ON A TEST FOR RANDOMNESS BASED ON SIGNS OF DIFFERENCES[1]

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1. Introduction. It has been pointed out by J. Wolfowitz [1] that we cannot expect a test for randomness to be most powerful with respect to every possible alternative. It is therefore necessary to find tests designed to distinguish a random sample of observations from the same population from a sample coming from some particular class Ω of distributions. Such a test need be consistent in the sense of Wald and Wolfowitz [2] only with respect to alternatives in the class Ω.

Let $x_1, \cdots, x_n$ be the measurable quality characteristics of n units of a manufactured article. We shall assume that the distribution of x_i is continuous. According to Shewhart the production process is termed "under statistical control" if $x_1, \cdots, x_n$ can be regarded as a random sample of n independent items each coming from the same population with known or unknown distribution function.

In a random sample $p_i = p(x_i > x_{i+1}) = \frac{1}{2}$, where $P(E)$ denotes the probability that E will hold. The class Ω of alternatives which we shall consider is described as follows. The cumulative distribution of x_i is f_i and the f_i, $i = 1, 2, \cdots$, are such that

$$p_i = \tfrac{1}{2} + \epsilon_i, \qquad \sum_{i=1}^{i=n} \epsilon_i = \lambda_n(n-1), \qquad \liminf_{n\to\infty} \lambda_n = \lambda > 0.$$

Such a situation may, for instance, obtain of the production process is under statistical control except for occasionally but not too infrequently occurring periods during which the quality of the product decreases, after which decrease statistical control is immediately restored. If the decreases in quality are sharp enough or the periods of decrease long enough, then the alternative will belong to the class Ω described before.

To give a practical example; consider a drill, which after some period of use will wear off so that the quality of the manufactured article will decrease until the drill is exchanged. After replacement of the drill by a new one, statistical control is immediately restored. Now, if the drill is not replaced in time, the periods of decrease in quality will be long and the rate of decrease will become rapid so that the sequence of distribution functions will satisfy the conditions of the class Ω. A similar situation occurs also in time studies. For instance, in the foregoing example, the time necessary for drilling one hole will tend to increase when the drill is too long in use.

The following test first proposed by Moore and Wallis [3] for the study of

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economic time series seems appropriate for our purpose: Let $x_1, \cdots, x_n$ be the sample and form the sequence $x_2 - x_1, \cdots, x_n - x_{n-1}$. Let S be the number of negative differences in this sequence. Clearly, the distribution of S is independent of the distribution of x_i provided the sample is an independent random sample from a continuous distribution. Under one of the alternatives of the class Ω, S will in a sample of n tend to be larger than in a random sample if $\lambda_n > 0$. Hence S may be used as a statistic to distinguish between randomness and any of the alternatives of the class Ω. The distribution of S was tabulated by Moore and Wallis [3] for $n \leq 12$. They also found empirically that S approaches a normal distribution. The asymptotic normality of the distribution of S can be proved rigorously in a way analogous to the proof of Theorem 1 of a paper by Wolfowitz [4]. The first four moments of S were obtained by Moore and Wallis. The fourth moment, however, only by empirical methods. In this paper we shall derive a formula which makes it possible to compute the moments of S recursively. With the help of this formula we shall indicate an alternative proof of the asymptotic normality of S using the method of moments. Finally, we shall derive a lower bound for the power of the S test with respect to alternatives in Ω valid for large n and depending only on λ_n.

2. The moments of S: Let $P_n(S)$ be the number of permutations in n variables with S negative differences. MacMahon [5] has shown that

$$P_n(S) = (S+1)P_{n-1}(S) + (n-S)P_{n-1}(S-1). \tag{1}$$

Using (1) Moore and Wallis [3] have tabulated $P\left(\left|S - \frac{n-1}{2}\right| \geq \left|\bar{S} - \frac{n-1}{2}\right|\right)$. In using their table for our purpose, one has to keep in mind that we are using a one tail region; therefore $P(S \geq \bar{S})$ is for $S \geq \frac{n-1}{2}$ one half of the value tabulated by Moore and Wallis.

