

CORRECTIONS TO A PAPER ON THE UNIQUENESS PROBLEM OF MOMENTS

BY M. G. KENDALL

London, England

I wish to make certain corrections in my paper on "Conditions for Uniqueness in the Problem of Moments" (*Annals of Math. Stat.*, Vol. 11 (1940), p. 402). I thought I had succeeded in improving on results given earlier by Stieltjes, Lévy and Carleman, but this is not so.

Theorem 1 of the paper stated that a set of moments determines a distribution uniquely if $\sum_{r=0}^{\infty} \frac{\nu_r t^r}{r!}$ converges for some real non-zero t , ν_r being the absolute moment of order r . This is true, and a similar result has been proved by Lévy, but my proof contained a small lacuna. It was shown that the characteristic function $\phi(t)$ has a Taylor expansion which, under the conditions of the theorem, is convergent; but it has also to be shown that it is equal to the sum of that expansion. This may be seen as follows:

We have

$$e^{itx} - \sum_{r=0}^{n-1} \frac{(itx)^r}{r!} \leq \theta \frac{|tx|^n}{n!}, \quad |\theta| \leq 1,$$

and hence, on taking mean values,

$$\left| \phi(t) - \sum_{r=0}^{n-1} \frac{(it)^r \mu_r}{r!} \right| \leq \frac{\nu_n t^n}{n!}.$$

Since by hypothesis $\frac{\nu_n t^n}{n!} \rightarrow 0$, $\phi(t)$ must be equal to the sum of its (convergent) Taylor expansion.

The principal error was a statement that $\nu_n^{1/n}/n$ must either tend to a limit or diverge. For this reason, the second theorem should run: a distribution determines a distribution uniquely if $\lim \nu_n^{1/n}/n$ is finite (not $\lim \nu_n^{1/n}/n$ as originally stated). Theorem 3 should also be restated with the upper limit substituted for the limit therein.

Theorem 4 stated that a set of moments uniquely determines a distribution if $\sum \frac{1}{\nu_n^{1/n}}$ diverges. A rigorous proof is as follows:

The characteristic function obeys the relation

$$|\phi^{(n)}(t)| \leq \nu_n, \quad n > 1$$

provided, of course, that ν_n exists. A theorem of Denjoy¹ states that if a function $f(x)$, defined in the segment (a, b) , possesses derivatives of all orders therein,

¹Arnaud Denjoy, "Sur les fonctions quasi-analytiques de variable réelle," *Comptes Rendus* Vol. 173 (1921), p. 1399.

if M_n is the maximum of $|f^{(n)}(x)|$ in the segment and if $\sum \frac{1}{M_n^{1/n}}$ is divergent, then $f(x)$ is completely determined by its value and that of its derivatives at a single point. $\phi(t)$ obeys the conditions of the theorem and by taking the point to be $t = 0$, theorem 4 follows.

I hope that this note will correct any misunderstandings that may have arisen on the main paper, and I regret that a number of circumstances, not the least of which is war, have made it impossible to forward the correction at an earlier date.

ANNOUNCEMENT CONCERNING COMPUTATION OF MATHEMATICAL TABLES

In the December, 1939, issue of the *Annals of Mathematical Statistics*, p. 399, there appeared an Announcement of the *Mathematical Tables Project*. This project is operated by the Work Projects Administration of New York City, as O. P. No. 265-2-97-11 under the technical supervision of Dr. A. N. Lowan. It is sponsored by the National Bureau of Standards, Dr. Lyman J. Briggs, Director.

In order to keep the readers of the *Annals* up-to-date on the progress of the work of the Project, information will be released from time to time.

The following list shows the status of work, as of October, 1941. The reader is referred to the December, 1939 issue of the *Annals* with respect to which n will denote the n^{th} item of Tables Published, Pn will denote the n^{th} item of Tables in Progress and Cn will denote the n^{th} item of Tables under Consideration.

Tables published. 1, 2, 3, $P1$, $P2$, $P3$, $P4$, $P6(b)$, $P6(c)$, $P6(d)$, $P6(e)$, $P7$, $C7$ and also

1. Table of Five-Point Lagrangian Interpolants for arguments ranging between 0 and 2 at intervals of 0.001.

2. Tables of Grid Coordinates (American Polyconic Projection) at 5 minute intervals of latitude and longitude for latitude from 70°N to 28°N and for latitude from 49°N to 72°N .

3. Table for Map Projections of Northwestern Extension of U. S.

Tables in process of reproduction. $P5$, $P6(a)$, $P8$ and $C1$ for $[0 (.001) 7 (.01) 50 (.1) 300 (1) 2,000 (10) 10,000; 12D]$ also

1. Tables of Section Moduli and Moments of Inertia for Structural Members used in Naval Architecture. (For the Bureau of Marine Inspection and Navigation.)

2. Tables of $Si(x)$ and $Ci(x)$ for x ranging from 10 to 100 at intervals of 0.001.