3. The special case. Let us now return to the special case mentioned at the end of 1—the application to the mean square successive difference.

There $p = 1$ and $B = (0)$, so that the "distribution" of ζ is concentrated at the point 0. Hence $\omega_B(\zeta)$ is an "improper" distribution, concentrated in the same way.[3] Using C and A as described at the end of 1, the above formula becomes (now $m = n - 2$, $p = 1$)

$$\text{(II)} \qquad \omega_{A+(0)}(\theta) = \frac{\Gamma[\frac{1}{2}(n-1)]}{\Gamma[\frac{1}{2}(n-2)]\Gamma[\frac{1}{2}]} \int_0^1 d\rho \cdot \omega_A\left(\frac{\theta}{\rho}\right) \rho^{\frac{1}{2}n-3}(1-\rho)^{-\frac{1}{2}}.$$

It would have been equally easy, of course, to establish (II) directly.

Putting $\rho = 1/t$ gives

$$\text{(III)} \qquad \omega_{A+(0)}(\theta) = \frac{\Gamma[\frac{1}{2}(n-1)]}{\Gamma[\frac{1}{2}(n-2)]\Gamma[\frac{1}{2}]} \int_1^\infty dt \cdot \omega_A(\theta t) t^{-\frac{1}{2}(n-3)}(t-1)^{-\frac{1}{2}}.$$

Since $\omega_A(\gamma)$ vanishes for $|\gamma| > \cos(\pi/n)$, we may replace this integral $\int_1^\infty$ by $\int_1^{\cos(\pi/n)/|\theta|}$

Formula (III) can be used for numerical work, and also to extend the formula (3) on p. 391, loc. cit., to even values of n.

CONVEXITY PROPERTIES OF GENERALIZED MEAN VALUE FUNCTIONS

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In an article appearing in the *Annals of Mathematical Statistics*[1] it was pointed out that while the mean value functions appearing below have been studied and used since 1840, there appeared to have been no attempt made to investigate the behavior of their second derivatives.

Consider (1) the unit weight or simple sample form

$$\varphi(t) \equiv \left(\frac{x_1^t + x_2^t + \cdots + x_n^t}{n}\right)^{1/t},$$

in which the x_i are positive numbers and in which t may take any real value; (2) the weighted sample form

$$\omega(t) \equiv \left(\frac{c_1 x_1^t + c_2 x_2^t + \cdots + c_n x_n^t}{c_1 + c_2 + \cdots + c_n}\right)^{1/t},$$

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[3] Dirac's famous "delta function." It could be described by a Stieltjes integral.

[1] Nilan Norris, "Convexity properties of generalized mean value functions," *Annals of Math. Stat.*, Vol. 8 (1937), pp. 118-120.

in which the c_i are positive numbers, and in which the x_i and t are restricted as in $\varphi(t)$; and (3) the integral form

$$\theta(t) \equiv \left(\frac{1}{x_2 - x_1} \int_{x_1}^{x_2} [f(x)]^t \, dx\right)^{1/t},$$

in which $f(x)$ is a positive continuous function for $x_1 \leqq x \leqq x_2$.

Since the analysis and results are essentially the same in all three cases, we restrict our attention to $\theta(t)$.

As is well known,[2] $\theta(t)$ is a monotone non-decreasing function which varies from the minimum of $f(x)$ to the maximum of $f(x)$ as t increases from $-\infty$ to $+\infty$. It is further of some importance to study the rate at which the rate of increase of this type bias is changing as t increases; the rate in question is given by the second derivative $\theta''(t)$.

The following points were made by Norris, loc. cit.: (1) Since, as we have pointed out, $\theta(t)$ has two horizontal asymptotes, $\theta(t)$ must have at least one inflection point. (2) Consideration of a simple example shows that there is not necessarily an inflection point at $t = 0$; $\theta''(0)$ can be made to take on any real value.