Clearly the first moment of S is $\frac{n-1}{2}$, since the expected value of $-$ signs equals the expected value of $+$ signs. To find higher moments we multiply (1) by $\left(S - \frac{n-1}{2}\right)^i$ divide by $n!$ and sum over S. Then we obtain

$$E_n\left[\left(S - \frac{n-1}{2}\right)^i\right] = \frac{1}{n} E_{n-1}\left[\left(S - \frac{n-1}{2}\right)^i (S+1)\right] + \frac{1}{n} E_{n-1}\left[(n-S-1)\left(S - \frac{n-1}{2} + 1\right)^i\right], \tag{2}$$

where $E_n[f(S)]$ denotes the expectation of $f(S)$ in permutations of n variables. From (2) we have

$$E_n\left[\left(S - \frac{n-1}{2}\right)^i\right] = \frac{1}{n} E_{n-1}\left[\left(S - \frac{n-2}{2} - \frac{1}{2}\right)^i \left(S - \frac{n-2}{2}\right)\right.$$
$$+ \tfrac{1}{2}E_{n-1}\left[\left(S - \frac{n-2}{2} - \frac{1}{2}\right)^i\right]$$
$$- \frac{1}{n} E_{n-1}\left[\left(S - \frac{n-2}{2} + \frac{1}{2}\right)^i \left(S - \frac{n-2}{2}\right)\right]$$
$$+ \frac{1}{2} E_{n-1}\left[\left(S - \frac{n-2}{2} + \frac{1}{2}\right)^i\right].$$

Putting $S - E(S) = x$ we obtain

$$(3) \qquad E_n(x^i) = \frac{1}{n} E_{n-1}[x(x - \tfrac{1}{2})^i - x(x + \tfrac{1}{2})^i] + \tfrac{1}{2}E_{n-1}[(x + \tfrac{1}{2})^i + (x - \tfrac{1}{2})^i].$$

From the symmetry of the distribution as well as from 3 it may be seen that all odd moments are 0 and therefore

$$\tfrac{1}{2}E[(x + \tfrac{1}{2})^{2i} + (x - \tfrac{1}{2})^{2i}] = E(x + \tfrac{1}{2})^{2i}$$
$$E[x(x - \tfrac{1}{2})^{2i} - x(x + \tfrac{1}{2})^{2i}] = -2E[(x + \tfrac{1}{2})^{2i+1}] + E(x + \tfrac{1}{2})^{2i}.$$

Hence we obtain from 2

$$(4) \qquad E_n(x^{2i+1}) = 0, \qquad i = 0, 1, \cdots$$
$$E_n(x^{2i}) = \frac{n+1}{n} E_{n-1}[(x + \tfrac{1}{2})^{2i}] - \frac{2}{n} E_{n-1}[(x + \tfrac{1}{2})^{2i+1}]$$

If all moments below the $2i$th moment are known (4) becomes a difference equation whose solution yields the $2i$th moment for $n \geq 2i$. Thus one obtains

$$\sigma_n^2(S) = E_n(x^2) = \frac{n+1}{12}, \qquad E_n(x^4) = \frac{5(n+1)^2 - 2(n+1)}{240},$$
$$E_n(x^6) = \frac{35(n+1)^3 - 42(n+1)^2 + 16(n+1)}{4032}.$$

It is not difficult to prove from (4) by induction that $\lim_{n\to\infty} \frac{E_n x^{2i}}{\sigma_n^{2i}(S)} = (2i-1)(2i-3) \cdots 3.1$. To do this one proves first by induction that $E_n(x^{2i})$ is for $n \geq 2i$ a polynomial in n of degree i. It can then be proved by induction that the first coefficient of this polynomial is $(2i - 1)(2i - 3) \cdots 3.1/12^i$ from which the assertion follows. Since $(2i - 1) \cdots 3.1$ are the moments of a normal distribution with variance 1 it follows that $\dfrac{\left(S - \frac{n-1}{2}\right)\sqrt{12}}{\sqrt{n+1}}$ is in the limit normally distributed with mean 0 and variance 1. This result follows, however, also easily from Theorem 2 of a paper by Wolfowitz [4].

It is also possible to show by induction from equation (4) that for $n \geq 2i$ the $2i$th moment of S is smaller than the corresponding moment of a normal distribution with variance $\frac{n+1}{12}$.

