ON THE FIRST TWO MOMENTS OF THE MEASURE OF A RANDOM SET

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1. Introduction. In a recent paper [3] H. E. Robbins derived general formulas for the moments of the measure of any random set X, and applied the formulas to find the mean and the variance of a random sum of intervals on a line. In subsequent papers, J. Bronowski and J. Neyman [1], using other methods, found the variance when X is a random sum of rectangles in the plane, and H. E. Robbins [4] found the variance when X is a random sum of n-dimensional intervals in n-dimensional euclidean space. In the latter paper Robbins solved also the corresponding problem for circles on the plane.

Using the methods of Robbins, our purpose in the present paper is to solve the following similar problems:

(i) Let R denote the rectangle consisting of all points (x,y) such that $0 \leq x \leq A_1$, $0 \leq y \leq A_2$, and let R' denote the larger rectangle for which $-\delta \leq x \leq A_1 + \delta$, $-\delta \leq y \leq A_2 + \delta$. Let ρ denote a rectangle of fixed dimensions, $a \times b$, but variable position in the plane. The position of ρ will be determined by the coordinates x, y of its center P and the angle φ between the side of length a and the x-axis. We suppose $(a^2 + b^2)^{\frac{1}{2}} \leq \min (A_1, A_2, \delta)$. Let a fixed number N of rectangles ρ be chosen independently with the probability density function for the coordinates (x, y, φ) of each rectangle constant and equal to $\frac{1}{2}\ \pi\ R'$ in the three-dimensional interval with base R' and height π and zero outside this interval. In section 3 we evaluate the first two moments of the measure of X, where X denotes the intersection of the set-theoretical sum of the N rectangles ρ with R.

(ii) Let R denote the n-dimensional interval consisting of all points $(x_1, x_2, x_3, \cdots, x_n)$ such that $0 \leq x_i \leq A_i$, $(i = 1, 2, \cdots, n)$, and let R' denote the larger interval for which $-\delta \leq x_i \leq A_i + \delta$. Let a fixed number N of n-dimensional spheres with radii r (such that $2r \leq \min (A_i, 2\delta)$) be chosen independently, with the probability density function for the centre of each n-sphere constant and equal to $1/R'$ in R' and zero outside this interval. Denoting by X the intersection of the set theoretical sum of the N n-spheres with R, we evaluate in section 4 the first two moments of the measure of X. This problem is a generalization to n-dimensional space of the case considered by Robbins for the plane $(n = 2)$ in [4].

2. Preliminary formulas. Let K be an indeformable plane convex figure of variable position in the plane. The position of K may be determined by the coordinates (x, y) of a point P fixed within K and the angle φ which measures the rotation of K about P. We shall call x, y, φ the coordinates of K. The measure of a set of figures congruent with K is defined as being the integral of the differential form

$$dK = dxdyd\varphi. \tag{2.1}$$

It is readily shown that this measure does not depend on the particular point P chosen to determined the position of K[5]. For instance, the measure of the set of figures K, each of which contains in its interior a fixed point Q, has the value $2\pi F$, where F denotes the area of K; that is,

$$\int_{Q \epsilon K} dK = 2\pi F. \tag{2.2}$$

Let P_1 and P_2 be two fixed points and let l be the distance P_1P_2. The measure of the set of figures congruent with K, each of which contains both points P_1 and P_2 in its interior, will be a function of K and l, say $\mu(K, l)$. If d is the diameter of K, that is, the maximal distance between two points of K, we have $\mu(K, l) = 0$ for $l \geq d$.

Examples. Let K be a rectangle ρ of fixed dimensions $a \times b$, and let us suppose $a \leq b$. The diameter d of ρ is $d = (a^2 + b^2)^{\frac{1}{2}}$. Let $P(x, y)$ be the centre of ρ and φ the angle which forms the side of length b with the segment line P_1P_2 of length l. If we keep first φ constant, then in order that there exist positions of ρ in which it contains the segment line P_1P_2 in its interior it is necessary that

$$a - l \sin \varphi \geq 0, \qquad b - l \cos \varphi \geq 0$$

and in this case the area covered by the centres P in all these positions has the value

$$(a - l \sin \varphi)\,(b - l \cos \varphi).$$

Integrating over all permissible values of φ, we obtain

$$\mu(\rho, l) = 4 \int_{\operatorname{arc\,cos}[b/l]_1}^{\operatorname{arc\,sin}[a/l]_1} (a - l \sin \varphi)(b - l \cos \varphi)\, d\varphi \tag{2.3}$$

where we define

$$[x]_1 = \begin{matrix} x \text{ if } x \leq 1 \\ 1 \text{ if } x \geq 1. \end{matrix}$$

