examination of the proof of Theorem 3 shows that it would go through with little change if equation (3) were replaced by the requirement that $|m(\alpha)|$ be bounded. We therefore obtain the following result: If for a doubly simple region there exists an unbiased estimate $p(\alpha)$ of p, not identically equal to $\hat{p}(\alpha)$, then not only is $p(\alpha)$ not proper, but also, no matter how large M, there exists a boundary point α such that $|p(\alpha)| > M$. The uselessness of such an estimate is manifest.

The author is of the opinion that freedom from bias is not necessarily an indispensable characteristic of an optimum estimate. In general there is no reason for requiring the first moment of the estimate rather than any other moment to be the unknown parameter. The justification in any particular case must be based on special conditions of the problem.

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DIFFERENTIATION UNDER THE EXPECTATION SIGN IN THE FUNDAMENTAL IDENTITY OF SEQUENTIAL ANALYSIS

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1. Introduction. Let $\{z_\alpha\}$ ($\alpha = 1, 2, \cdots$, ad inf.) be a sequence of random variables which are independently distributed with identical distributions. Let a be a positive, and b a negative constant. For each positive integral value m, let Z_m denote the sum $z_1 + \cdots + z_m$. Denote by n the smallest integral value for which Z_n does not lie in the open interval (b, a). For any random variable u, let the symbol $E(u)$ denote the expected value of u. The following identity, which plays a fundamental role in sequential analysis, has been proved in [1].

$$E[e^{Z_n t}\varphi(t)^{-n}] = 1, \tag{1.1}$$

where

$$\varphi(t) = E(e^{zt}) \tag{1.2}$$

and the distribution of z is equal to the common distribution of $z_1, z_2, \cdots$, etc. Identity (1.1) holds for all points t in the complex plane for which $\varphi(t)$ exists and $|\varphi(t)| \geq 1$.

The purpose of this paper is to formulate conditions under which we may differentiate (1.1) with respect to t under the expectation sign. This is of interest, since various results in sequential analysis can easily be established by differentiating (1.1) under the expectation sign. For example, the formula for $E(n)$ can immediately be obtained by differentiating (1.1) at $t = 0$. The derivative of $e^{Z_n t}\varphi(t)^{-n}$ at $t = 0$ is given by

$$Z_n - \frac{\varphi'(0)}{\varphi(0)} n = Z_n - E(z)n \tag{1.3}$$

where $\varphi'(t)$ denotes the derivative of $\varphi(t)$. Hence, if we may differentiate (1.1) under the expectation sign, we obtain the basic formula

$$E(Z_n) = E(z)E(n). \tag{1.4}$$

If $E(z) \neq 0$, the above equation has been used [2] to derive lower and upper limits for $E(n)$. If, however, $E(z) = 0$, formula (1.4) is of little value. It will be shown in section 3 that

$$E(n) = \frac{E(Z_n^2)}{E(Z^2)} \quad \text{when} \quad E(z) = 0. \tag{1.5}$$

This result is obtained, as will be seen in section 3, by differentiating identity (1.1) twice at $t = 0$.

2. A sufficient condition for the differentiability of (1.1) under the expectation sign. In what follows, the parameter t in (1.1) will be restricted to real values, even if this is not stated explicitly. For any random variable u and any relation R, the symbol $E(u \mid R)$ will denote the conditional expected value of u under the restriction that R holds. In this section we shall establish the following theorem.

THEOREM 2.1. *If $\varphi(t)$ exists for all real values t, identity* (1.1) *may be differentiated under the expectation sign any number of times with respect to t at any value t in the domain $\varphi(t) \geq 1$.*

PROOF: First we shall derive an upper bound for $E(e^{tZ_n} \mid n = m)$ for any given integral value m. Consider the case when $t > 0$. Then

$$E(e^{tZ_n} \mid n = m) \leq E(e^{tZ_n} \mid Z_n \geq a,\ n = m) \qquad (t > 0). \tag{2.1}$$

Clearly,

$$E(e^{tZ_n} \mid Z_n \geq a, n = m, e^{tZ_{n-1}} = \rho e^{at}) = e^{at}\rho E\left(e^{zt} \mid e^{zt} \geq \frac{1}{\rho}\right). \tag{2.2}$$