3. The power of the S test. Let us assume now that one of the alternatives of the class Ω is true. This is to say $p_i = P(x_i > x_{i+1}) = \frac{1}{2} + \epsilon_i$, $\sum \epsilon_i = \lambda_n(n-1)$, $\liminf \lambda_n = \lambda > 0$. Let

$$z_i = \begin{cases} 1 \text{ if the } i\text{th sign is } -, \\ 0 \text{ if the } i\text{th sign is } +. \end{cases}$$

We shall show that

$$P(z_{i+1} = 1 \mid z_i = 1) \leqq P(z_{i+1} = 1).$$

We have

$$\left[\int_{\infty}^{x_1} df_2(x_2) \int_{\infty}^{x_2} df_3(x_3)\right] \int_{x_1}^{\infty} df_2(x_2) \leq \int_{\infty}^{x_1} df_2(x_2) \int_{x_1}^{\infty} df_2(x_2) \int_{\infty}^{x_1} df_3(x_3)$$
$$\leq \int_{\infty}^{x_1} df_2(x_2) \left[\int_{x_1}^{\infty} df_2(x_2) \int_{\infty}^{x_2} df_3(x_3).\right.$$

Adding $\int_{\infty}^{x_1} df_2(x_2) \left[\int_{\infty}^{x_1} df_2(x_2) \int_{\infty}^{x_2} df_3(x_3)\right]$ to both sides of this inequality we have

$$\int_{\infty}^{x_1} df_2(x_2) \int_{\infty}^{x_2} df_3(x_3) \leq \int_{\infty}^{x_1} df_2(x_2) \left[\int_{\infty}^{+\infty} df_2(x_2) \int_{\infty}^{x_2} df_3(x_3).\right.$$

Integrating both sides with respect to x_1, we obtain

$$\int_{\infty}^{+\infty} df_1(x_1) \int_{\infty}^{x_1} df_2(x_2) \int_{\infty}^{x_2} df_3(x_3)$$
$$\leq \left[\int_{\infty}^{+\infty} df_1(x_1) \int_{\infty}^{x_1} df_2(x_2)\right]\left[\int_{-\infty}^{\infty} df_2(x_2) \int_{\infty}^{x_2} df_3(x_3)\right]$$

or

$$P(z_1 = 1 \quad \text{and} \quad z_2 = 1) \leq P(z_1 = 1) \cdot P(z_2 = 1).$$

From this it follows that $\sigma_{z_i z_{i+1}} \leq 0$. Since $\sigma_{z_i}^2 = \frac{1}{4} - \epsilon_i^2$ we have $\sigma_s^2 \leq \frac{n-1}{4} - \sum_{i=1}^{i=n-1} \epsilon_i^2 \leqq \frac{n-1}{4}(1 - 4\lambda_n^2)$. Moreover $E(S) = \frac{n-1}{2} + \lambda_n(n-1)$.

Let $\lambda' = \lambda$ if $\lambda < \frac{1}{2}$ and $0 \leq \lambda' < \lambda$ if $\lambda = \frac{1}{2}$. The critical region is for sufficiently large n given approximately by $S > \frac{n-1}{2} + t\sqrt{\frac{n+1}{12}}$, where t

depends on the level of significance α and must be chosen so that $\frac{1}{\sqrt{2\pi}}\int_t^\infty e^{-\frac{1}{2}x^2}\,dx = \alpha$. Hence, if we can show that under any alternative H of the class Ω and for any $\epsilon > 0$

$$(5)\qquad P\left(S \geq E(S) - \frac{t}{2}\sqrt{(n-1)(1-4\lambda'^2)}\right) \leq \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-t} e^{-\frac{1}{2}x^2}\,dx + \epsilon$$

for every $t \geq \bar{t} > 0$, $n \geq N(\epsilon, H, \bar{t})$, then we shall be able to give a lower bound for the power of the S test. The power of the S test is approximately given by $P\left(S \geq \frac{n-1}{2} + t\sqrt{\frac{n+1}{12}}\right)$.