Carrying out the obvious integration in (2.3) we have

$$\mu(\rho, l) = \begin{cases} 2\pi ab - 4l(a + b) + 2l^2 & \text{for } l \leq a \leq b \\ 4(ab \operatorname{arc\,sin}(a/l) - \frac{1}{2}a^2 - bl + b(l^2 - a^2)^{\frac{1}{2}}) & \\ & \text{for } a \leq l \leq b \\ 4(ab \operatorname{arc\,sin}(a/l) - \operatorname{arc\,cos}(b/l) + b(l^2 - a^2)^{\frac{1}{2}} & \\ + a(l^2 - b^2)^{\frac{1}{2}} - \frac{1}{2}(a^2 + b^2) - \frac{1}{2}l^2) & \text{for } a \leq b \leq l. \end{cases} \tag{2.4}$$

As another example, let R be the rectangle consisting of all points (x, y) such that $0 \leq x \leq A_1$, $0 \leq y \leq A_2$ and let R' be the rectangle consisting of all points (x, y) such that

$$-\delta \leq x \leq A_1 + \delta, \quad -\delta \leq y \leq A_2 + \delta, \quad (a^2 + b^2)^{\frac{1}{2}} \leq \min (A_1, A_2, \delta).$$

Let us consider the set of rectangles ρ whose centers belong to R' and do not contain either P_1 or P_2, P_1 and P_2 being two fixed points which belong to R. Let l be the distance P_1P_2. According to (2.2) and the definition of $\mu(\rho, l)$ the measure of the set of rectangles ρ under consideration is

$$2\,\pi R' - 2.2\,\pi\rho + \mu(\rho, l), \tag{2.5}$$

where $R' = (A_1 + 2\,\delta)\,(A_2 + 2\,\delta)$ and $\rho = ab$.

Let K be a plane convex figure of fixed position in its plane. Let us suppose K to be translated a distance l in the direction θ, and let $F(Km, l, \theta)$ be the area of the intersection of K with the translated figure. Obviously if d is the diameter of K, $F(K, l, \theta) = 0$ for $l \geq d$. In what follows we shall consider the function

$$\Phi(K, l) = \int_0^{2\pi} F(K, l, \theta)\, d\theta. \tag{2.6}$$

Example. Let K be a rectangle R of sides A_1, A_2. Let the symbol $[x]$, as in [1], be defined by

$$[x] = \begin{matrix} x \text{ if } x \geq 0 \\ 0 \text{ if } x \leq 0. \end{matrix}$$

It is then readily seen that

$$F(R, l, \theta) = [A_1 - l \sin \theta]\,[A_2 - l \cos \theta]. \tag{2.7}$$

For our purpose the case in which $l \leq \min (A_1, A_2)$ is of interest. In this case, carrying out the immediate integrations, we obtain

$$\Phi(R, l) = 2\,\pi\,A_1A_2 - 4\,l(A_1 + A_2) + 2\,l^2. \tag{2.8}$$

Let $S_{n,r}$ be an n-dimensional sphere of radius r. $S_{n,r}$ will denote also the volume of this sphere, that is, as is known, (see [2, p. 109]),

$$S_{n,r} = \frac{(\pi r^2)^{n/2}}{\Gamma\left(\frac{n}{2} + 1\right)}. \tag{2.9}$$

Let us call the measure of a set of spheres $S_{n,r}$ the measure of the set of their centers. That is, if the point $P(x_1, x_2, \cdots, x_n)$ is the center of $S_{n,r}$ the measure of a set of spheres $S_{n,r}$ equals the integral extended over the set, of the differential form

$$dP = dx_1dx_2 \cdots dx_n. \tag{2.10}$$

For instance, the measure of the set of spheres $S_{n,r}$, each of which contains a fixed point Q in its interior, has the value

$$\int_{Q \epsilon S_{n,r}} dP = S_{n,r} \tag{2.11}$$

where $S_{n,r}$ is given by (2.9).

The measure $\mu(S_{n,r}, l)$ of the set of spheres $S_{n,r}$, each of which contains totally in its interior a segment of length $l(l \leq 2r)$, equals the volume of the intersection of two-spheres $S_{n,r}$ whose centers are placed at the end points of the given segment. That is, $\mu(S_{n,r}, l)$ equals twice the volume of the spherical segment of an n-sphere of radius r and semiangle $\alpha = \text{arc cos } (l/2r)$. We will represent the volume of this spherical segment by $S_{n,r}(\alpha)$ and it may be calculated in the following way: The intersection of the n-sphere with a hyperplane at a distance x from the center is an $(n - 1)$-dimensional sphere of radius $r' = (r^2 - x^2)^{\frac{1}{2}}$. Let $S_{n-1,r'}$ denote the volume of this $(n - 1)$-dimensional sphere (given by the general formula (2.9)). The volume of the spherical segment, whose base has the radius $h = r \cos \alpha$, will be

$$S_{n,r}(\alpha) = \int_h^r S_{n-1,r'}\, dx.$$

Putting $x = r \cos \theta$ and substituting for $S_{n-1,r'}$ the expression given in (2.9), we obtain

$$S_{n,r}(\alpha) = \frac{\pi^{(n-1)/2} r^n}{\Gamma\left(\frac{n+1}{2}\right)} \int_0^\alpha \sin^n \theta \, d\theta = r S_{n-1,r} \int_0^\alpha \sin^n \theta \, d\theta.$$