Thus it is *not* true that $\theta''(t)$ must be positive for all $t < 0$ and negative for all $t > 0$. On the other hand, we shall give simple bounds for $\theta''(t)$ in the other direction; namely, we shall give a positive upper bound of $\theta''(t)$ for $t < 0$ and a lower bound for $t > 0$. These bounds are precise in the sense that they are actually taken on in the special case $f(x) \equiv$ const. Their main advantage lies in the fact that while the expression for $\theta''(t)$ is quite involved, these bounds are simple expressions in the quantities $\theta(t)$ and $\theta'(t)$ which might already have been computed.

Let

$$\lambda(t) \equiv \log \theta(t).$$

Differentiating, we obtain

$$t^2\lambda'(t) \equiv t^2 \frac{\theta'(t)}{\theta(t)} = \frac{\int_{x_1}^{x_2} [f(x)]^t \log [f(x)]^t \, dx}{\int_{x_1}^{x_2} [f(x)]^t \, dx} - \log \left(\frac{1}{x_2 - x_1} \int_{x_1}^{x_2} [f(x)]^t \, dx\right).$$

It follows[3] that

$$\lambda'(t) \geqq 0$$

and

$$\theta'(t) \geqq 0.$$

Let

$$\mu(t) \equiv t^2\lambda'(t).$$

[2] See for instance, G. Pólya und G. Szegö, *Aufgaben und Lehrsätze aus der Analysis* (Berlin, 1925), Vol. 1, pp. 54–55 and 210–211.

[3] See G. Pólya und G. Szegö, loc. cit., p. 210.

Curiously, while $\lambda''(t)$ and $\theta''(t)$ appear to be rather formidable, the closely related quantity $\mu'(t)$ is made relatively simple by the fact that two of the terms obtained by formal differentiation are negatives of each other; and Schwarz' inequality can be applied to the remaining terms, as follows.

We obtain

$$\left(\int_{x_1}^{x_2} [f(x)]^t\,dx\right)^2 \mu'(t) = t\left[\left(\int_{x_1}^{x_2} [f(x)]^t\,dx\right)\left(\int_{x_1}^{x_2} [f(x)]^t[\log f(x)]^2\,dx\right) - \left(\int_{x_1}^{x_2} [f(x)]^t \log f(x)\,dx\right)^2\right].$$

By Schwarz' inequality,[4] it follows that

$$\mu'(t) = t\pi(t),$$

with

$$\pi(t) \geqq 0,$$

the sign of equality holding if and only if $f(x) \equiv$ const.

From the definition of $\mu(t)$ we obtain

$$\mu'(t) = t[2\lambda'(t) + t\lambda''(t)] = \frac{t}{\theta(t)}\left[2\theta'(t) + t\theta''(t) - \frac{t[\theta'(t)]^2}{\theta(t)}\right],$$

whence

$$2\lambda'(t) + t\lambda''(t) = \frac{1}{\theta(t)}\left[2\theta'(t) + t\theta''(t) - \frac{t[\theta'(t)]^2}{\theta(t)}\right] = \pi(t) \geqq 0;$$

that is,

$$t\lambda''(t) \geqq -2\lambda'(t), \qquad t\theta''(t) \geqq \frac{t[\theta'(t)]^2}{\theta(t)} - 2\theta'(t).$$

It follows that for $t < 0$, we have

$$\lambda''(t) \leqq -2\lambda'(t)/t$$

and

$$\theta''(t) \leqq \frac{[\theta'(t)]^2}{\theta(t)} - \frac{2\theta'(t)}{t};$$

while for $t > 0$, we have

$$\lambda''(t) \geqq -2\lambda'(t)/t$$

and

$$\theta''(t) \geqq \frac{[\theta'(t)]^2}{\theta(t)} - \frac{2\theta'(t)}{t}.$$

[4] See G. Pólya und G. Szegö, loc. cit., p. 54.