY}' = 0$, to test $\gamma = \gamma'$.
H_2' : Assuming $\rho_{XY} = 0$, $\xi = 0$, and $\rho_{XY}' = 0, \xi' = 0$, to test $\gamma = \gamma'$.

3. **The derivation of the criteria.** Let $(x_{1i}, x_{2i})(x_{1j}', x_{2j}')$ be the measurements of the characters on the ith and jth individuals in the two samples from their respective populations. After transformation, the corresponding measurements become (X_i, Y_i) and (X_j', Y_j'). Let $p(E)$ denote the joint elementary proba-

bility law of the N and N' observations, $E = (X_1, \cdots, X_N, Y_1, \cdots, Y_N; X'_1, \cdots, X'_{N'}, Y', \cdots, Y'_{N'}$.

Following Neyman and Pearson, we shall use Ω to designate the class of admissible populations under conditions which can be assumed to be satisfied in any case; and ω to designate a subclass of Ω under conditions which are satisfied only if the hypothesis to be tested is true.

Thus for H', Ω specifies for $\rho_{XY} = \rho'_{XY} = 0$, any real values of ξ, η, ξ', η' and any positive values of σ_X, σ_Y, σ'_X, σ'_Y; ω specifies $\rho_{XY} = \rho'_{XY} = 0$, any real values of ξ, η, ξ', η' and any positive values of σ_Y and γ which are defined by (9). While for H', Ω specifies $\rho_{XY} = \rho'_{XY} = 0$, $\xi = \xi' = 0$, any real values of η and η' and any positive values of σ_X, σ_Y, σ'_X, σ'_Y; ω specifies $\rho_{XY} = \rho'_{XY} = 0$, $\xi = \xi' = 0$, any real values of η and η', and any positive values of σ_Y and γ which are defined by (9).

For our hypothesis H'_1, the values of the parameters required to make $p(\Omega)$ a maximum are:

$$\hat{\xi} = \bar{X}, \qquad \hat{\eta} = \bar{Y}, \qquad \hat{\sigma}_X = s_X, \qquad \hat{\sigma}_Y = s_Y$$
$$\hat{\xi}' = \bar{X}', \qquad \hat{\eta}' = \bar{Y}', \qquad \hat{\sigma}'_X = s'_X, \qquad \hat{\sigma}'_Y = s'_Y.$$

Thus $p(\Omega \max) = \left(\frac{1}{2\pi}\right)^{N+N'} \frac{1}{s_X^N s_Y^N s_X'^{N'} s_Y'^{N'}} e^{-N-N'}$

To obtain $p(\omega \max)$, let us define, according to the notation in the writer's previous paper [7],

$$R_1 = \frac{2Ys_1s_2}{s_1^2 + s_2^2} \qquad \text{and} \qquad R'_1 = \frac{2Y's'_1s'_2}{s_1'^2 + s_2'^2}$$

and

$$u = \frac{s_Y^2}{s_X^2} = \frac{1 + R_1}{1 - R_1} \qquad u' = \frac{s_Y'^2}{s_X'^2} = \frac{1 + R'_1}{1 - R'_1}.$$

Then the values making $p(\omega)$ a maximum are:

$$\hat{\xi} = \bar{X}, \qquad \hat{\eta} = \bar{Y}, \qquad \sigma_Y^2 = \tfrac{1}{2}s_X^2(\hat{\gamma} + u)$$
$$\hat{\xi}' = \bar{X}', \qquad \hat{\eta}' = \bar{Y}', \qquad \sigma_Y'^2 = \tfrac{1}{2}s_X'^2(\hat{\gamma} + u')$$

and $\hat{\gamma}$ is the positive root of the equation

$$(N + N')\gamma^2 - (N - N')(u - u')\gamma - (N + N')uu' = 0$$

or

$$\hat{\gamma} = \frac{(N - N')(u - u') + \sqrt{(N - N')^2(u - u')^2 + 4(N + N')uu'}}{2(N + N')} \tag{11}$$

$$= \gamma_1, \text{ say.}$$

Then

$$p(\omega \max) = \left(\frac{1}{2\pi}\right)^{N+N'} \left[\frac{2\sqrt{\gamma_1}}{(\gamma_1 + u)s_X^2}\right]^N \left[\frac{2\sqrt{\gamma_1}}{(\gamma_1 + u')s_X'^2}\right]^{N'} e^{-N-N'},$$