Consequently we can write

$$\mu(S_{n,r}, l) = 2S_{n,r}(\alpha) = 2rS_{n-1,r} \int_0^\alpha \sin^n \theta \, d\theta, \tag{2.12}$$

where $S_{n-1,r}$ is the volume of the $(n - 1)$-dimensional sphere of radius r and $\alpha = \text{arc cos } (l/2r)$.

In (2.12) we may substitute

$$\begin{aligned} \int_0^\alpha \sin^n \theta \, d\theta = {} & \frac{(n-1)(n-3)\cdots 3.1}{n(n-2)\cdots 4.2} \text{arc cos } (l/2r) \\ & - \frac{l}{2r}\left\{\frac{1}{n}\left(1 - \frac{l^2}{4r^2}\right)^{(n-1)/2} + \frac{(n-1)}{n(n-2)}\left(1 - \frac{l^2}{4r^2}\right)^{(n-3)/2} \right. \\ & \left. + \cdots + \frac{(n-1)(n-3)\cdots 3.1}{n(n-2)\cdots 4.2}\left(1 - \frac{l^2}{4r^2}\right)^{\frac{1}{2}}\right\} \end{aligned} \tag{2.13}$$

for n even, and

$$(2.14)\quad \int_0^\alpha \sin^n \theta\, d\theta\, \frac{(n-1)(n-3)\cdots 4.2}{n(n-2)\cdots 3} - \frac{l}{2r}\left\{\frac{1}{n}\left(1-\frac{l^2}{4r^2}\right)^{(n-1)/2} + \frac{n-1}{n(n-2)}\left(1-\frac{l^2}{4r^2}\right)^{(n-3)/2} + \cdots + \frac{(n-1)(n-3)\cdots 4.2}{n(n-2)\cdots 5.3}\right\}$$

for n odd.

In particular, for $n = 2, 3$ we have

$$(2.15)\quad \mu(S_{2,r}, l) = 4r^2 \int_0^\alpha \sin^2\theta\, d\theta = 2r^2 \arccos(l/2r) - \frac{1}{2} l(4r^2 - l^2)^{\frac{1}{2}}$$

$$(2.16)\quad \mu(S_{3,r}, l) = 2\pi r^3 \int_0^\alpha \sin\theta\, d\theta = \frac{4}{3}\pi r^3 - \pi r^2 l + \frac{1}{12}\pi l^3.$$

We shall now generalize the formula (2.8) to n-space.

A direction in n-space may be given by the corresponding point on the surface of the n-dimensional sphere of unit radius, that is, by the end point of the radius which is parallel to the given direction. The parametric equations of the n-sphere $\sum_1^n \xi_i^2 = 1$ are

$$(2.17)\quad \begin{aligned} \xi_1 &= \cos\varphi_1 \\ \xi_2 &= \sin\varphi_1 \cos\varphi_2 \\ \xi_3 &= \sin\varphi_1 \sin\varphi_2 \cos\varphi_3 \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \xi_{n-1} &= \sin\varphi_1 \sin\varphi_2 \cdots \sin\varphi_{n-2} \cos\varphi_{n-1} \\ \xi_n &= \sin\varphi_1 \sin\varphi_2 \cdots \sin\varphi_{n-2} \sin\varphi_{n-1}, \end{aligned}$$

where $0 \le \varphi_i \le \pi$ for $i < n - 1$ and $0 \le \varphi_{n-1} \le 2\pi$. The element of area of this n-sphere has the value (see, [2, p. 109])

$$(2.18)\quad d\sigma = \sin^{n-2}\varphi_1 \sin^{n-3}\varphi_2 \cdots \sin\varphi_{n-2}\, d\varphi_1 d\varphi_2 \cdots d\varphi_{n-1}.$$

A direction in n-dimensional space may then be given by the $n - 1$ parameters $\varphi_1, \varphi_2, \cdots, \varphi_{n-1}$.