Let $l(t)$ denote the least upper bound of the expression

$$\rho E\left(e^{zt} \mid e^{zt} \geq \frac{1}{\rho}\right) \tag{2.3}$$

with respect to ρ over the interval $(e^{-(a-b)|t|}, 1)$. The existence of $\varphi(t)$ implies that $l(t)$ is finite. It follows from (2.1) and (2.2) that

$$E(e^{tZ_n} \mid n = m) \leq e^{at}l(t) \qquad (t > 0) \tag{2.4}$$

and, therefore, also

$$E(e^{tZ_n}) \leq e^{at}l(t) \qquad (t > 0). \tag{2.5}$$

If $t < 0$, one can show in a similar way that

$$E(e^{tZ_n} \mid n = m) \leq e^{bt}l(t) \qquad (t < 0) \tag{2.6}$$

and

$$E(e^{tZ_n}) \leq e^{bt}l(t) \qquad (t < 0). \tag{2.7}$$

To prove Theorem 2.1, it is sufficient to show that the following two propositions hold.[1]

PROPOSITION 2.1. *All derivatives of $e^{Z_n t}\varphi(t)^{-n}$ with respect to t exist in the domain $\varphi(t) \geq 1$.*

PROPOSITION 2.2. *For any positive integral value r and for any finite interval I in which $\varphi(t) \geq 1$, it is possible to find a function $D(Z_n, n)$ such that*

$$D(Z_n, n) \geq \left| \frac{d^r}{dt^r} [e^{Z_n t} \varphi(t)^{-n}] \right| \tag{2.8}$$

for all values t in I and

$$E[D(Z_n, n)] < \infty. \tag{2.9}$$

Proposition 2.1 is clearly true, if all derivatives of $\varphi(t)$ exist. The existence of these derivatives follows from the existence of $\varphi(t)$ for all values t.

Since $\frac{d^r}{dt^r} e^{Z_n t}\varphi(t)^{-n}$ is equal to the sum of a finite number of terms of the type $Z_n^{r_1} n^{r_2} e^{Z_n t}\varphi(t)^{-n}$, Proposition 2.2 is proved if we can show that for any given integral values r_1 and r_2 there exists a function $D_{r_1 r_2}(Z_n, n)$ such that

$$D_{r_1 r_2}(Z_n, n) \geq | Z_n^{r_1} n^{r_2} e^{Z_n t}\varphi(t)^{-n} | \tag{2.10}$$

for all t in I and

$$E[D_{r_1 r_2}(Z_n, n)] < \infty. \tag{2.11}$$

Clearly, since $\varphi(t) \geq 1$ in I,

$$| Z_n^{r_1 r_2} e^{Z_n t}\varphi(t)^{-n} | \leq | Z_n^{r_1} | n^{r_2} e^{|Z_n| t_0} \tag{2.12}$$

where t_0 is an upper bound of $|t|$ in I. Let t_1 be a value $> t_0$. Then for a properly chosen constant C we have

$$| Z_n^{r_1} | e^{|Z_n| t_0} < C e^{|Z_n| t_1}. \tag{2.13}$$

[1] See, for example, E. J. McShane, *Integration*, Princeton University Press (1944), p. 216, 217 and 276.

Hence, it follows from (2.12) and (2.13) that

$$|Z_n^{r_1}n^{r_2}e^{Z_nt}\varphi(t)^{-n}| \leq Cn^{r_2}e^{|Z_n|t_1} \leq Cn^{r_2}(e^{Z_nt_1} + e^{-Z_nt_1}) \tag{2.14}$$

for all t in I.

We put

$$D_{r_1r_2}(Z_n, n) = Cn^{r_2}(e^{Z_nt_1} + e^{-Z_nt_1}). \tag{2.15}$$

We have

$$E[D_{r_1r_2}(Z_n, n)] = C\sum_{m=1}^{\infty} p_m m^{r_2}[E(e^{Z_nt_1} \mid n = m) + E(e^{-Z_nt_1} \mid n = m)] \tag{2.16}$$

where p_m denotes the probability that $n = m$.