and the likelihood ratio criterion for the hypothesis H_1' is

$$\begin{aligned}\lambda = \frac{p(\omega \max)}{p(\Omega \max)} &= \left[\frac{2\sqrt{\gamma_1}\, s_Y}{(\gamma_1 + u)s_X}\right]^N \left[\frac{2\sqrt{\gamma_1}\, s_Y'}{(\gamma_1 + u')s_X}\right]^{N'} \\ &= \left[\frac{2\sqrt{\gamma_1 u}}{\gamma_1 + u}\right]^N \left[\frac{2\sqrt{\gamma_1 u'}}{\gamma_1 + u'}\right]^{N'}. \end{aligned} \tag{12}$$

For H_2', the values the parameters to make $p(\omega)$ a maximum are:

$$\hat{\eta} = \bar{Y}, \qquad \hat{\sigma}_X^2 = \frac{1}{N}\Sigma X^2 \qquad \hat{\sigma}_Y = s_Y$$

$$\hat{\eta}' = \bar{Y}', \qquad \hat{\sigma}_X'^2 = \frac{1}{N'}\Sigma X'^2 \qquad \hat{\sigma}_Y' = s_Y'.$$

Thus

$$p(\Omega \max) = \left(\frac{1}{2\pi}\right)^{N+N'} \frac{\sqrt{NN'}}{(\Sigma X^2)^{N/2}(\Sigma X'^2)^{N'/2} s_Y^N s_Y'^{N'}} e^{-N-N'}.$$

Similarly, if we write

$$R_2 = \frac{2\gamma s_1 s_2 - \frac{1}{2}(\bar{x}_1 - \bar{x}_2)^2}{s_1^2 + s_2^2 + \frac{1}{2}(\bar{x}_1 - \bar{x}_2)^2}, \qquad R_2' = \frac{2\gamma' s_1' s_2' - \frac{1}{2}(\bar{x}_1' - \bar{x}_2')^2}{s_1'^2 + s_2'^2 + \frac{1}{2}(\bar{x}_1' - \bar{x}_2')^2},$$

and

$$v = \frac{Ns_Y^2}{\Sigma X^2} = \frac{s_Y^2}{s_X^2 + \bar{x}^2} = \frac{1 + R_2}{1 - R_2}, \qquad v' = \frac{Ns_Y'^2}{\Sigma X'^2} = \frac{1 + R_2'}{1 - R_2'},$$

the values to make $p(\omega)$ a maximum are:

$$\hat{\eta} = \bar{Y}, \qquad \sigma_Y^2 = \frac{1}{2N}\Sigma X^2(\hat{\gamma} + v)$$

$$\hat{\eta}' = \bar{Y}', \qquad \sigma_Y'^2 = \frac{1}{2N'}\Sigma X'^2(\hat{\gamma} + v)$$

$$\hat{\gamma} = \frac{(N - N')(v - v') + \sqrt{(N - N')^2(v - v')^2 + 4(N + N')^2 uu'}}{2(N + N')} \tag{13}$$

$$= \gamma_2, \text{ say.}$$

Then

$$p(\omega \max) = \left(\frac{1}{2\pi}\right)^{N+N'} \left[\frac{2N\sqrt{\gamma_2}}{(\gamma_2 + v)\Sigma X^2}\right]^N \left[\frac{2N'\sqrt{\gamma_2}}{(\gamma_2 + v')\Sigma X'^2}\right]^{N'},$$

and the likelihood ratio criterion for the hypothesis H_2' is

$$\lambda_2 = \frac{p(\omega \max)}{p(\Omega \max)} = \left[\frac{2\sqrt{N\gamma_2}\, s_Y}{(\gamma_2 + v)\sqrt{\Sigma X^2}}\right]^N \left[\frac{2\sqrt{N'\gamma_2}\, s_Y'}{(\gamma_2 + v')\sqrt{\Sigma X'^2}}\right]^{N'} \tag{14}$$
$$= \left[\frac{2\sqrt{\gamma_2 v}}{\gamma_2 + v}\right]^N \left[\frac{2\sqrt{\gamma_2 v'}}{\gamma_2 + v'}\right]^{N'}.$$