Given the n-dimensional interval R consisting of all points $(x_1, x_2, x_3, \cdots, x_n)$ such that $0 \le x_i \le A_i$ $(i = 1, 2, 3, \cdots n)$, and suppose that R is translated a distance $l(l \le \min(A_1, A_2, A_3, \cdots, A_n))$ in the direction $(\varphi_1, \varphi_2, \cdots, \varphi_{n-1})$, the intersection of the translated interval with R is a new interval whose volume has the value $\prod_1^n (A_i - x_i)$, where $x_i = l\xi_i$ (ξ_i given by (2.17)).

Our purpose is to evaluate the integral

$$\Phi(R, l) = \int_{E_n} \prod_{1}^{n} (A_i - x_i)\, d\sigma \tag{2.19}$$

extended over the surface E_n of the n-dimensional sphere of radius unity. We shall denote by E_m either the surface of the m-dimensional sphere of radius unity or its area, given, as is known [2, p. 110] by

$$E_m = \frac{2\pi^{m/2}}{\Gamma\left(\frac{m}{2}\right)}. \tag{2.20}$$

Because of the symmetry, the coefficients of all the products $A_{i_1}A_{i_2}A_{i_3} \cdots A_{i_{n-k}}$ have the same value

$$\alpha_k = (-1)^k \int_{E_n} x_1 x_2 \cdots x_k \, d\sigma.$$

The integral extended over the whole surface E_n equals 2^n times the integral extended over the portion for which $\xi_i \geq 0$. Hence, taking into account (2.17) and (2.18) we get

$$\begin{aligned} \alpha_k &= (-1)^k 2^k l^k E_{n-k} \int_0^{\pi/2} \cdots \int_0^{\pi/2} \sin^{n+k-3}\varphi_1 \cos\varphi_1 \sin^{n+k-5}\varphi_2 \cos\varphi_2 \\ &\qquad \cdots \sin^{n-k-1}\varphi_k \cos\varphi_k \, d\varphi_1\, d\varphi_2 \cdots d\varphi_k \\ &= (-1)^k \frac{2^k l^k E_{n-k}}{(n+k-2)(n+k-4)\cdots(n+k-2k)} \end{aligned} \tag{2.21}$$

for $k = 1, 2, \cdots, n-1$. For $k = n$ we find that

$$\begin{aligned} \alpha_n &= (-1)^n 2^n l^n \int_0^{\pi/2} \cdots \int_0^{\pi/2} \sin^{2n-3}\varphi_1 \cos\varphi_1 \\ &\qquad \cdots \sin\varphi_{n-1} \cos\varphi_{n-1} \, d\varphi_1\, d\varphi_2 \cdots d\varphi_{n-1} \\ &= (-1)^n \frac{2^n l^n}{(2n-2)(2n-4)\cdots 4.2}. \end{aligned} \tag{2.22}$$

Hence, we have the following general formula

$$\begin{aligned} \Phi(R, l) = {} & A_1 A_2 \cdots A_n E_n + (-1)^n \frac{2^n l^n}{(2n-2)(2n-4)\cdots 4.2} \\ & + \sum_{k=1}^{n-1} (-1)^k \Big(\sum_{i_1, i_2, \cdots, i_{n-k}} A_{i_1} A_{i_2} \cdots A_{i_{n-k}} \Big) \\ & \qquad \frac{2^k l^k E_{n-k}}{(n+k-2)(n+k-4)\cdots(n+k-2k)}. \end{aligned} \tag{2.23}$$

In particular, for $n = 2$ this result coincides with (2.8). For $n = 3$ we have

$$\Phi(R, l) = 4\pi A_1A_2A_3 - l^3 - 2\pi l(A_1A_2 + A_1A_3 + A_2A_3) + \tfrac{8}{3}l^2(A_1 + A_2 + A_3). \tag{2.24}$$

3. First problem. We can now solve the first problem (i) stated in the introduction. Denoting by the same letters either sets or their measures, we consider, as in [1] and [4], the set Y of points of R that do not belong to X. We have identically:

$$X + Y = R. \tag{3.1}$$

The general method of Robbins [3] taking into account (2.2), gives immediately the first moments

$$E(Y) = R\left(1 - \frac{\rho}{R'}\right)^N, \qquad E(X) = R\left\{1 - \left(1 - \frac{\rho}{R'}\right)^N\right\}, \tag{3.2}$$

where $R = A_1A_2$, $R' = (A_1 + 2\delta)(A_2 + 2\delta)$, $\rho = ab$.

