PROBABILITY OF COINCIDENCE FOR TWO PERIODICALLY RECURRING EVENTS[1]

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Summary. This paper contains a study of the following problem: Each of two events recurs with definitely known period and duration, while the starting time of each event is unknown. It is desired that, before the elapse of a certain time, the events occur simultaneously and that this "overlap" be of at least a given minimum duration.

The probability of this satisfactory coincidence is first evaluated, and it is found that the solution, while mathematically adequate, is of no value for practical application. This circumstance arises from the possibility that, with certain rational ratios of the periods, the events may "lock in step". Accordingly, an attempt is made to smooth the probability function with respect to small variations in the ratio of the periods. Due to difficulties in manipulating the number-theoretic expressions involved, this smoothing is carried through only by the use of certain approximations. Moreover, because of these same difficulties, an averaged value of the probability itself is not obtained, but, in its stead, there is derived a formula for that fraction of randomly related repeated trials in which the original probability will be less than one-half.

Thus, the original problem is not completely solved. The results obtained, however, do allow one to compare the relative advantages of different situations and to make a rough estimate of the likelihood of success. Generally speaking, the analysis is applicable whenever the ratio of "on time" to "off time" is small for each event.

1. Introduction. Our problem may be represented schematically as follows: Consider two pulse waves (Fig. 1) of periods T_1, T_2, pulse widths t_1, t_2, and phases ϕ_1, ϕ_2. It is desired that these pulses overlap at least once within a given time interval; moreover, an overlap is not satisfactory unless its duration is at least as great as some assigned t_m. The starting phases ϕ_1 and ϕ_2 are unknown for *both* waves. Our problem, then, would appear to be to calculate as a function of time the probability of at least one overlap of duration at least t_m.

This probability will be calculated later, and, while mathematically adequate, is totally useless for practical application. This rather unusual occurrence in applied mathematics arises from sources generally kept in mind only by experimental physicists. Namely, the very nature of the science of measurement, involving as it always does at some stage, the use of the human senses, precludes

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the availability of mathematically exact values of the parameters of the problem. In other words, although experimental error can sometimes be made amazingly small, it can *never* be eliminated.

Now, as might be expected from the possibility that the waves may "lock in step", our probability is extremely erratic with respect to very minute changes in the periods T_1, T_2. For example, let $T_1 = T_2 = 100t_1 = 100t_2$ ($t_m = 0$); a simple direct calculation then shows that, for all times greater than $T_1 = T_2$, the desired probability is 0.03. Now if we let $T_1 = T_2 + \epsilon$, one wave will "creep up" on the other, and eventually (for times greater than T_1T_2/ϵ) the probability is unity! Thus it may very well happen in a practical application that the parameters are known to an accuracy essentially sufficient only to give the obvious result: $0 < P \leq 1$.

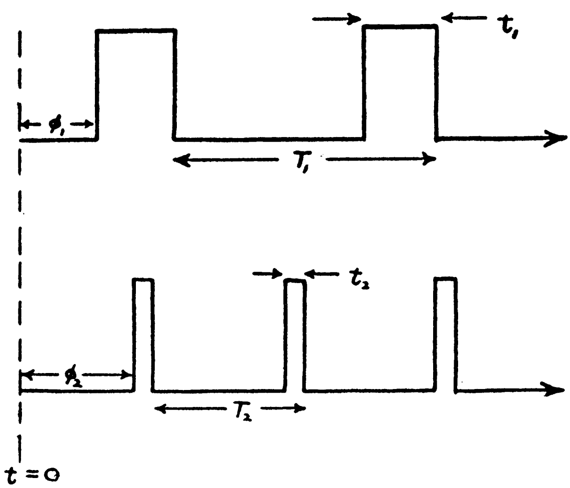

FIG. 1

In the practical problem originally considered, uncertainty in the data arose not only from experimental error but also from slight instability of equipment. Thus some means of averaging over variations in the periods had to be found if the analysis was to be of any practical value whatsoever.

For reasons which will appear in the later analysis, this smoothing entails difficulties which the author was unable to overcome with any great success; the nature of the results which have been obtained is discussed in the next section. These results involve several approximations which, generally speaking, are based on the assumption that the ratios t_i/T_i are both small.

It might be noted finally that the obviously favorable situations $t_1 > T_2$ or $t_2 > T_1$ often cannot be used because of numerous practical difficulties.

2. Results. In this section, we shall summarize the results of the later analysis for the benefit of those readers not interested in the latter. At the end of this section, there is an outline of the practical application of the formulas.