The case $N = N'$. The above criteria λ_1 and λ_2 cannot in general be expressed simply, but when $N = N'$, by (11) and (13)

$$\gamma_1 = \sqrt{uu'}, \qquad \gamma_2 = \sqrt{vv'},$$

and

$$\lambda_1 = \left[\frac{4\sqrt{uu'}}{(\sqrt{u} + \sqrt{u'})^2}\right]^N, \qquad \lambda_2 = \left[\frac{4\sqrt{vv'}}{(\sqrt{v} + \sqrt{v'})^2}\right]^N,$$

or we may express as monotonic functions of λ_1 and λ_2,

$$L_1 = \lambda_1^{2/(N+N')} = \lambda_1^{1/N} = \frac{4}{\left(\sqrt[4]{\frac{u}{u'}} + \sqrt[4]{\frac{u'}{u}}\right)^2}, \tag{15}$$

$$L_2 = \lambda_2^{1/N} = \frac{4}{\left(\sqrt[4]{\frac{v}{v'}} + \sqrt[4]{\frac{v'}{v}}\right)^2}. \tag{16}$$

Thus, λ's, L's, or their functions $\frac{u}{u'}, \frac{v}{v'}$, may be used as the criteria in the present case.

Furthermore, if we introduce,

$$z = \tfrac{1}{2}\log u, \quad \text{and} \quad z' = \tfrac{1}{2}\log u', \tag{17}$$

we have

$$\tfrac{1}{2}(z - z') = \tfrac{1}{4}\log\frac{u}{u'} \quad \text{or} \quad \sqrt[4]{\frac{u}{u'}} = e^{\frac{1}{2}(z-z')}.$$

Thus L_1 can be written in terms of z and z'

$$L_1 = 4/(e^{\frac{1}{2}(z-z')} + e^{-\frac{1}{2}(z-z')}) = 1/\cosh^2 \tfrac{1}{2}(z - z') = \operatorname{sech}^2 \tfrac{1}{2}(z - z'), \tag{18}$$

and $z - z' = w$, say, may be used also as a criterion for H_1.

We shall now proceed to obtain the distributions of some of these statistics.

4. **The distributions of u/u' and v/v'.** Since Ns_Y^2/σ_Y^2 and Ns_X^2/σ_X^2 have independently the χ^2 distribution with $N - 1$ degrees of freedom,

$$u = \frac{s_Y^2}{s_X^2} = \frac{\sigma_Y^2 \chi_2^2}{\sigma_X^2 \chi_1^2} = \frac{\gamma \chi_2^2}{\chi_1^2}$$

and u/γ has the F distribution with degrees of freedom $f_1 = N - 1, f_2 = N - 1$.

Similarly, $u'/\gamma' = \chi_2'^2/\chi_1'^2$ has the F distribution with the same numbers of degrees of freedom (since $N = N'$, in the present case).

If the hypothesis H_1' is true (i.e., $\gamma = \gamma'$)

$$\frac{u}{u'} = \frac{\chi_2^2 \chi_1'^2}{\chi_1^2 \chi_2'^2} = \frac{\theta_1'\theta_2}{\theta_1\theta_2'} = \frac{z_1}{z_2}, \tag{19}$$

where $\theta_i(-\frac{1}{2}\chi_i^2)$ or θ_1' is distributed as

$$\frac{1}{\Gamma(a_i)}\,\theta_i^{a_i-1}e^{-\theta_i}\,d\theta_i, \tag{20}$$

with $a_i = \frac{1}{2}(N - 1)$, and $z_1(= \theta_1'\theta_2)$, $z_2(= \theta_1\theta_2')$ follow independently the Wilks' z-distribution, [14], which we shall study in detail for the present case.

Distribution of z when $p = 2$: Consider

$$z = B\theta_1\theta_2 \cdots \theta_p .$$

Wilks has succeeded in integrating the distribution of z for the case $p = 2$ for special values of a's, e.g., $a_1 = \frac{1}{2}(N - 1)$, $a_2 = \frac{1}{2}(N - 2)$. Now we want the distribution of z when $p = 2$ and for any values of a, and then for $a_1 = a_2 = \frac{1}{2}(N - 1)$.