Our remaining problem is that of evaluating the second moment of X. Let x_i, y_i, φ_i $(i = 1, 2, 3, \cdots, N)$ be the coordinates of the N rectangles ρ (section 2) and let us put, as in (2.1), $d\rho_i = dx_i dy_i d\varphi_i$. Let $P(x, y)$ and $P_0(x_0, y_0)$ be two points which belong to R and let us put $dP = dx\, dy$, $dP_0 = dx_0 dy_0$. Let us consider the following multiple integral

$$J = \int \frac{dP\, dP_0\, d\rho_1\, d\rho_2 \cdots d\rho_N}{(2\pi R')^N} \tag{3.3}$$

extended over the sets of rectangles ρ_i (congruent with ρ) such that x_i, y_i belongs to R', $0 \le \varphi_i \le 2\pi$, and do not contain either P or P_0. That is, the domain of integration of J is defined by

$$\begin{gathered} -\delta \le x_i \le A_1 + \delta, \qquad -\delta \le y_i \le A_2 + \delta, \qquad 0 \le \varphi_i \ge 2\pi, \\ P \in R, \qquad P_0 \in R, \qquad P \notin \rho_i, \qquad P_0 \notin \rho_i, \qquad (i = 1, 2, \cdots, N). \end{gathered} \tag{3.4}$$

In order to calculate J, we can first keep the rectangles ρ_i fixed; the points P and P_0 can then vary independently over the set of points Y. That gives

$$J = \int_{(x_i, y_i) E R'} \frac{Y^2\, d\rho_1\, d\rho_2 \cdots d\rho_N}{(2\pi R')^N} = E(Y^2). \tag{3.5}$$

We can now reverse the order of integration, an operation which is obviously justified in this case. Keeping P and P_0 fixed, we can vary each rectangle ρ_i over the set of positions in which it does not contain either P or P_0; letting l denote the distance PP_0, we have, according to (2.5),

$$J = \int_{P \in R, P_0 \in R} \left(1 - \frac{4\pi\rho - \mu(\rho, l)}{2\pi R'}\right)^N dP dP_0. \tag{3.6}$$

In order to evaluate this integral we divide it into two parts $J = J_1 + J_2$, according as $0 \leq l \leq d$ or $d \leq l \leq D$, where $d = (a^2 + b^2)^{\frac{1}{2}}$ and $D = (A_1^2 + A_2^2)^{\frac{1}{2}}$. In the interval $0 \leq l \leq d$ we introduce the new variables of integration l. θ related to x, y, x_0, y_0 by

$$x_0 = x + l \cos \theta, \qquad y_0 = y + l \sin \theta \tag{3.7}$$

whence

$$\frac{\partial(x, y, x_0, y_0)}{\partial(x, y, l, \theta)} = l.$$

In terms of the new variables we have

$$J_1 = \int \left(1 - \frac{4\pi p - \mu(p, l)}{(2\pi R')}\right)^N l \, dl \, dP \, d\theta.$$

In this integral the point P can vary over the intersection of R with the figure obtained by translating R a distance l in the direction θ; that is, the integration of dP gives the function $F(R, l, \theta)$ defined in section 2. According to (2.6) we therefore have

$$J_1 = \int_0^d \left(1 - \frac{4\pi\rho - \mu(\rho, l)}{2\pi R'}\right)^N \Phi(R, l) l \, dl, \tag{3.8}$$

where $\mu(\rho, l)$ is given by (2.4) and $\Phi(R, l)$ by (2.8).

In order to evaluate J_2 we observe that in the interval $d \leq l \leq D$ $\mu(\rho, l) = 0$ and we have

$$J_2 = \left(1 - \frac{2\rho}{R'}\right)^N \int_{d \leq l \leq D} dP \, dP_0 = \left(1 - \frac{2\rho}{R'}\right)^N \left\{ \int^{0 \leq l \leq d} dP \, dP_0 - \int_{0 \leq l \leq d} dP \, dP_0 \right\}.$$

Further we have

$$\int_{0 \leq l \leq D} dP \, dP_0 = R^2 \tag{3.9}$$

and with the change of variables (3.7) and the formula (2.8) we find that

$$\int_{0 \leq l \leq d} dP \, dP_0 = \int_0^d \Phi(R, l) l \, dl = \pi A_1 A_1 d^2 - \frac{4}{3} (A_1 + A_2) d^3 + \frac{1}{2} d^4. \tag{3.10}$$

Collecting (3.8), (3.9), (3.10) and taking into account (3.5) we have

$$\begin{aligned} E(Y^2) &= \int_0^d \left(1 - \frac{4\pi\rho - \mu(\rho, l)}{2\pi R'}\right)^N \Phi(R, l) l \, dl \\ &\quad + \left(1 - \frac{2\rho}{R'}\right)^N \{R^2 - \pi A_1 A_2 d^2 + \tfrac{4}{3}(A_1 + A_2) d^3 - \tfrac{1}{2} d_4\}, \end{aligned} \tag{3.11}$$

where $\rho = ab$, $R = A_1A_2$, $R' = (A_1 + 2\delta)(A_2 + 2\delta)$, $\mu(\rho, l)$ is given by (2.4) and $\Phi(R, l)$ by (2.8).

