ON THE MEASURE OF A RANDOM SET. II

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1. Introduction. In a recent paper[1] the author derived general formulas for the moments of the measure of any random set X, and applied the formulas to find the mean and variance of a random sum of intervals on the line. In a subsequent paper[2] J. Bronowski and J. Neyman, using other methods, found the variance when X is a random sum of rectangles in the plane, and raised the question of finding the variance when X is a random sum of n-dimensional intervals in n-space. This will be done in the present paper, independently of the work of Bronowski and Neyman, using the methods of (I). The corresponding problem for circles in the plane will also be solved.

2. n-dimensional intervals, N fixed. Let the random set X be defined as follows. Let A_i, a_i (the range of the subscript i throughout this paper will be from 1 to n) and δ be fixed positive numbers such that $a_i \leq 2\,\delta$. Let R denote the n-dimensional interval consisting of all points $(x_1, \cdots, x_n)$ such that $0 \leq x_i \leq A_i$, and let R' denote the larger interval for which $-\delta \leq x_i \leq A_i + \delta$ (and also its measure $\Pi(A_i + 2\,\delta)$). Let a fixed number N of intervals with sides a_i parallel to the axes be chosen independently, with the probability density function for the center of each interval constant and equal to $1/R'$ in R'. The set X is the intersection of the set-theoretical sum of the N intervals with R. The set Y consists of those points of R that do not belong to X. We have identically

$$X + Y = R, \tag{1}$$

where capital letters denote either sets or their measures.

From (I), equation (15), we have

$$E(Y) = \int_0^{A_n} \cdots \int_0^{A_1} p(x_1, \cdots, x_n)dx_1 \cdots dx_n, \tag{2}$$

where, setting $r = \Pi a_i$, we have

$$p(x_1, \cdots, x_n) = Pr((x_1, \cdots, x_n)\epsilon Y) = \left(1 - \frac{r}{R'}\right)^N. \tag{3}$$

Hence

$$E(Y) = R\left(1 - \frac{r}{R'}\right)^N. \tag{4}$$

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[1] H. E. Robbins. "On the measure of a random set," *Annals of Math. Stat.* Vol. 15 (1944), pp. 70–74. We shall refer to this paper as (I).

[2] J. Bronowski and J. Neyman. "On the variance of a random set." *Annals of Math. Stat.* Vol. 16 (1945), pp. 330–341. We shall refer to this paper as (BN).

From (1) it follows that

$$E(X) = R\left\{1 - \left(1 - \frac{r}{R'}\right)^n\right\}. \tag{5}$$

From (I), equation (21), we have

$$E(Y^2) = \int_0^{A_n} \cdots \int_0^{A_1} \int_0^{A_n} \cdots \int_0^{A_1} p(x_1, \cdots x_n, y_1, \cdots, y_n) \cdot dx_1 \cdots dx_n\, dy_1 \cdots dy_n, \tag{6}$$

where

$$p(x_1, \cdots, x_n, y_1, \cdots, y_n) = Pr((x_1, \cdots, x_n) \epsilon Y \text{ and } (y_1, \cdots, y_n) \epsilon Y). \tag{7}$$

It is clear from the symmetry of the problem that the distribution of Y will be unchanged if we assume that for all i, $x_i \leq y_i$. Hence, since there are 2^n possible sets of n inequalities each, we can write

$$E(Y^2) = 2^n \int_0^{A_n} \cdots \int_0^{A_1} \int_0^{y_n} \cdots \int_0^{y_1} p\, dx_1 \cdots dx_n\, dy_1 \cdots dy_n. \tag{8}$$

We now introduce the new variables of integration

$$u_i = x_i, \qquad v_i = y_i - x_i \tag{9}$$

for which

$$\frac{\partial(u_1, \cdots, u_n, v_1, \cdots, v_n)}{\partial(x_1, \cdots, x_n, y_1, \cdots, y_n)} = 1. \tag{10}$$

In terms of the new variables we have

$$p = f(v_1, \cdots, v_n) = \begin{cases} \left(1 - \dfrac{2r}{R'}\right)^N & \text{if } v_i \geq a_i \text{ for } \textit{some } i, \\ \left(1 - \dfrac{2r - \Pi(a_i - v_i)}{R'}\right)^N & \text{if } v_i \leq a_i \text{ for } \textbf{\textit{all i}}. \end{cases} \tag{11}$$

Equation (8) now becomes

$$\begin{aligned} E(Y^2) &= 2^n \int_0^{A_n} \cdots \int_0^{A_1} \int_0^{A_n - v_n} \cdots \int_0^{A_1 - v_1} f\, du_1 \cdots du_n dv_1 \cdots dv_n \\ &= 2^n \int_0^{A_n} \cdots \int_0^{A_1} f \Pi(A_i - v_i)\, dv_1 \cdots dv_n. \end{aligned} \tag{12}$$