By (20) the joint distribution of θ_1 and θ_2 is

$$\frac{1}{\Gamma(a_1)\Gamma(a_2)}\,\theta_1^{a_1-1}e^{-\theta_1}\theta_2^{a_2-1}e^{-\theta_2}\,d\theta_1\,d\theta_2.$$

Applying the transformation $z = B\theta_1\theta_2$, $v_1 = \theta_1$, the joint distribution of v_1, z is

$$\frac{1}{\Gamma(a_1)\Gamma(a_2)}\,v_1^{a_1-1}e^{-v_1}\left(\frac{z}{Bv_1}\right)^{a_2-1}e^{-z/Bv_1}\,\frac{dv_1\,dz}{Bv_1}.$$

Integrating v_1 from $v_1 = 0$ to $v_1 = \infty$, we have the distribution of z, viz.,

$$\frac{z^{a_2-1}\,dz}{B^{a_2}\Gamma(a_1)\Gamma(a_2)}\int_0^\infty v^{a_1-a_2-1}e^{-v_1-z/Bv_1}\,dv_1. \tag{21}$$

In order to evaluate the integral of (20), consider the transformation $v_1 = y^2$, $dv_1 = 2y\,dy$, we have

$$I_0 = 2\int_0^\infty y^{2(a_1-a_2)-1}e^{-y^2-z/By^2}\,dy. \tag{22}$$

To evaluate I_0 for any a's, by putting $y = 1/x$, $dy = -dx/x^2$, we have

$$I_0 = 2\int_0^\infty \frac{e^{-zx^2/B-1/x^2}}{x^{2(a_1-a_2)+1}}\,dx. \tag{23}$$

Consider

$$\frac{\Gamma(a_1 - a_2 + \frac{1}{2})}{x^{2(a_1-a_2)+1}} = \int_0^\infty e^{-x^2y}y^{a_1-a_2-\frac{1}{2}}\,dy. \tag{24}$$

Then

$$I_0\Gamma(a_1 - a_2 + \tfrac{1}{2}) = 2\int_0^\infty e^{-(zx^2/B+1/x^2)}\,dx \int_0^\infty e^{-x^2y}y^{a_1-a_2-\frac{1}{2}}\,dy$$
$$= 2\int_0^\infty y^{a_1-a_2-\frac{1}{2}}\,dy \int_0^\infty e^{-[(z/B+y)x^2+1/x^2]}\,dx$$
$$= \sqrt{\pi}\int_0^\infty e^{-2\sqrt{z/B+y}}\,y^{a_1-a_2-\frac{1}{2}}\,\frac{dy}{\sqrt{z/B+y}}.$$

Since by the substitution $\sqrt{\frac{z}{B}+y} = \sqrt{\frac{z}{B}}+y$ or $y = x^2 + 2\sqrt{\frac{z}{B}}x$, $dy = 2\left(x + \sqrt{\frac{z}{B}}\right)dx$ and therefore

$$I_0\Gamma(a_1 - a_2 + \tfrac{1}{2}) = 2\sqrt{\pi}\int_0^\infty e^{-2(\sqrt{z/B}+x)}\left(x^2 + 2x\sqrt{\frac{z}{B}}\right)^{a_1-a_2-\frac{1}{2}}dx,$$

$$(25)\qquad I_0 = \frac{\sqrt{\pi}\,e^{-2\sqrt{z/B}}}{\Gamma(a_1 - a_2 + \frac{1}{2})}\int_0^\infty e^{-2(\sqrt{z/B}+x)}\left(x^2 + 2x\sqrt{\frac{z}{B}}\right)^{a_1-a_2-\frac{1}{2}}dx.$$

Hence, z is distributed as

$$(26)\qquad \frac{2\sqrt{\pi}\,z^{a_2-1}e^{-2\sqrt{z/B}}}{B^{a_2}\Gamma(a_1)\Gamma(a_2)\Gamma(a_1 - a_2 + \frac{1}{2})}\int_0^\infty e^{-2x}\left(2\sqrt{\frac{z}{B}} + x\right)^{a_1-a_2-\frac{1}{2}}x^{a_1-a_2-\frac{1}{2}}\,dx.$$

We infer from this distribution that when $2(a_1 - a_2)$, i.e., the difference of degrees of freedom, is odd, the integral can be expressed as a terminated series; but for even values of $2(a_1 - a_2)$, the series is infinite.