For the variance of X and of Y, we have by (3.1) and (3.2)

$$\begin{aligned}\sigma_X^2 &= E(X^2) - E^2(X) = E(Y^2) - E^2(Y) \\ &= \int_0^d \left(1 - \frac{4\pi\rho - \mu(\rho, l)}{2\pi R'}\right)^N \Phi(R, l)l\,dl + \left(1 - \frac{2\rho}{R'}\right)^N \\ &\quad \cdot \{R^2 - \pi A_1A_2d^2 + \tfrac{4}{3}(A_1A_2)d^3 - \tfrac{1}{2}d^4\} - R^2\left(1 - \frac{\rho}{R'}\right)^{2N},\end{aligned}$$

which completes the solution of our first problem stated in the introduction.

4. Second problem. In order to solve the second problem (ii) stated in the introduction we will follow the same method of the preceding section.

Let X be the intersection of the set theoretical sum of the N n-dimensional spheres $S_{n,r}$ of radius r with the n-interval R. Let us call Y the set of those points of R that do not belong to X, that is,

$$X + Y = R. \tag{4.1}$$

The general method of Robbins gives immediately

$$E(Y) = R\left(1 - \frac{S_{n,r}}{R'}\right)^N, \qquad E(X) = R\left\{1 = \left(1 - \frac{S_{n,r}}{R'}\right)^N\right\} \tag{4.2}$$

where $R = \prod_1^n A_i$, $R' = \prod_1^n (A_i + 2\delta)$, and $S_{n,r}$ is given by (2.9).

We now proceed to calculate $E(Y^2)$. For this purpose let $Q_1(y_1^1, y_2^1, \cdots, y_n^1)$ and $Q_2(y_1^2, y_2^2, \cdots, y_n^2)$ be two points which belong to R and $P_i(x_1^i, x_2^i, \cdots, x_n^i)$ be the centers of the N spheres $S_{n,r}$. Let us put

$$dQ_i = dy_1^i dy_2^i \cdots dy_n^i, \ (i = 1, 2), \quad dP_i = dx_1^i dx_2^i \cdots dx_n^i, \ (i = 1, 2, \cdots, N). \tag{4.3}$$

Consider the integral

$$J = \int \frac{dQ_1 dQ_2 dP_1 dP_2 \cdots dP_N}{R'^N} \tag{4.4}$$

extended over the domain defined by

$$Q_1 \in R, \quad Q_2 \in R, \quad P_i \in R', \quad \overline{Q_1P_i} > r, \quad \overline{Q_2P_i} > r, \qquad (i = 1, 2, \cdots, N).$$

If we keep $P_1, P_2, P_3, \cdots, P_N$ fixed, each point Q_1, Q_2 can vary independently over the set Y; consequently we have

$$J = \int_{P_i \in R'} \frac{Y^2 dP_1 dP_2 \cdots dP_N}{R'^N} = E\,(Y^2). \tag{4.5}$$

On the other hand, if we keep Q_1 and Q_2 fixed, the integral of each dP_i gives

$R' - 2\, S_{n,r} + \mu(S_{n,r}\,,\, l)$ where $\mu(S_{n,r}\,,\, l)$ is given by (2.12) and $l = \overline{Q_1Q_2}$. Hence we have

$$J = \int_{Q_1 \epsilon R, Q_2 \epsilon R} \left(1 - \frac{2S_{n,r} - \mu(S_{n,r}\,,\, l)}{R'}\right)^N dQ_1\, dQ_2. \tag{4.6}$$

In order to calculate this integral we split it into two parts $J = J_1 + J_2$, according as $0 \leq l \leq 2r$ or $2r \leq l \leq D$, where $D = (\sum_1^n A_i^2)^{\frac{1}{2}}$. In the interval $0 \leq l \leq 2r$ we introduce the new variables of integration $l, \varphi_1, \varphi_2, \cdots, \varphi_{n-1}$ related to $\overset{1}{y}_1, \overset{1}{y}_2, \cdots \overset{1}{y}_n, \overset{2}{y}_1, \overset{2}{y}_2, \cdots, \overset{2}{y}_n$ by

$$\overset{2}{y}_i = \overset{1}{y}_i + l\xi_i, \qquad (i = 1, 2, \cdots, n), \tag{4.7}$$

where ξ_i is given in (2.17). It is found that

$$\frac{\partial(\overset{1}{y}_1, \overset{1}{y}_2, \cdots, \overset{1}{y}_n, \overset{2}{y}_1, \overset{2}{y}_2, \cdots, \overset{2}{y}_n)}{\partial(\overset{1}{y}_1, \overset{1}{y}_2, \cdots, \overset{1}{y}_n, l, \varphi_1, \cdots \varphi_{n-1})} = l^{n-1} \sin^{n-2} \varphi_1 \sin^{n-3} \varphi_2 \cdots \sin \varphi_{n-2}.$$

Hence we have,

$$dQ_1 dQ_2 = l^{n-1}\, dQ_1 d\sigma dl, \tag{4.8}$$

where $d\sigma$ denotes the element of area of the n-dimensional sphere of unit radius, given by (2.18). The same method used in section 3 gives

$$J_1 = \int_0^{2r} \left(1 - \frac{2S_{n,r} - \mu(S_{n,r}\,,\, l)}{R'}\right)^N \Phi(R, l) l^{n-1}\, dl, \tag{4.9}$$

where $\Phi(R, l)$ is given by (2.23).