We shall continue to use the notation already introduced:

$$(1)\qquad \begin{aligned} t_1, t_2 &= \text{durations of the events;} \\ T_1, T_2 &= \text{periods of the events;} \\ t_m &= \text{minimum satisfactory duration of coincidence; and} \\ P &= \text{probability of at least one satisfactory coincidence.} \end{aligned}$$

We shall also use the (at present) rather arbitrary notation:

$$(2)\qquad \begin{aligned} t &= (\text{time} - t_m) \\ P_0 &= (t_1 - t_m)(t_2 - t_m)/T_1T_2 \\ w &= (t_1 + t_2 - 2t_m)/T_1T_2. \end{aligned}$$

The probability function for short time intervals is:

$$(3)\qquad P = P_0 + wt, \qquad \text{for } t \leq \operatorname{Max}(T_1, T_2).$$

In any case:

$$(4)\qquad P \leq P_0 + wt.$$

As already explained, the functional dependence of P for large t is of no practical use due to its extremely erratic variation with small changes in the periods T_1, T_2.

For reasons which will later become apparent, the only type of averaging which has yet been carried to completion is the following. Consider that many trials of equal length are made and that in each individual trial, all the parameters are, by some mysterious device, held constant with absolute, mathematical exactitude. Assume for definiteness that $T_2 \leq T_1$. Between different trials, let t_2 and T_2 vary in such a way that T_1/T_2 takes all values within a range of $\frac{1}{2}$ with equal probability. (In the original problem, the ratios t_i/T_i necessarily remained constant.) The quantity f given below then represents that fraction of the trials in which the rigorous probability is *less* than an assigned value $= P_0 + Q$. Thus the *smaller* f is, the greater are the chances of success.

It must be admitted that this method assumes several things which are not true in practice. First, the parameters of the problem probably vary by at least a percent even within a single trial. More serious, the required variation in T_1/T_2 may, in the extreme case $T_1 = T_2$, demand as much as 33% variation in T_2. While considerable variation does occur, it is doubtful that it attains this magnitude. Finally, the method assumes that T_1 stays fixed as T_2 varies, whereas actually T_1 and T_2 vary simultaneously.

Despite these drawbacks, it was felt that the results were meaningful for the practical problem. In any case, they must serve until a more adequate analysis can be carried through.

The reader will notice that the final results have the form of a "probability of a probability". It would thus seem that a simple integration would yield a true probability, but, unfortunately, the formulas for f are reasonably accurate only for $Q \leq \frac{1}{2}$. The final formula for f = fraction of trials in which $P < P_0 + Q$ is:

$$(5) \quad f = \begin{cases} 1 & \text{for } tw < Q, \\ 1.216\, Q\left\{1 + \left(\frac{tw}{Q} - 1\right)\log\left(1 - \frac{Q}{tw}\right)\right\}, & \text{for } tw > Q, \quad Q \leq 1/2. \end{cases}$$

This expression is subject to error from several sources. First it is an approximation to a number-theoretic formula given in (31); this approximation is best for t and Q/w large compared to $\text{Max}(T_1, T_2)$. A *completely general* comparison of (31) and (5) = (33) is given in Fig. 2, where the agreement will be seen to be quite adequate even for relatively small t and Q/w. (The dotted contours are straight lines passing through the origin.) When t and Q/w are small this first source of error can be eliminated by using the solid contours of Fig. 2 in place of (5).

Secondly, formula (31) itself is an approximation and involves the use of simplified probability formulas and an assumption that P_0 and w are constant as T_2 varies. The maximum possible magnitude of these errors in (31) is given by (parentheses indicate functional dependence):

$$(6) \qquad f(t\bar{w}, Q - p_0 - q) \leq f(tw, Q) \leq f(t\underline{w}, Q + p_0 + q),$$

where, as T_2 varies,

$$\underline{w}, \bar{w} = \text{minimum, maximum values of } w$$

$$p_0 = \text{change in } P_0$$

$$q = \text{maximum value of } w^2 T_1 T_2.$$

Generally speaking, these errors are small if t_i/T_i are small and if t is large compared to $\text{Max}(T_1, T_2)$. Also, there is considerable possibility that certain errors will cancel in such a way as to make (6) correct with $q = 0$.

We shall now outline the practical use of these results. Given nominal values of the parameters defined in (1), choose a convenient value for $Q \leq \frac{1}{2}$ (usually $Q = \frac{1}{2}$), and substitute into (2) to find tw/Q. From (5), one may then determine f = fraction of trials in which $P < P_0 + Q$. (Low values of f are thus desirable.) For computational convenience, (5) has been plotted in Fig. 3, while, above the range of Fig. 3, the following lies within 1% of (5).

$$(7) \qquad f = 0.608(Q^2/tw) \qquad \text{for} \qquad tw > 10Q.$$

Note also that (4) may often be of considerable use in quickly eliminating cases of very poor probability, and recall also that (3) will give the true, directly meaningful probability whenever t is no greater than $\text{Max}(T_1, T_2)$.