When $B = \frac{1}{A}$, $a_1 = \frac{1}{2}(N - 1)$, $a_2 = \frac{1}{2}(N - 2)$, (26) is reduced to

$$(27)\qquad \frac{\sqrt{\pi}\,A^{a_2}z^{a_2-1}e^{-2\sqrt{Az}}}{\Gamma(a_1)\Gamma(a_2)},$$

which is Wilks' ξ distribution, [15], for $p = 2$.

When $B = 1$ and $a_1 = a_2 = \frac{1}{2}(N - 1)$, it becomes

$$(28)\qquad \frac{2\sqrt{\pi}\,z^{a_2-1}e^{-2\sqrt{z}}}{\Gamma(a_1)\Gamma(a_2)}\int_0^\infty e^{-2x}(2\sqrt{z} + x)^{-\frac{1}{2}}x^{-\frac{1}{2}}\,dx,$$

which is the distribution of z involved in (19).

Since (28) can apparently not be simplified, I have been unable thus far to find in manageable form the distribution of the ratio z_1/z_2 and therefore of u/u' in this case. However, it would be simpler to use the alternative criterion $w = z - z'$ for the hypothesis H_1. The distribution of w will be taken up in a later section.

The distribution of v/v': Since Ns_Y^2/σ_Y^2 and $\Sigma X^2/\sigma_X^2$ have independently the χ^2 distribution with $N - 1$ and N degrees of freedom respectively, therefore,

$$v = \frac{NS_Y^2}{\Sigma X^2} = \frac{\sigma_Y^2 \chi_2^2}{\sigma_X^2 \chi_1^2} = \frac{\gamma \chi_2^2}{\chi_1^2},$$

and $\frac{v}{\gamma} \Big/ \frac{N-1}{N}$ has the F-distribution with $f_1 = N - 1$ degrees of freedom and $f_2 = N$.

Similarly $\frac{v'}{\gamma'} \Big/ \frac{N-1}{N}$ has the F-distribution with degrees of freedom f_1 and f_2 as above.

If the hypothesis H_2 is true (i.e., $\gamma = \gamma'$),

$$\frac{v}{v'} = \frac{\chi_2^2 \chi_1'^2}{\chi_1^2 \chi_2'^2} = \frac{\theta_1' \theta_2}{\theta_1 \theta_2'} = \frac{z_1}{z_2},$$

where each θ_i is distributed as in (19), but with $a_1 = \frac{1}{2}N$ and $a_2 = \frac{1}{2}(N - 1)$. We can infer from (27) that $t_1 = 4\sqrt{z_1}$ and $t_2 = 4\sqrt{z_2}$ have independently the χ^2-distribution each with $4a_2$ or $2(N - 1)$ degrees of freedom, and $t_1/t_2 = \sqrt{z_1/z_2} = \sqrt{v/v'}$ follows the F-distribution with degrees of freedom $f_1 = f_2 = 2(N - 1)$. The 5% and 1% points of the $F = v/v'$ may be obtained from Snedecor's table ([12], p. 174).

5. **The distribution of** $y = \log z$. Wald [13] has suggested that the distribution of $z = B\theta_1\theta_2 \cdots \theta_p$ for any a_i's $(i = 1, \cdots, p)$ may also be obtained indirectly with the aid of the characteristic function. A similar method has been applied in a recent paper by Wald and Brookner [14]. Consider the transformation

$$y = \log t = \log B\theta_1\theta_2 \cdots \theta_p. \tag{29}$$

The characteristic function of y is

$$\begin{aligned} \varphi_y(t) &= E(e^{ty}) = E\{(B\theta_1\theta_2 \cdots \theta_p)^t\} \\ &= \frac{B^t \Gamma(a_1 + t)\Gamma(a_2 + t) \cdots \Gamma(a_p + t)}{\Gamma(a_1)\Gamma(a_2) \cdots \Gamma(a_p)}. \end{aligned} \tag{30}$$

Thus the distribution $f(y)\,dy$ is given by

$$f(y) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{-ty} \varphi_y(t)\,dt = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} B^t e^{-ty} \prod_{i=1}^{p} \frac{\Gamma(a_i + t)}{\Gamma(a_i)}\,dt. \tag{31}$$

Without loss of generality, we may take $a_1 \geqq a_2 \geqq \cdots \geqq a_p > 0$ and let $a_p + t \doteq -t'$, then

$$f(y) = \frac{c_p}{2\pi i} \int_{-a_p - i\infty}^{-a_p + i\infty} e^{yt'} B^{-t'} \prod_{i=1}^{p} \Gamma(a_i - a_p - t')\,dt', \tag{32}$$

where $c_p = e^{a_p y} B^{-a_p} \Big/ \prod_{i=1}^{p} \Gamma(a_i)$.

