characteristic equation of that matrix. Since A and B are real and symmetric, the roots under consideration are real. Thus Q_1 and Q_2 have independent Chi-Square distributions with r_1 and r_2 degrees of freedom respectively.

This theorem can likewise be extended to any finite number of these quadratic forms.

Of special interest is the case of, say k, quadratic forms for which the sum of the k matrices is the identity matrix. Thus $A_1 + A_2 + \cdots + A_k = I$. By Theorem 1, it is both necessary and sufficient for the mutual independence of the k forms that $A_u A_v = 0$, $u \neq v$.

Now

$$A_i = I - A_1 - \cdots - A_{i-1} - A_{i+1} - \cdots - A_j - \cdots - A_k$$

and

$$A_i A_j = A_j - A_1 A_j - \cdots - A_{i-1} A_j - A_{i+1} A_j - \cdots - A_j^2 - \cdots - A_k A_j,$$

so that $A_j = A_j^2$. In this particular case it is to be seen that the mutual independence of the forms implies that their several distributions are of the Chi-Square type.

A CHARACTERIZATION OF THE NORMAL DISTRIBUTION

By Irving Kaplansky

Harvard University

In 1925 R. A. Fisher gave a geometric derivation of the joint distribution of mean and variance in samples from a normal population (*Metron*, Vol. 5, pp. 90–104). On examining the argument however, we find that an (apparently) more general result is actually established: if $f(x_1) \cdots f(x_n)$ is a function $g(m, s)$ of the sample mean m and standard deviation s, then the probability density of m and s in samples of n from the population $f(x)$ is $g(m, s)s^{n-2}$. This condition on $f(x)$ is of course satisfied if $f(x)$ is normal; in this note we shall conversely show that for $n \geqq 3$ it characterizes the normal distribution. In the proof it will be assumed that $g(m, s)$ possesses partial derivatives of the first order, although a weaker assumption would probably suffice.

Let us for the moment restrict the variables x_i to values such that $f(x_i) > 0$. After a change of notation we have

$$\phi(x_1) + \cdots + \phi(x_n) = h(u, v),$$

where $\phi = \log f$, $u = x_1 + \cdots + x_n$, $v = \frac{1}{2}(x_1^2 + \cdots + x_n^2)$. A differentiation yields

$$\phi'(x_i) = h_u + h_v x_i .$$

Solving two of these equations for h_v, we find

$$h_v = \frac{\phi'(x_i) - \phi'(x_j)}{x_i - x_j}, \qquad (i \neq j), \tag{1}$$

and, for $n \geqq 3$, it follows that the right member of (1) is a constant, say 2A. Then

$$\phi'(x_i) - 2Ax_i = \phi'(x_j) - 2Ax_j = a \text{ constant } B.$$

$$\phi(x) = Ax^2 + Bx + C.$$

We now have $f(x) = e^{\phi(x)}$ whenever $f(x) > 0$; but since $f(x)$ is continuous, this implies $f(x) = e^{\phi(x)}$ everywhere.