A RECIPROCITY PRINCIPLE FOR THE NEYMAN-PEARSON THEORY OF TESTING STATISTICAL HYPOTHESES

By Louis M. Court

In contrasting the tested hypothesis H_1 with the alternative H_2, i.e., in comparing the probability distribution $p(x_1, \cdots, x_n \mid H_1)$ associated with the first hypothesis with the distribution $p(x_1, \cdots, x_n \mid H_2)$ associated with the second, Neyman and Pearson select the *best* critical region R^* from the infinite set of critical regions R of a specified size α by minimizing the probability

$$\int_{S-R} p(x_1, \cdots, x_n \mid H_2)\, dx_1 \cdots dx_n \tag{1}$$

of accepting H_1 when H_2 is true (Type II Errors) subject to the constancy of the probability of rejecting H_1 when H_1 is correct (Type I Errors),

$$\int_{R} p(x_1, \cdots, x_n \mid H_1)\, dx_1 \cdots dx_n = \alpha. \tag{2}$$

S in (1) represents the whole of variate $(x_1, \cdots, x_n)$ space and $S - R$, the complement of R relative to S.

Obviously (1) is conditionally minimized when

$$\int_{R} p(x_1, \cdots, x_n \mid H_2)\, dx_1 \cdots dx_n \tag{3}$$

is maximized subject to (2). Neyman and Pearson have shown that if one or more members of the one parameter (λ) family of regions $R^*(\lambda)$ defined by the inequalities

$$p(x_1, \cdots, x_n \mid H_2) \geqq \lambda p(x_1, \cdots, x_n \mid H_1) \tag{4}$$

satisfy the "side" condition (2), they will be best critical regions maximizing (3) subject to (2) or minimizing (1) subject to (2)[1]. As suggested by the notation, the family $R^*(\lambda)$ depends upon λ and, if sufficient restrictions are imposed upon $p(x_1, \cdots, x_n \mid H_1)$ and $p(x_1, \cdots, x_n \mid H_2)$, there is *one and only one* region for every value of λ lying in an interval contained in the positive half-axis. λ, itself, is clearly a function $\lambda(\alpha)$ of α. Consequently, $R^*(\lambda)$ depends upon α and may be written as $R^*[\alpha]$. The best critical region for a preassigned size $\bar{\alpha}$ is given by $R^*[\bar{\alpha}]$.

Will we get the *same* best critical region if among the regions T that fix the probability of Type II Errors,

$$\int_{S-T} p(x_1, \cdots, x_n \mid H_2)\, dx_1 \cdots dx_n = 1 - \beta, \tag{5}$$

---

[1] For a full exposition, see Neyman and Pearson, *Stat. Res. Memoirs*, Vol. 1 (1936).

we find the one that minimizes the integral in (2) with R replaced by T, i.e. if we find the one that minimizes the probability of Type I Errors? We shall call this turnabout of the usual process the *reversed* Neyman-Pearson principle. To discover the answer, we note that $\int_S p(x_1, \cdots, x_n \mid H_2)\, dx_1 \cdots dx_n$ is equal to unity and (5) may be rewritten as

$$\int_T p(x_1, \cdots, x_n \mid H_2)\, dx_1 \cdots dx_n = \beta. \tag{6}$$

The regions that minimize the left side of (2) with R replaced by T subject to (6) are obviously identical with those that maximize the negative of this left side subject to (6) multiplied through by -1. The latter problem is formally identical with the one referred to in the second paragraph of this note and, invoking Neyman and Pearson's result, we conclude that the said conditional maximization is effected by the one parameter (μ) family of regions $T^*(\mu^{-1})$ defined by the inequalities

$$-p(x_1, \cdots, x_n \mid H_1) \geqq -\mu p(x_1, \cdots, x_n \mid H_2) \tag{7}$$

or

$$p(x_1, \cdots, x_n \mid H_2) \geqq \frac{1}{\mu}\, p(x_1, \cdots, x_n \mid H_1).$$

μ^{-1} in $T^*(\mu^{-1})$ denotes the reciprocal of μ. It is clear from (4) and (7) that the families of regions $R^*(\lambda)$ and $T^*(\mu^{-1})$, satisfying the direct and reversed Neyman-Pearson processes, coincide.

As before, μ is some function $\mu(\beta)$ of β. Hence, β is a function $\beta(\mu)$[2] of μ and, accordingly, a function $\beta\left[\frac{1}{\lambda(\alpha)}\right]$ of α. Consequently, if the level at which the probability of Type II Errors in the *reversed* Neyman-Pearson process is held constant is taken equal to $1 - \beta\left[\frac{1}{\lambda(\alpha)}\right]$ in terms of the level α at which the probability of Type I Errors is fixed in the *direct* Neyman-Pearson method, the reversed and direct processes yield the *same* best critical region. *This is the reciprocity principle alluded to in the title of this note in its full completeness.*

---

[2] $\beta(\mu^{-1})$ will generally be distinct in form from $\alpha(\lambda)$, although the second line in (7) coincides upon the substitution of λ for μ^{-1} with (4), since the integrand in (5) is $p(x_1, \cdots, x_n \mid H_2)$ whereas that in (2), regarded as a constraint in the direct process, is $p(x_1, \cdots, x_n \mid H_1)$.