The integration can be carried out by the method of residue along the contour C, bounded by the line $x = -a_p$ and that part of the circle with center at origin and radius r, which lies to the right of the line $x = -a_p$. The integral of the function $e^{t'y}B^{-t'}\prod_{i=1}^{p}\Gamma(a_i - a_p - t')$ along the arc converges to zero as the radius of the circle tends to infinity (Kullback, [8]). Hence the integrals along the vertical line $x + a_p = 0$ and along the closed contour C are equal. Then we may write

$$f(y) = -\frac{c_p}{2\pi i}\int_C e^{yt'}B^{-t'}\prod_{i=1}^{p}\Gamma(a_i - a_p - t')\,dt', \tag{33}$$

and its value is c_p times the sum of the residues at the poles within the contour C.

For the present purpose, $p = 2$, we have

$$f(y) = \frac{c_2}{2\pi i}\int_{-a_2+i\infty}^{-a_2+i\infty} e^{yt'}\Gamma(a_1 - a_2 - t')\Gamma(-t')\,dt'. \tag{34}$$

We shall study the integral of (34) in more detail in the following cases:

(i) $a_1 - a_2 = \frac{1}{2}$. By the duplication formula

$$\Gamma(\tfrac{1}{2} - t')\Gamma(-t') = 2^{1+2t'}\sqrt{\pi}\,\Gamma(-2t'),$$

and the function

$$\Gamma(-2t') = \lim_{N\to\infty}\frac{N!\,N^{-2t'}}{(-2t')(-2t'+1)\cdots(-2t'+N)},$$

has simple poles at the points $0, \frac{1}{2}, 1, 3/2, \cdots$. The residue at $t' = m/2$, where m is zero or a positive integer, is $(-1)^{m+1}/2\cdot m!$ and (34) becomes

$$\begin{aligned} f(y) &= \sqrt{\pi}\,c_2\left(1 - 2e^{\frac{1}{2}y} + \frac{1}{2!}2^2e^{y} - \frac{1}{3!}2^3e^{3y/2} + \cdots\right) \\ &= \sqrt{\pi}\,c_2e^{-2e^{\frac{1}{2}y}}. \end{aligned} \tag{35}$$

The distribution of $z = e^y$ is

$$\frac{2\sqrt{\pi}\,z^{a_2-1}e^{-2\sqrt{z}}}{\Gamma(a_1)\Gamma(a_2)}\,dz. \tag{27 bis}$$

(ii) $a_1 - a_2 = m + \frac{1}{2}$. The function

$$\begin{aligned} \Gamma(a_1 - a_2 - t')\Gamma(-t') &= (m - \tfrac{1}{2} - t')(m - \tfrac{3}{2} - t')\cdots(\tfrac{1}{2} - t')\Gamma(\tfrac{1}{2} - t')\Gamma(-t') \\ &= 2^{1+2t'}\sqrt{\pi}(m - \tfrac{1}{2} - t')(m - \tfrac{3}{2} - t')\cdots(\tfrac{1}{2} - t')\Gamma(-2t') \end{aligned}$$

has simple poles at 0, m, $m + \frac{1}{2}$, $m + 1$, $\cdots$, and

$$f(y) = \sqrt{\pi}\, c_2 \left[\frac{(2m-1)!}{2^{2m-1}(m-1)!} - \frac{1}{2^m(2m)} (2^2 e^y)^m + \frac{1}{2^m(2m+1)} (2^2 e^y)^{m+\frac{1}{2}} - \frac{1}{2^m(2m+2)\cdot 2\cdot 1} (2^2 e^y)^{m+1} + \cdots \right]$$

$$= \sqrt{\pi}\, c_2 \left[\frac{(2m-1)!}{2^{2m-1}(m-1)!} - \frac{1}{2^m} \sum_{\gamma=0}^{\infty} \frac{1}{(2m+\gamma)\gamma!} (2^2 e^y)^{m+\gamma/2} \right].$$

This agrees with the expansion of (26) when we put $a_1 - a_2 - \frac{1}{2} = m$.

