ON THE DISTRIBUTION OF THE RADIAL STANDARD DEVIATION

By Frank E. Grubbs[1]

Aberdeen Proving Ground

1. Introduction. Of interest in the field of ballistics is a measure of the accuracy of bullets. In acceptance tests of small arms ammunition lots, for example, a sample of rounds from each lot is fired from a fixed rifle at a vertical target placed a specified distance from the rifle. The accuracy of the bullets is taken to be some measure of the scattering (or lack of scattering) of the bullet holes on the target. The purpose of such a test would be to determine whether or not the lot under consideration differs significantly in accuracy from (a) standard values or (b) its predecessors.

One useful measure of accuracy is the radial standard deviation which is defined by the relation

$$Z = \sqrt{\frac{1}{N}\{\Sigma(x_i - \bar{x})^2 + \Sigma(y_i - \bar{y})^2\}}, \tag{1}$$

where x_i and y_i are respectively the abscissa and ordinate of any point measured from an arbitrary origin and N is the sample size.

It will be the purpose of the present discussion to call attention to a series expansion for the distribution of the statistic Z in samples of N assuming that the distribution of all rounds of the lot on the target follow the bivariate normal population law

$$f(x, y) = \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{x^2}{2\sigma_1^2} - \frac{y^2}{2\sigma_2^2}}, \qquad (x \text{ and } y \text{ statistically independent}) \tag{2}$$

where σ_1^2 and σ_2^2 are the parent variances of x and y respectively. In the above probability density function, the population means are taken to be zero since the statistic Z is quite independent of the origin selected.

2. Moment generating function of Z^2. The distribution of $s_1^2 = \frac{1}{N}\Sigma(x_i - \bar{x})^2$ in samples of N from a normal population is given by the well-known law,

$$dF(s_1^2) = \frac{\frac{N}{2\sigma_1^2}}{\Gamma\left(\frac{N-1}{2}\right)} \left(\frac{Ns_1^2}{2\sigma_1^2}\right)^{\frac{1}{2}(N-3)} e^{-\frac{Ns_1^2}{2\sigma_1^2}} ds_1^2, \qquad s_1^2 \geq 0. \tag{3}$$

The moment generating function of s_1^2 may be found (in a neighborhood of $t = 0$) by straightforward integration:

$$M_{s_1^2}(t) = E(e^{s_1^2 t}) = \int_0^\infty e^{s_1^2 t} dF(s_1^2) = \left\{1 - \frac{2\sigma_1^2 t}{N}\right\}^{-\frac{1}{2}(N-1)} \tag{4}$$

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[1] Captain, Ordnance Department, Ballistic Research Laboratory, Aberdeen Proving Ground, Md.

Likewise, for $s_2^2 = \frac{1}{N} \Sigma(y_i - \bar{y})^2$, we have

$$M_{s_2^2}(t) = \left\{1 - \frac{2\sigma_2^2 t}{N}\right\}^{-\frac{1}{2}(N-1)} \tag{5}$$

Now $M_{Z^2}(t) = M_{s_1^2+s_2^2}(t) = E\{e^{s_1^2t+s_2^2t}\} = E(e^{s_1^2t})\cdot E(e^{s_2^2t})$ since x and y are independent. Thus,

$$M_{Z^2}(t) = M_{s_1^2}(t)\cdot M_{s_2^2}(t) = \left\{1 - \frac{2\sigma_1^2 t}{N}\right\}^{-\frac{1}{2}(N-1)}\left\{1 - \frac{2\sigma_2^2 t}{N}\right\}^{-\frac{1}{2}(N-1)}$$

3. Distribution function of Z^2. Making use of the Fourier theorem, we have

$$f(Z^2) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{1 - \frac{2\sigma_1^2 it}{N}\right\}^{-\frac{1}{2}(N-1)}\left\{1 - \frac{2\sigma_2^2 it}{N}\right\}^{-\frac{1}{2}(N-1)} e^{-iZ^2t}\,dt, \tag{6}$$

at all points of continuity of $f(Z^2)$.