Let $z_i = \min(a_i, A_i)$. Then from (11) and (12) we obtain

$$\begin{aligned} E(Y^2) = {} & 2^n \int_0^{z_n} \cdots \int_0^{z_1} \left(1 - \frac{2r - \Pi(a_i - v_i)}{R'}\right)^N \Pi(A_i - v_i)\, dv_1 \cdots dv_n \\ & + 2^n\left(1 - \frac{2r}{R'}\right)^N \left\{\int_0^{A_n} \cdots \int_0^{A_1} \Pi(A_i - v_i)\, dv_1 \cdots dv_n \right. \\ & \left. - \int_0^{z_n} \cdots \int_0^{z_1} \Pi(A_i - v_i)\, dv_1 \cdots dv_n\right\}. \end{aligned} \tag{13}$$

Let the symbol $[x]$, as in (BN), be defined by

$$[x] = \begin{cases} x \text{ if } x \geq 0, \\ 0 \text{ if } x \leq 0. \end{cases} \tag{14}$$

In the integral in the first line of (13) we introduce the new variables of integration $w_i = a_i - v_i$, while in the two integrals in the second line we introduce the variables $s_i = A_i - v_i$. The result is

$$\begin{aligned} E(Y^2) &= 2^n \int_{[a_n - A_n]}^{a_n} \cdots \int_{[a_1 - A_1]}^{a_1} \left(1 - \frac{2r - \Pi w_i}{R'}\right)^N \\ &\qquad \cdot \Pi(w_i + A_i - a_i)\, dw_1 \cdots dw_n \\ &\qquad + 2^n \left(1 - \frac{2r}{R'}\right)^N \left\{ \int_0^{A_n} \cdots \int_0^{A_1} \Pi s_i \, ds_1 \cdots ds_n \right. \\ &\qquad \left. - \int_{[A_n - a_n]}^{A_n} \cdots \int_{(A_1 - a_1]}^{A_1} \Pi s_i \, ds_1 \cdots ds_n \right\} \\ &= 2^n \int_{[a_n - A_n]}^{a_n} \cdots \int_{[a_1 - A_1]}^{a_1} \left(1 - \frac{2r - \Pi w_i}{R'}\right)^N \\ &\qquad \cdot \Pi(w_i + A_i - a_i)\, dw_1 \cdots dw_n \\ &\qquad + \left(1 - \frac{2r}{R'}\right)^N \{\Pi A_i^2 - \Pi(A_i^2 - [A_i - a_i]^2)\}. \end{aligned} \tag{15}$$

From (1) we see that $\sigma_X^2 = E(X^2) - E^2(X) = E(Y^2) - E^2(Y)$. Thus from (4) and (5) we have

$$\begin{aligned} \sigma_X^2 &= 2^n \int_{[a_n - A_n]}^{a_n} \cdots \int_{[a_1 - A_1]}^{a_1} \left(1 - \frac{2r - \Pi w_i}{R'}\right)^N \\ &\qquad \cdot \Pi(w_i + A_i - a_i)\, dw_1 \cdots dw_n \\ &\qquad + \left(1 - \frac{2r}{R'}\right)^N \{\Pi A_i^2 - \Pi(A_i^2 - [A_i - a_i]^2)\} \\ &\qquad - R^2 \left(1 - \frac{r}{R'}\right)^{2N}. \end{aligned} \tag{16}$$

3. n-dimensional intervals, N variable. Now let X and Y be defined as before except that the number N is taken as a random variable, capable of assuming the values $0, 1, \cdots$ with respective probabilities $p_0, p_1, \cdots$, and with generating function

$$\varphi(t) = \sum_0^\infty p_N t^N. \tag{17}$$

Then from (5) we have

$$E(X) = \sum_{0}^{\infty} p_N R\left\{1 - \left(1 - \frac{r}{R'}\right)^N\right\} = R\left\{1 - \varphi\left(1 - \frac{r}{R'}\right)\right\}, \tag{18}$$

while from (15) we have

$$\begin{aligned} \sigma_X^2 = E(Y^2) - E^2(Y) = 2^n \int_{[a_n - A_n]}^{a_n} \cdots \int_{[a_1 - A_1]}^{a_1} \varphi\left(1 - \frac{2r - \Pi w_i}{R'}\right) \\ \Pi(w_i + A_i - a_i)\, dn_1 \cdots dw_n \\ + \varphi\left(1 - \frac{2r}{R'}\right)\{\Pi A_i^2 - \Pi(A_i^2 - [A_i - a_i]^2)\} - R^2 \varphi^2\left(1 - \frac{r}{R'}\right). \end{aligned} \tag{19}$$

In particular, suppose that, as in (BN), N has a Poisson distribution with a parameter λ,

$$p_N = e^{-\lambda R'} \cdot \frac{(\lambda R')^N}{N!}, \tag{20}$$

so that

$$\varphi(t) = e^{\lambda R'(t-1)}. \tag{21}$$

Then (18) becomes

$$E(X) = R\{1 - e^{-\lambda r}\}, \tag{22}$$

while (19) becomes

$$\begin{aligned} \sigma_X^2 = 2^n \cdot e^{-2\lambda r} \int_{[a_n - A_n]}^{a_n} \cdots \int_{[a_1 - A_1]}^{a_1} \left\{\sum_{0}^{\infty} \frac{(\lambda \Pi w_i)^N}{N!}\right\} \\ \cdot \{\Pi(w_i + A_i - a_i)\}\, dw_1 \cdots dw_n \\ + e^{-2\lambda r}\{\Pi A_i^2 - \Pi(A_i^2 - [A_i - a_i]^2)\} - R^2 e^{-2\lambda r}. \end{aligned} \tag{23}$$