(iii) $a_1 - a_2 = 0$. The function

$$[\Gamma(-t')]^2 = \lim_{N\to\infty} \frac{(N!)^2 N^{-2t'}}{(-t')^2(-t'+1)^2 \cdots (-t'+N)^2},$$

has poles of the second order at the points 0, 1, 2, 3, $\cdots$ and

$$f(y) = c_2 \sum_{\gamma=0}^{\infty} \frac{d}{dt'} \{(t'-\gamma)^2 e^{t'y} [\Gamma(-t')]^2\}_{t'=\gamma}$$

(iv) $a_1 - a_2 = m$. The function

$$\Gamma(m-t')\Gamma(-t') = (m-1-t')(m-2-t') \cdots (1-t')(-t')[\Gamma(-t')]^2,$$

has finite simple poles at 1, 2, $\cdots$, $m-1$ and poles of the second order at m, $m+1$, $\cdots$, and

$$f(y) = c_2 \sum_{\gamma=0}^{m-1} \{(t-\gamma) e^{t'y} \Gamma(m-t')\Gamma(-t')\}_{t'=\gamma} + c_2 \sum_{\gamma=m}^{\infty} \left\{ \frac{d}{dt'} (t'-\gamma)^2 e^{t'y} \Gamma(m-t')\Gamma(-t') \right\}_{t'=\gamma}.$$

6. **The distribution of** $w = z - z'$ **or** $\psi = \cosh w$. Since the distribution of u is given in [7] as

$$\text{(39)} \qquad \frac{1}{\gamma \mathrm{B}[\frac{1}{2}(N-1), \frac{1}{2}(N-1)]} \left(\frac{u}{\gamma}\right)^{\frac{1}{2}N-3} \left(1 + \frac{u}{\gamma}\right)^{-(N-1)} du,$$

therefore, by transformation (17), we have that the distribution of z for a given $\zeta = \frac{1}{2}\log\gamma = \frac{1}{2}\log\frac{1+\rho}{1-\rho}$ is

$$\text{(40)} \qquad \frac{1}{\mathrm{B}\left(\frac{1}{2}, \frac{n}{2}\right)} \operatorname{sech}^n (z-\zeta)\, dz,$$

where $n = N - 1$. The distribution of z has been given by R. A. Fisher [3] for $n = 1$ and by Delury [2]. Similarly, the distribution of z' for a given ζ' is

$$\frac{1}{B\left(\frac{1}{2}, \frac{n'}{2}\right)} \operatorname{sech}^{n'} (z' - \zeta')\, dz', \tag{41}$$

where $n' = N' - 1$.

In case $n = n'$, the joint distribution of z and z' for a given common ζ is

$$C \operatorname{sech}^n (z - \zeta) \operatorname{sech}^n (z' - \zeta)\, dz\, dz' = \frac{C\, dz\, dz'}{\cosh^n (z - \zeta) \cosh^n (z' - \zeta)}, \tag{42}$$

where $1/C = \left[B\left(\frac{1}{2}, \frac{n}{2}\right)\right]^2$.

By the transformation $\bar{z} = \frac{1}{2}(z + z')$, $w = z - z'$, we have the joint distribution of $\bar{z}$ and w,

$$\frac{C\, d\bar{z}\, dw}{[\cosh^n (z - \zeta) \cosh^n (z' - \zeta)]} = \frac{2^n C\, d\bar{z}\, dw}{[\cosh 2(\bar{z} - \zeta) + \cosh w]^n}. \tag{43}$$

Integrating with respect to $\bar{z}$ from $-\infty$ to ∞, we have

$$\begin{aligned} 2^n C\, dw \int_{-\infty}^{\infty} \frac{d\bar{z}}{[\cosh 2(\bar{z} - \zeta) + \cosh w]^n} &= 2^n C\, dw \int_0^{\infty} \frac{2\, d\bar{z}}{[\cosh 2(\bar{z} - \zeta) + \cosh w]^n} \\ &= 2^n C\, dw I_n, \text{ say.} \end{aligned} \tag{44}$$