The discussion will be divided preferably into the two cases: Case I: $\sigma_1^2 = \sigma_2^2$, and Case II: $\sigma_1^2 \neq \sigma_2^2$.

Case I: $\sigma_1^2 = \sigma_2^2 = \sigma^2$.

In this case the distribution of Z^2 reduces to

$$f(Z^2) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{1 - \frac{2\sigma^2 it}{N}\right\}^{-(N-1)} e^{-iZ^2t}\,dt. \tag{7}$$

It will simplify the algebra to find first the distribution of $u^2 = \frac{NZ^2}{2\sigma^2}$ and then that of Z^2. Since $M_{u^2}(t) = \{1 - t\}^{-(N-1)}$,

$$f(u^2) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\{1 - it\}^{-(N-1)} e^{-iu^2t}\,dt. \tag{8}$$

This integral may be evaluated easily by the calculus of residues since the integrand has only a single pole of order $(N - 1)$ at $t = -i$. We will, however, make use of the following method.

Put $-v = u^2 - iu^2t$; then

$$\begin{aligned} f(u^2) &= \frac{1}{2\pi}\int_{-u^2-i\infty}^{-u^2+i\infty}\left(-\frac{v}{u^2}\right)^{-(N-1)} e^{-v-u^2}\frac{dv}{iu^2} \\ &= \frac{-e^{-u^2}(u^2)^{N-2}}{2\pi i}\int_{-u^2+i\infty}^{-u^2-i\infty} e^{-v}(-v)^{-(N-1)}\,dv. \end{aligned} \tag{9}$$

The integral in the last expression is Hankel's integral [1]; namely,

$$\frac{1}{\Gamma(Z)} = \frac{i}{2\pi}\int_{-a+i\infty}^{-a-i\infty} e^{-t}(-t)^{-Z}\,dt, \qquad R(Z) > 0, \qquad a > 0. \tag{10}$$

Therefore $$f(u^2) = \frac{1}{\Gamma(N-1)} e^{-u^2}(u^2)^{N-2},$$

and $$dF(Z^2) = \frac{\frac{N}{2\sigma^2}}{\Gamma(N-1)} e^{-\frac{NZ^2}{2\sigma^2}} \left(\frac{NZ^2}{2\sigma^2}\right)^{N-2} dZ^2; \text{ from which}$$

$$dF(Z) = \frac{2\left(\frac{N}{2\sigma^2}\right)^{N-1}}{\Gamma(N-1)} e^{-\frac{NZ^2}{2\sigma^2}} Z^{2N-3}\, dZ. \tag{11}$$

(Note that $f(Z)$ is continuous over $0 \leq Z \leq \infty$.)

This expected result has been obtained by Reno and Mowshowitz [2] who employed an extension of the famous Helmert distribution.

Actually, the result is an obvious one and may be argued as follows: Ns_1^2/σ^2 is distributed as χ^2 with $N - 1$ degrees of freedom and Ns_2^2/σ^2 is also distributed as χ^2 with $N - 1$ degrees of freedom. Hence, the statistic $\frac{N}{\sigma^2}(s_1^2 + s_2^2)$ is, from the additive property of χ^2, distributed like χ^2 with $2N - 2$ degrees of freedom.

We now turn to the general

Case II: $\sigma_1^2 \neq \sigma_2^2$

No generality will be lost by taking $\sigma_1^2 < \sigma_2^2$. In fact, the present attack will hold with obvious modifications provided $\sigma_1^2 < 2\sigma_2^2$.

Recall that

$$f(Z^2) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{1 - \frac{2\sigma_1^2 it}{N}\right\}^{-\frac{1}{2}(N-1)}\left\{1 - \frac{2\sigma_2^2 it}{N}\right\}^{-\frac{1}{2}(N-1)} e^{-iZ^2t}\, dt, \tag{12}$$

at all continuity points of $f(Z^2)$.