In the interval $2r \leq l \leq D$ $\mu(S_{n,r}\,,\, l) = 0$ and we have

$$\begin{aligned} J_2 = \left(1 = \frac{2S_{n,r}}{R'}\right)^N \int_{2r \leq l \leq D} dQ_1 dQ_2 &= \left(1 = \frac{2S_{n,r}}{R'}\right)^N \\ &\cdot \left\{\int_{0 \leq l \leq D} dQ_1 dQ_2 - \int_{0 \leq l \leq 2r} dQ_1 dQ_2\right\}. \end{aligned} \tag{4.10}$$

Now we have

$$\int_{0 \leq l \leq D} dQ_1\, dQ_2 = R^2 \tag{4.11}$$

and with the change of variables (4.7) we readily find that

$$\int_{0 \leq l \leq 2r} dQ_1\, dQ_2 = \int_0^{2r} \Phi(R, l) l^{n-1}\, dl. \tag{4.12}$$

Collecting (4.9), (4.10), (4.11), (4.12) and taking into account (4.5) and (2.23) we have

$$
(4.13)\quad
\begin{aligned}
E(Y^2) = &\int_0^{2r} \left(1 - \frac{2S_{n,r} - \mu(S_{n,r}, l)}{R'}\right)^N \Phi(R, l) l^{n-1}\, dl \\
&+ \left(1 - \frac{2S_{n,r}}{R'}\right)^N \Bigg\{ R^2 - \frac{2^n r^n}{n} RE_n - (-1)^n \frac{2^{3n} r^{2n}}{2n(2n-2)\cdots 4.2} \\
&- \sum_{k=1}^{n-1} (-1)^k \Big(\sum_{i_1, i_2, \cdots, i_{n-k}} A_{i_1} A_{i_2} \cdots A_{i_{n-k}} \Big) \\
&\qquad \cdot \frac{2^{n+2k} E_{n-k} r^{n+k}}{(n+k)(n+k-2)\cdots(n+k-2k)} \Bigg\}
\end{aligned}
$$

where $R = \prod_1^n A_i$, $R' = \prod_1^n (A_i + 2\delta)$; $S_{n,r}$ is given by (2.9), E_m by (2.20), $\mu(S_{n,r}, l)$ by (2.12) and $\Phi(R, l)$ by (2.23). In particular, for $n = 2$, we obtain the value given by Robbins [3, (30)], by use of (2.8), (2.15) and the equations $S_{2,n} = \pi r^2$, $E_i = 2$. For $n = 3$, the case of ordinary space it follows from (2.16), (2.24) and the equations $S_{3,r} = \frac{4}{3}\pi r^3$, $E_3 = 4\pi$, $E_2 = 2\pi$, that

$$
(4.14)\quad
\begin{aligned}
E(Y^2) = &\int_0^{2r} \left(1 - \frac{16\pi r^3 + 12\pi r^2 l - \pi l^3}{12R'}\right)^N \Big(4\pi R - l^3 - 2\pi(A_1A_2 + A_1A_3 \\
&+ A_2A_3)l + \frac{8}{3}(A_1 + A_2 + A_3)l^2\Big) l^2\, dl + \left(1 - \frac{8\pi r^3}{3R'}\right)^N \Big\{ R^2 \\
&- \frac{32}{3}\pi R r^3 + 8\pi(A_1A_3 + A_2A_3 + A_2A_3)r^4 \\
&- \frac{256}{15}(A_1 + A_2 + A_3)r^5 + \frac{32}{3} r^6 \Big\}.
\end{aligned}
$$

In this case the exact evaluation is easy if one expands the binomial under the sign of the integral and integrates term by term.

From (4.1) we see that $\sigma_X^2 = E(X^2) - E^2(X) = E(Y^2) - E^2(Y)$. Thus, from (4.2) and (4.13) we obtain immediately the second moment $E(X^2)$ and the variance σ_X^2 of X.