Applying the transformation $\phi = 2(\bar{z} - \zeta)$, $\psi = \cosh w$, the integral of (34) becomes

$$I_n = \int_0^{\infty} \frac{d\phi}{(\cosh \phi + \psi)^n}.$$

Substituting $\cosh \phi + \psi = \dfrac{1 + \psi}{\theta}$, we have

$$\begin{aligned} I_n &= \int_0^1 \left(\frac{\theta}{1 + \psi}\right)^n \frac{1}{\theta} \frac{d\theta}{\sqrt{\left(1 - \frac{\psi - 1}{\psi + 1}\theta\right)(1 - \theta)}} \\ &= \frac{1}{(\psi + 1)^n} \int_0^1 \theta^{n-1} (1 - \theta)^{-\frac{1}{2}} \left(1 - \frac{\psi - 1}{\psi + 1}\theta\right)^{-\frac{1}{2}} d\theta. \end{aligned} \tag{45}$$

Comparing (35) with the hypergeometric function

$$I = \int_0^1 \theta^{b-1} (1 - \theta)^{c-b-1} (1 - \theta x)^{-a}\, d\theta = \frac{\Gamma(b)\Gamma(c - b)}{\Gamma(c)} F(a, b, c, x), \tag{46}$$

we have $b = n$, $c - b = \frac{1}{2}$, $a = \frac{1}{2}$, and therefore (35) can be expressed in terms of a hypergeometric series as

$$I_n = \frac{\Gamma(n)\Gamma(\frac{1}{2})}{\Gamma(n + \frac{1}{2})} \frac{1}{(\psi + 1)^n} F\left(\tfrac{1}{2}, n, n + \tfrac{1}{2}, \frac{\psi - 1}{\psi + 1}\right). \tag{47}$$

The series (37) is convergent since $\frac{\psi - 1}{\psi + 1}$ is less than unity. Thus the distribution of w, from (34), is

$$\frac{2^n C\Gamma(n)\Gamma(\frac{1}{2})}{\Gamma(n + \frac{1}{2})} \frac{1}{(\cosh w + 1)^n} F\left(\tfrac{1}{2}, n, n + \tfrac{1}{2}, \frac{\cosh w - 1}{\cosh w + 1}\right) dw, \tag{48}$$

and the distribution of $\psi = \cosh w$ is

$$\frac{2^{n+1} C\Gamma(n)\Gamma(\frac{1}{2})}{\Gamma(n + \frac{1}{2})} \frac{1}{(\psi + 1)^{n+\frac{1}{2}}(\psi - 1)^{\frac{1}{2}}} F\left(\tfrac{1}{2}, n, n + \tfrac{1}{2}, \frac{\psi - 1}{\psi + 1}\right) d\psi. \tag{49}$$

We notice that the distribution of ψ expressed in (39) is very similar to the r-distribution expressed in terms of hypergeometric series, except that in the first case the argument is $\frac{\psi - 1}{\psi + 1}$, while in the second case it is $\frac{1 - p}{1 + p}$ where $p = \rho r$. Hotelling [5] has obtained a very rapidly convergent hypergeometric series for the distribution of the correlation coefficient since $| p | < 1$. But for the distribution of ψ, we cannot obtain a more rapidly convergent series than (39), since the values of ψ lie between 1 and ∞.

7. Summary and remark. Two hypotheses concerning the comparison of correlation coefficients of two samples from bivariate normal populations have been considered. The appropriate test criteria for each hypothesis have been derived by the use of a transformation of the variates. The distributions of certain of the criteria have been obtained in the special case where $N = N'$. Incidentally the distribution of Wilks' z for $p = 2$ and any values of a_1 and a_2 has been derived.

Again though we assume throughout the paper that $\sigma_1 = \sigma_2$ and $\sigma_1' = \sigma_2'$, the tests can be generalized to fit the case where the ratios $\sigma_1/\sigma_2 = k$, $\sigma_1'/\sigma_2' = k'$ are known, but are different from unity. In the latter case we can apply the transformation

$$y_1 = w_1 x_1, \qquad y_2 = w_2 x_2;$$
$$y_1' = w_1' x_1', \qquad y_2' = w_2' x_2';$$

where

$$w_1 k_1 = w_2 k_2 = 1, \qquad w_1' k_1' = w_2' k_2' = 1,$$

so that after transformation the variances of each pair of y's are equal.

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