In a manner analogous to that employed by Hsu [3], we replace

$$\left(1 - \frac{2\sigma_2^2 it}{N}\right) \text{ by } \frac{\sigma_2^2}{\sigma_1^2}\left(1 - \frac{2\sigma_1^2 it}{N}\right)\left\{1 - \frac{1 - \sigma_1^2/\sigma_2^2}{1 - 2\sigma_1^2 it/N}\right\}.$$

Further, since

$$\left|\frac{1 - \sigma_1^2/\sigma_2^2}{1 - 2\sigma_1^2 it/N}\right| < 1,$$

we may write

$$\left\{1 - \frac{2\sigma_2^2 it}{N}\right\}^{-\frac{1}{2}(N-1)} = \left(\frac{\sigma_1^2}{\sigma_2^2}\right)^{\frac{1}{2}(N-1)}\left(1 - \frac{2\sigma_1^2 it}{N}\right)^{-\frac{1}{2}(N-1)} \sum_{r=0}^{\infty} \frac{\Gamma\left(\frac{N-1}{2} + r\right)}{\Gamma\left(\frac{N-1}{2}\right)\Gamma(r+1)} \cdot \frac{\left(1 - \frac{\sigma_1^2}{\sigma_2^2}\right)^r}{\left(1 - \frac{2\sigma_1^2 it}{N}\right)^r}.$$

Thus,

$$(13)\qquad f(Z^2) = \frac{\left(\frac{\sigma_1^2}{\sigma_2^2}\right)^{\frac{1}{2}(N-1)}}{2\pi} \int_{-\infty}^{\infty} \sum_{r=0}^{\infty} \frac{\left(1 - \frac{\sigma_1^2}{\sigma_2^2}\right)^r}{r\beta\left(\frac{N-1}{2}, r\right)} \left\{1 - \frac{2\sigma_1^2 it}{N}\right\}^{-(N+r-1)} e^{-iZ^2t}\, dt,$$

with the understanding that $r\beta\left(\frac{N-1}{2}, r\right) = 1$ for $r = 0$.

We note that the moduli of the terms of the above series are for all t not greater than the corresponding terms of the following convergent series of positive terms:

$$\sum_{r=0}^{\infty} \frac{\left(1 - \frac{\sigma_1^2}{\sigma_2^2}\right)^r}{r\beta\left(\frac{N-1}{2}, r\right)}.$$

Therefore, uniform convergence over $(-\infty, \infty)$ is established. To show that we may integrate over the infinite interval term by term, we observe that $| S(t) - S_r(t) | \leq \epsilon\varphi(t)$ for all t and all large r, where

$$S(t) = \left\{1 - \frac{2\sigma_1^2 it}{N}\right\}^{-\frac{1}{2}(N-1)} \left\{1 - \frac{2\sigma_2^2 it}{N}\right\}^{-\frac{1}{2}(N-1)},$$

$S_r(t)$ = the sum of the first $r + 1$ terms of the series, and the function $\varphi(t) = \left|1 - \frac{2\sigma_1^2 it}{N}\right|^{-2}$ which is integrable over $(-\infty, \infty)$. That is, $S_r(t)$ converges to $S(t)$ uniformly relative to $\varphi(t)$.[2] Hence,

$$(14)\qquad f(Z^2) = \frac{\left(\frac{\sigma_1^2}{\sigma_2^2}\right)^{\frac{1}{2}(N-1)}}{2\pi} \sum_{r=0}^{\infty} \frac{\left(1 - \frac{\sigma_1^2}{\sigma_2^2}\right)^r}{r\beta\left(\frac{N-1}{2}, r\right)} \int_{-\infty}^{\infty} \left\{1 - \frac{2\sigma_1^2 it}{N}\right\}^{-(N+r-1)} e^{-iZ^2t}\, dt.$$