From (5) we have

$$(6)\qquad \begin{aligned} &P\left(S \geq \frac{n-1}{2} + t\sqrt{\frac{n+1}{12}}\right) \geq \frac{1}{\sqrt{2\pi}}\int_{\frac{t\sqrt{n+1}-2\lambda_n(n-1)\sqrt{3}}{\sqrt{(3n-3)(1-4\lambda'^2)}}}^{\infty} e^{-\frac{1}{2}x^2}\,dx - \epsilon \\ &\text{for}\quad \frac{t\sqrt{n+1} - 2\lambda_n(n-1)\sqrt{3}}{\sqrt{3(n-1)(1-4\lambda'^2)}} < -\bar{t} < 0, \qquad n \geq N(\epsilon, H, \bar{t}). \end{aligned}$$

The author considers it safe to assume that (6) holds with a fairly small ϵ for $n \geq 12$ if λ' in (6) is replaced by λ_n' where $\lambda_n' = \lambda_n$ if $\lambda_n < \frac{1}{2}$ and $\lambda_n' < \frac{1}{2}$ if $\lambda_n = \frac{1}{2}$ and if λ_n' is not too close to $\frac{1}{2}$. He bases this belief on the rapidity with which the distribution of S approaches normality under the null hypothesis of randomness, and on the fact that at least under the 0 hypothesis the moments of S are smaller than the corresponding moments of a normal distribution. It may also be seen from the following derivation of (6) that in many cases the power of the S test will be considerably above the lower bound given in (6).

To prove (5), we need the following two lemmas

LEMMA 1. *Let $P(x \leqq t) = f(t)$. Let further $E(z) = 0$, $E(z^2) = \epsilon$. Then for every $\delta > 0$*

$$(7)\qquad f(t+\delta) + \frac{\epsilon}{\delta^2} \geq P(x + z \leq t) \geq f(t-\delta) - \frac{\epsilon}{\delta^2}.$$

PROOF: Applying Tschebycheff's inequality we have

$$\begin{aligned} P(x + z \leq t) &\leq P(x \leq t + \delta) + P(x \geq t + \delta \text{ and } z \leq -\delta) \\ &\leq P(x \leq t + \delta) + P(z \leq -\delta) \leq f(t+\delta) + \frac{\epsilon}{\delta^2}, \end{aligned}$$

$$\begin{aligned} P(x + z \leq t) &\geq P(x \leq t - \delta \text{ and } z \leq \delta) \\ &\geq P(x \leq t - \delta) - P(z \geq \delta) \geq f(t-\delta) - \frac{\epsilon}{\delta^2}. \end{aligned}$$

LEMMA 2. *Let $\{x_i\}$, $i = 1, 2, \cdots$ be a sequence of independent random variables with mean 0 bounded kth absolute moment, $k > 2$, and variance σ_i^2. Let $M > 0$ and* $\limsup_{n\to\infty} \frac{\sum \sigma_i^2}{n} \leq M^2$. *Form the sequence of random variables* $y_n = \frac{x_1 + \cdots + x_n}{M\sqrt{n}}$ *then for any $\epsilon > 0$ and any $t > \bar{t} > 0$*

$$P(y_n \leq -t) \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-t} e^{-\frac{1}{2}x^2}\,dx + \epsilon \quad \text{for} \quad n \geq N(\epsilon, \bar{t}). \tag{8}$$

PROOF. Form a sequence m_α with $\lim_{\alpha\to\infty} m_\alpha = 0$. Let $y_n = x_n^\alpha + z_n^\alpha$

where

$$x_n^\alpha = \frac{\sum^\alpha x_i}{M\sqrt{n}}, \qquad z_n^\alpha = \frac{\sum x_i - \sum^\alpha x_i}{M\sqrt{n}}.$$

$\sum^\alpha$ denotes summation over all i for which $\sigma_i^2 \geq m_\alpha$ and all sums extend from one to n.

Let f_n^α be the distribution of x_n^α then by LEMMA 1

$$f_n^\alpha(-t + \delta) + \frac{m_\alpha^2}{M\delta^2} \geq P(y_n \leq -t) \geq f_n^\alpha(-t - \delta) - \frac{m_\alpha^2}{M\delta^2}.$$

Now we distinguish two cases.

1st Case. The number of integers i with $\sigma_i^2 \geq m_\alpha$ is for some α of order n. In this case $\{f_n^\alpha\}$ differs arbitrarily little from a sequence of normal distributions with mean 0 and the upper limit of the variances at most 1.