Hence, because of (2.4) and (2.6), we obtain

$$\begin{aligned} E[D_{r_1r_2}(Z_n, n)] &\leq C(e^{at_1}l(t_1) + e^{-bt_1}l(-t_1))[\Sigma p_m m^{r_2}] \\ &= C[e^{at_1}l(t_1) + e^{-bt_1}l(-t_1)]E(n^{r_2}). \end{aligned} \tag{2.17}$$

Since all moments of n are finite,[2] Proposition 2.2 is proved. This completes the proof of Theorem 2.1.

3. The expected value of n when $E(z) = 0$. It will be shown in this section that

$$E(n) = \frac{E(Z_n^2)}{E(z^2)} \quad \text{when} \quad E(z) = 0, \tag{3.1}$$

if identity (1.1) can be differentiated twice under the expectation sign at $t = 0$. The second derivative of $e^{tZ_n}\varphi(t)^{-n}$ with respect to t is given by

$$\left\{\left[Z_n - n\frac{\varphi'(t)}{\varphi(t)}\right]^2 - n\frac{\varphi''(t)\varphi(t) - [\varphi'(t)]^2}{[\varphi(t)]^2}\right\} e^{Z_nt}\varphi(t)^{-n} \tag{3.2}$$

where $\varphi'(t)$ denotes the first, and $\varphi''(t)$ the second derivative of $\varphi(t)$.

Since $\varphi(0) = 1$, $\varphi'(0) = E(z) = 0$ and $\varphi''(0) = E(z^2)$, putting $t = 0$, expression (3.2) becomes

$$Z_n^2 - n\varphi''(0) = Z_n^2 - nE(z^2) \tag{3.3}$$

Hence, if (1.1) may be differentiated twice under the expectation sign at $t = 0$, we obtain

$$E[Z_n^2 - nE(z^2)] = 0 \tag{3.4}$$

from which (3.1) follows.

An approximate value of $E(n)$ can be obtained from (3.1) by neglecting the excess of Z_n over the boundaries. Then Z_n can take only the values a and b. Hence

$$E(Z_n^2) \sim a^2P(Z_n \geq a) + b^2P(Z_n \leq b) \tag{3.5}$$

where the sign $\sim$ denotes approximate equality.

[2] See the paper by C. Stein, "A note on cumulative suns," in this issue of the *Annals of Mathematical Statistics*.

It was shown in [1] (equation 28) that neglecting the excess of Z_n over the boundaries, the approximation formula

$$(3.6) \qquad P(Z_n \geq a) \sim \frac{1 - e^{bh}}{e^{ah} - e^{bh}}$$

holds, where h is the non-zero root of the equation $\varphi(t) = 1$. This formula was derived there under the assumption that $E(z) \neq 0$. If $E(z)$ approaches zero, $h \to 0$ and the right hand member of (3.6) converges to $\frac{-b}{a - b}$.

Putting $P(Z_n \geq a) = \frac{-b}{a - b}$ and $P(Z_n \leq b) = 1 - \frac{-b}{a - b} = \frac{a}{a - b}$, we obtain from (3.5)

$$(3.7) \qquad E(Z_n^2) \sim a^2 \left(\frac{-b}{a - b}\right) + b^2 \frac{a}{a - b} = -ab.$$

Hence[3]

$$(3.8) \qquad E(n) \sim \frac{-ab}{E(z^2)}.$$

Limits for $E(n)$ can be obtained by deriving limits for $E(Z_n^2)$. Let r be a non-negative real variable. One can verify that

$$(3.9) \qquad a^2 \leqq E(Z_n^2 \mid Z_n \geq a) \leqq \underset{0<r<a-b}{\text{l.u.b.}} E[(a - r + z)^2 \mid z \geq r]$$

and

$$(3.10) \qquad b^2 \leqq E(Z_n^2 \mid Z_n \leqq b) \leq \underset{0<r<a-b}{\text{l.u.b.}} E[(b + r + z)^2 \mid z + r \leq 0].$$

We have

$$(3.11) \qquad E(Z_n^2) = P(Z_n \geq a)E(Z_n^2 \mid Z_n \geq a) + P(Z_n \leq b)E(Z_n^2 \mid Z_n \leq b).$$

Limits for $E(Z_n^2)$ can be obtained by replacing the conditional expected values in the right hand member of (3.11) by their limits given in (3.9) and (3.10).

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