We have already carried out the integration under Case I with the exception that $(N - 1)$ should now be replaced by $(N + r - 1)$. The distribution of Z^2 will then be given by

$$(15)\qquad \begin{aligned} dF(Z^2) = \left(\frac{\sigma_1^2}{\sigma_2^2}\right)^{\frac{1}{2}(N-1)} \sum_{r=0}^{\infty} \frac{\left(1 - \frac{\sigma_1^2}{\sigma_2^2}\right)^r}{r\beta\left(\frac{N-1}{2}, r\right)} \frac{\frac{N}{2\sigma_1^2}}{\Gamma(N + r - 1)} \\ \cdot e^{-\frac{NZ^2}{2\sigma_1^2}} \left(\frac{NZ^2}{2\sigma_1^2}\right)^{N+r-2} d(Z^2). \end{aligned}$$

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[2] The author is indebted to Prof. E. J. McShane for this definition which is due to Prof. E. H. Moore. It may be shown easily that $\lim_{r\to\infty} \int_{-\infty}^{\infty} S_r(t)\, dt = \int_{-\infty}^{\infty} \lim_{r\to\infty} S_r(t)\, dt$.

Finally, the distribution function of Z is

$$(16)\quad dF(Z) = 2\left(\frac{\sigma_1^2}{\sigma_2^2}\right)^{\frac{1}{2}(N-1)} \sum_{r=0}^{\infty} \frac{\left(1 - \frac{\sigma_1^2}{\sigma_2^2}\right)^r}{r\beta\left(\frac{N-1}{2}, r\right)} \cdot \frac{\left(\frac{N}{2\sigma_1^2}\right)^{N+r-1}}{\Gamma(N+r-1)} e^{-\frac{NZ^2}{2\sigma_1^2}} Z^{2N+2r-3}\, dz.$$

We remark that the above series expansion holds, of course, for N odd or even. In case N is odd it may be shown that the distribution function may be expressed as a finite series of Incomplete Gamma Functions.[3] However, the finite expansion for N odd appears to offer no marked advantage since for computational purposes the infinite series expansion converges quite rapidly (N either odd or even) and may be put into a convenient form given below.

4. Computational form for the distribution function. In deciding whether or not an observed value of Z is significant and likewise in control chart procedure, one is interested in the percentage points of $f(Z)$. For example, it may be desired to find the value of k such that $P\{Z \le k\sqrt{\sigma_1^2 + \sigma_2^2}\} = .995$, say, for various sample sizes N. In this connection it will be convenient to work with the distribution of Z^2, for $P\{Z \le k\sqrt{\sigma_1^2 + \sigma_2^2}\} = P\{Z^2 \le k^2(\sigma_1^2 + \sigma_2^2)\}$ also. Now,

$$P\{Z^2 \le k^2(\sigma_1^2 + \sigma_2^2)\} = \int_0^{k^2(\sigma_1^2+\sigma_2^2)} dF(Z^2)$$

$$(18)\quad = \left(\frac{\sigma_1^2}{\sigma_2^2}\right)^{\frac{1}{2}(N-1)} \sum_{r=0}^{\infty} \frac{\left(1 - \frac{\sigma_1^2}{\sigma_2^2}\right)^r}{r\beta\left(\frac{N-1}{2}, r\right)} \cdot \frac{\left(\frac{N}{2\sigma_1^2}\right)}{\Gamma(N+r-1)} \cdot \int_0^{k^2(\sigma_1^2+\sigma_2^2)} e^{-\frac{NZ^2}{2\sigma_1^2}} \left(\frac{NZ^2}{2\sigma_1^2}\right)^{N+r-2} d(Z^2),$$

since we may integrate the series term by term over the entire range of Z^2 or any part of it [5]. In the terminology of Karl Pearson's Incomplete Gamma Function [3],