5. Remark. In the second problem we can substitute the n-intervals R and R' by concentric n-dimensional spheres. The problem may then be stated as follows:

Let $S_{n,a}$ denote a fixed n-dimensional sphere of radius a and $S_{n,a+\delta}$ the concentric n-dimensional sphere of radius $a + \delta$. $S_{n,a}$ and $S_{n,a+\delta}$ shall also denote the corresponding volumes. Let a fixed number N of n-dimensional spheres with radii r ($r \le \min(a, \delta)$) be chosen independently with the probability density function for the center of each $S_{n,r}$ constant and equal to $1/S_{n,a+\delta}$ in $S_{n,a+\delta}$ and zero outside this n-sphere. Let X denote the intersection of the set-theoretical sum of the N n-spheres with $S_{n,a}$; we wish to evaluate the first two moments of the measure of X.

It suffices to observe that in this case we have

$$\Phi(S_{n,a}, l) = \mu(S_{n,a}, l)E_n = 2a\, S_{n-1,a}E_n \int_0^\alpha \sin^n \theta\, d\theta \tag{5.1}$$

where $S_{n-1,a}$ is the volume of the $(n - 1)$-dimensional sphere of radius a and $\alpha = \text{arc cos } (l/2a)$.

The same method used in section 4 gives

$$E(Y) = S_{n,a}\left(1 - \frac{S_{n,r}}{S_{n,a+\delta}}\right)^N, \qquad E(X) = S_{n,a}\left\{1 - \left(1 - \frac{S_{n,r}}{S_{n,a+\delta}}\right)^N\right\}, \tag{5.2}$$

$$\begin{aligned} E(Y^2) &= \int_0^{2r} \left(1 - \frac{2S_{n,r} - \mu(S_{n,r}, l)}{S_{n,a+\delta}}\right)^N \Phi(S_{n,a}, l)l^{n-1}\, dl \\ &\quad + \left(1 - \frac{2S_{n,r}}{S_{n,a+\delta}}\right)^N \left\{S_{n,a}^2 - \int_0^{2r} \Phi(S_{n,a}, l)l^{n-1}\, dl\right\}, \end{aligned} \tag{5.3}$$

where $\Phi(S_{n,a}, l)$ is given by (5.1).

In particular, for $n = 2$, by use of (5.1), (2.15) and the indefinite integrals

$$\int \text{arc cos } (l/2a)l\, dl = (\tfrac{1}{2}l^2 - a^2) \text{ arc cos } (l/2a) - \tfrac{1}{4}\, l(4a^2 - l^2)^{\frac{1}{2}} + \text{constant},$$

$$\begin{aligned} \int l^2(4a^2 - l^2)^{\frac{1}{2}}\, dl = -\tfrac{1}{4}l(4a^2 - l^2)^{\frac{3}{2}} + \tfrac{1}{2}a^2l(4a^2 - l^2)^{\frac{1}{2}} \\ + 2a^4 \text{ arc sin } (l/2a) + \text{constant} \end{aligned}$$

we find that

$$\begin{aligned} E(Y^2) = 2\pi \int_0^{2r} &\left(1 - \frac{2\pi r^2 - 2r^2 \text{ arc cos } (l/2r) + \frac{1}{2}l(4r^2 - l^2)^{\frac{1}{2}}}{\pi(a + \delta)^2}\right. (2a^2 \text{ arc cos}(l/2a) \\ &- \tfrac{1}{2}l(4a^2 - l^2)^{\frac{1}{2}}\Big)l\, dl + \left(1 - \frac{2r^2}{(a + \delta)^2}\right)^N \left\{\pi^2 a^4 - 2\pi\left(2a^2(2r^2 - a^2) \text{ arc cos}\left(\frac{r}{a}\right)\right.\right. \\ &- 3a^2\, r(a^2 - r^2)^{\frac{1}{2}} + \pi a^4 + 2r(a^2 - r^2)^{\frac{3}{2}} - a^4 \text{ arc sin}(r/a)\Big)\Big\}. \end{aligned}$$

For $n = 3$, we have by (5.1) and 2.16)

$$\begin{aligned} E(Y^2) = 4\pi \int_0^{2r} &\left(1 - \frac{16r^3 + 12r^2 l - l^3}{16(a + \delta)^3}\right)^N (\tfrac{4}{3}\pi a^3 - \pi a^2 l + \tfrac{1}{12}\pi l^3)l^2\, dl \\ &+ 4\pi\left(1 - \frac{2r^3}{(a + \delta)^3}\right)^N \left\{\tfrac{4}{9}\pi a^6 - \tfrac{32}{9}\pi a^3 r^3 + 4\pi a^2 r^4 - \tfrac{8}{9}\pi r^6\right\}. \end{aligned}$$

From (5.2) and (5.3) with the use of the relation $\sigma_X^2 = E(X^2) - E^2(X) = E(Y^2) - E^2(Y)$ we obtain immediately the second moment $E(X^2)$ and the variance σ_X^2 of X.

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