Integrating term by term and simplifying the resulting expression, we obtain finally

$$\begin{aligned} \sigma_X^2 = r \cdot 2^n \cdot e^{-2\lambda r} \sum_{1}^{\infty} \left\{\frac{(\lambda r)^N}{N!\{(N+1)(N+2)\}^n}\right. \\ \left. \cdot \Pi\left\{(N+2)A_i - a_i + [a_i - A_i]\left(1 - \frac{A_i}{a_i}\right)^{N+1}\right\}\right\}. \end{aligned} \tag{24}$$

4. Circles in the plane. Let the random set X be defined as follows. Let A_1, A_2, a, and δ be fixed positive numbers such that $2a \leq \min(A_1, A_2, 2\delta)$. Let R denote the rectangle consisting of all points (x_1, x_2) such that $0 \leq x_1 \leq A_1$, $0 \leq x_2 \leq A_2$, and let R' denote the larger rectangle for which $-\delta \leq x_1 \leq A_1 + \delta$, $-\delta \leq x_2 \leq A_2 + \delta$. Let a fixed number N of circles with radii a and areas $b = \pi a^2$ be chosen independently, with the probability density function for the center of each circle constant and equal to $1/R'$ in R'. The set X is the intersection of the set-theoretical sum of the N circles with R. The set Y consists of those points of R that do not belong to X. Equation (1) holds as before. The analogue of (4) is

$$E(Y) = \int_0^{A_2} \int_0^{A_1} p(x_1, x_2)\, dx_1\, dx_2 = R\left(1 - \frac{b}{R'}\right)^N, \tag{25}$$

while (8) becomes

$$E(Y^2) = 4 \int_0^{A_2} \int_0^{A_1} \int_0^{v_2} \int_0^{v_1} p(x_1, x_2, y_1, y_2)\, dx_1\, dx_2\, dy_1\, dy_2, \tag{26}$$

where

$$p(x_1, x_2, y_1, y_2) = Pr((x_1, x_2) \epsilon Y \text{ and } (y_1, y_2) \epsilon Y). \tag{27}$$

Introducing the new variables (9) we obtain the analogue of (12),

$$E(Y^2) = 4 \int_0^{A_2} \int_0^{A_1} f(A_2 - v_2)(A_1 - v_1)\, dv_1\, dv_2, \tag{28}$$

where, setting $r = (v_1 + v_2)^{\frac{1}{2}}$,

$$f(v_1, v_2) = \begin{cases} \left(1 - \dfrac{2b}{R'}\right)^N & \text{if } r \geq 2a, \\[2ex] \left\{1 - \dfrac{2b - 2a^2 \arccos\left(\dfrac{r}{2a}\right) + \dfrac{r}{2}\sqrt{4a^2 - r^2}}{R'}\right\}^N & \text{if } r \leq 2a. \end{cases} \tag{29}$$

Introducing polar coördinates r, θ in the v_1, v_2-plane and carrying out the obvious integrations, we obtain

$$\begin{aligned} E(Y^2) = \left(1 - \frac{2b}{R'}\right)^N &\left\{R^2 + \frac{32}{3} a^3 (A_1 + A_2) - 8a^4 - 4bR\right\} \\ &+ 8a^2 \int_0^1 (\pi R t + 4a^2 t^3 - 4a(A_1 + A_2)t^2) \\ &\cdot \left(1 - \frac{2b - 2a^2 \arccos t + 2a^2\, t\sqrt{1 - t^2}}{R'}\right)^N dt. \end{aligned} \tag{30}$$

If now N is a random variable with generating function (17), then (25) becomes

$$E(Y) = R\varphi\left(1 - \frac{b}{R'}\right), \tag{31}$$

and hence

$$E(X) = R\left\{1 - \varphi\left(1 - \frac{b}{R'}\right)\right\}, \tag{32}$$

while

$$
\begin{aligned}
\sigma_X^2 &= E(X^2) - E^2(X) = E(Y^2) - E^2(Y) \\
&= \varphi\left(1 - \frac{2b}{R'}\right)\left\{R^2 + \frac{32}{3}\,a^3(A_1 + A_2) - 8a^4 - 4bR\right\} \\
&\qquad - R^2\,\varphi^2\left(1 - \frac{b}{R'}\right) + 8a^2\int_0^1 (\pi Rt + 4a^2\,t^3 - 4a(A_1 + A_2)t^2) \\
&\qquad\qquad \cdot\,\varphi\left(1 - \frac{2b - 2a^2 \arccos t + 2a^2\,t\sqrt{1 - t^2}}{R'}\right) dt.
\end{aligned} \tag{33}
$$