$$(19)\quad I(u, p) = \frac{1}{\Gamma(p+1)} \int_0^{u\sqrt{p+1}} e^{-v} v^p\, dv,$$

we may write the above series in the form

$$P\{Z^2 \le k^2(\sigma_1^2 + \sigma_2^2)\}$$

$$(20)\quad = \left(\frac{\sigma_1^2}{\sigma_2^2}\right)^{\frac{1}{2}(N-1)} \sum_{r=0}^{\infty} \frac{\left(1 - \frac{\sigma_1^2}{\sigma_2^2}\right)^r}{r\beta\left(\frac{N-1}{2}, r\right)} I\left\{\frac{Nk^2\left(1 + \frac{\sigma_2^2}{\sigma_1^2}\right)}{2\sqrt{N+r-1}}, N+r-2\right\}.$$

[3] Prof. C. C. Craig kindly pointed out this fact to the author.

It is indeed convenient and enlightening that the result is a function of the ratio, σ_1^2/σ_2^2, and not σ_1^2 and/or σ_2^2 explicitly.

Hence, for a given sample size and ratio of σ_1^2/σ_2^2, we may find k by inverse interpolation such that $P\{Z \leq k\sqrt{\sigma_1^2 + \sigma_2^2}\} = \alpha$, any desired level of probability.

5. Moments and percentage points for Case I. For the case met many times in practice, i.e. $\sigma_1^2 = \sigma_2^2 = \sigma^2$, we will give a table of the mean and standard deviation and also several probability levels which are obtainable directly from the percentage points of the χ^2 distribution [6].

From (11), we have

$$E(Z^k) = \frac{2\left(\frac{N}{2\sigma^2}\right)^{N-1}}{\Gamma(N-1)} \int_0^\infty e^{-\frac{NZ^2}{2\sigma^2}} Z^{2N+k-3}\, dZ \tag{21}$$

$$= \frac{\Gamma(N-1+k/2)}{\Gamma(N-1)} \left(\frac{2\sigma^2}{N}\right)^{\frac{1}{2}k}.$$

Thus,

$$\mu'_{1:z} = \frac{\Gamma(N-1/2)}{\Gamma(N-1)} \sqrt{\frac{2}{N}}\,\sigma, \tag{22}$$

$$\mu'_{2:z} = \frac{2(N-1)}{N}\sigma^2, \tag{23}$$

and

$$\mu_{2:z} = \frac{2}{N}\left\{N - 1 - \left[\frac{\Gamma(N-1/2)}{\Gamma(N-1)}\right]^2\right\}\sigma^2. \tag{24}$$

In the table below, the mean and standard deviation are given as a multiple of $\sqrt{2}\sigma$ and $k_{.95}$, for example, is that value of k such that $P\{Z \leq k\sqrt{2}\sigma\} = .95$.

TABLE I

N	Mean	Standard Deviation	Percentage Points			
			$k_{.005}$	$k_{.05}$	$k_{.95}$	$k_{.995}$
2	.6267	.3276	.0501	.1602	1.2239	1.6276
3	.7675	.2786	.1857	.3442	1.2575	1.5738
4	.8308	.2443	.2906	.4521	1.2546	1.5226
5	.8670	.2198	.3667	.5227	1.2453	1.4817
6	.8904	.2014	.4239	.5730	1.2351	1.4488
7	.9068	.1869	.4686	.6110	1.2255	1.4218
8	.9189	.1752	.5046	.6408	1.2167	1.3991
9	.9282	.1653	.5345	.6651	1.2087	1.3798
10	.9356	.1569	.5597	.6852	1.2014	1.3630
11	.9416	.1498	.5813	.7023	1.1949	1.3483
12	.9466	.1434	.6001	.7170	1.1889	1.3353
13	.9508	.1378	.6166	.7298	1.1835	1.3237
14	.9544	.1330	.6313	.7411	1.1784	1.3132
15	.9575	.1285	.6445	.7512	1.1738	1.3038

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