2nd Case. The number of integers i with $\sigma_i^2 \geq m_\alpha$ is for every α of smaller order than n. In this case x_n^α converges stochastically to 0. In both cases (8) holds true since m_α can be chosen arbitrarily small.

We can now prove (5). It follows easily from Tschebycheff's theorem that (5) is true if $\lambda = \frac{1}{2}$. Hence we may assume $\lambda < \frac{1}{2}$. Let z_i be defined as at the beginning of this section. Form

$$v_j^k = \sum_{(j-1)k+1}^{jk-1} \frac{2(z_i - E(z_i))}{\sqrt{(n-1)(1-4\lambda^2)}},$$

$$u_j^k = \frac{2(z_{jk} - E(z_{jk}))}{\sqrt{(n-1)(1-4\lambda^2)}}, \qquad \bar{v}_n^k = \sum_{i=m'+1}^{i=n-1} \frac{2(z_i - E(z_i))}{\sqrt{(n-1)(1-4\lambda^2)}}$$

where $m' = gk$ is the largest integer multiple of k which does not exceed $(n - 1)$. We form further

$$x_n^k = \sum_{j=1}^{j=g} v_j^k, \qquad z_n^k = \sum_{j=1}^{j=g} u_j^k.$$

Since $\sigma_{z_n^k}^2 \leq \frac{\frac{1}{4}(g-1)}{\frac{1}{4}(n-1)(1-4\lambda^2)} \leq \frac{1}{k(1-4\lambda^2)}$ it follows from LEMMA 1 that the distribution of $\frac{2(S - E(S))}{\sqrt{(n-1)(1-4\lambda^2)}}$ differs arbitrarily little from the distribution of x_n^k for sufficiently large n and k. The second and the third absolute moment of $\sqrt{n-1}\, v_j^k$ are bounded. Hence $\sqrt{n-1}\, v_j^k$ fulfills the conditions of LEMMA 2. The application of LEMMA 2 yields (5) and consequently 6.

The integer $N(\epsilon, H, \bar{t})$ is independent of t provided the lower limit of the integral does not exceed $-\bar{t}$. Hence we have proved

THEOREM. *Let $t_1, t_2, \cdots$ be any sequence of numbers satisfying the condition*

$$-\bar{t}_n = \frac{t_n\sqrt{n+1} - 2(n-1)\lambda_n\sqrt{3}}{\sqrt{(3n-3)(1-4\lambda'^2)}} \leq -\bar{t} < 0,$$

where $\lambda' = \liminf_{n\to\infty} \lambda_n$ if $\liminf_{n\to\infty} \lambda_n < \frac{1}{2}$ and $0 \leq \lambda' < \frac{1}{2}$ otherwise. Let $P_n(S, H)$ be the power of the S test with respect to the alternative H and critical region $S \geq \frac{n-1}{2} + t_n\sqrt{\frac{n+1}{12}}$. Then

$$\liminf_{n\to\infty}\left[P_n(S_1H) \Big/ \frac{1}{\sqrt{2\pi}}\int_{-\bar{t}_n}^{\infty} e^{-\frac{1}{2}x^2}\,dx\right] \geq 1. \tag{9}$$

It is worthwhile to remark that (9) is sharp. That is to say there exist alternatives for which the left side of (9) is equal to (1). This is obviously the case for any alternative with $P(x_i > x_{i+1}) = \frac{1}{2} + \lambda$ and $P(z_i = 1 \text{ and } z_{i+1} = 1) = P(z_i = 1)\cdot P(z_{i+1} = 1)$. These conditions are, for instance fulfilled by the alternative given by $P(x_{i+1} = a - \delta - \cdots - \delta^i) = \frac{1}{2} + \lambda$, $P(x_{i+1} = C + \delta + \cdots + \delta^i) = \frac{1}{2} - \lambda$, $i = 1, 2, \cdots$ where $(a - c) > \frac{2\delta}{1-\delta} > 0$.

If $t_n = t$ for every n then (9) implies the consistency of the test if the order of λ_n is larger than $1/\sqrt{n}$. It may also be seen that the test is not consistent with respect to alternatives for which λ_n is of order at most equal to $\frac{1}{\sqrt{n}}$. This remark refers of course only to alternatives for which x_i is independent of x_j for $i \neq j$.

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