Evaluation of the maximum possible error in f as so obtained is more complicated. If t and Q/w are small, Fig. 2 may be used to eliminate inexactness due to the approximation of (31) by (5) = (33). Otherwise, this error may safely be assumed to be negligible (less than 0.025; (31) may be employed directly, but this is laborious unless Q/w is small). The remaining errors, given by (6), may change depending on how T_2 is assumed to vary. To make these bounds as close as possible, it is best to choose $T_2 = \mathrm{Min}(T_1, T_2)$ and then let

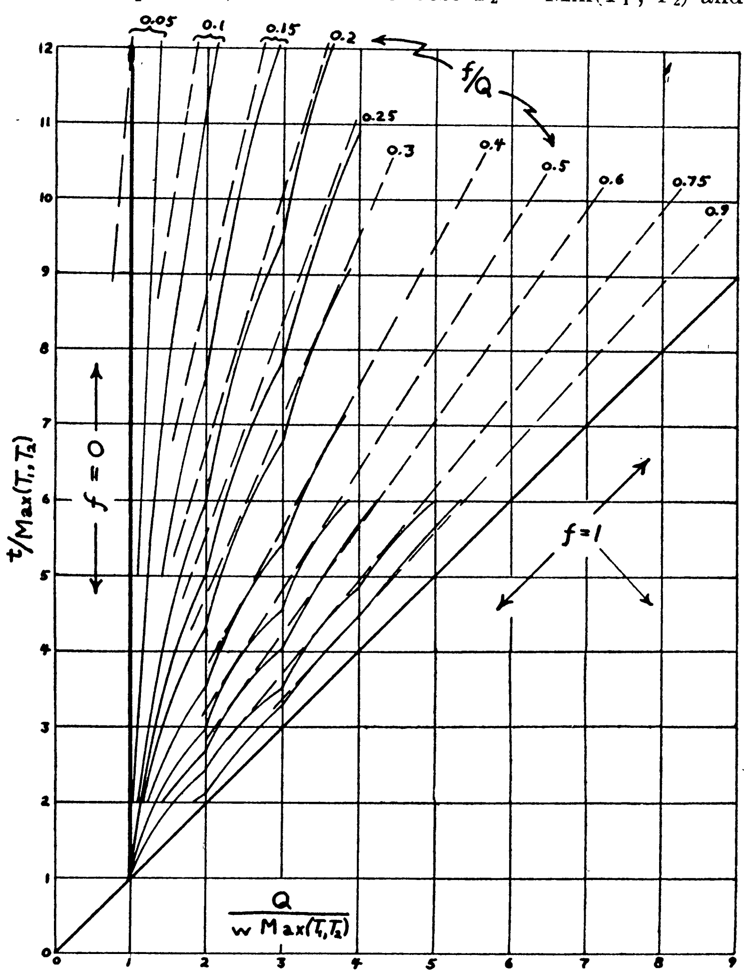

FIG. 2. Contours of f/Q: —— (31); — — (33)

T_2 decrease from its nominal value by an amount sufficient to cause T_1/T_2 to increase by $\frac{1}{2}$.

The reader may have noticed that f has a jump discontinuity as t passes through the value Q/w. This is not the result of approximations; it occurs also in the number-theoretic formula (excepting only when $\mathrm{Max}(T_1, T_2) = \frac{1}{2}w$ and $Q = \frac{1}{2}$) and merely means that the "lock in" phenomena are suddenly able to have an effect when t becomes greater than Q/w.

3. The probability function. Our problem has already been represented by the pulse waves of Fig. 1. The starting phases ϕ_1, ϕ_2 of the waves are random, and we desire the probability P of at least one overlap of duration at least t_m within a given time interval. Manifestly $P = 0$ until time t_m; hence we shall give t the meaning already assigned in (2).

Consider any sub-interval of width t_m. The range of phases favorable to satisfactory coincidence on this interval is easily seen to be a rectangle with sides $(t_1 - t_m)$, $(t_2 - t_m)$ in the phase plane (ϕ_1, ϕ_2). By proper choice of the (arbitrary) zero-phase reference, the small rectangle favorable to coincidence on $(0, t_m)$ can be made to fall in the lower left corner of the phase plane (Fig. 4).

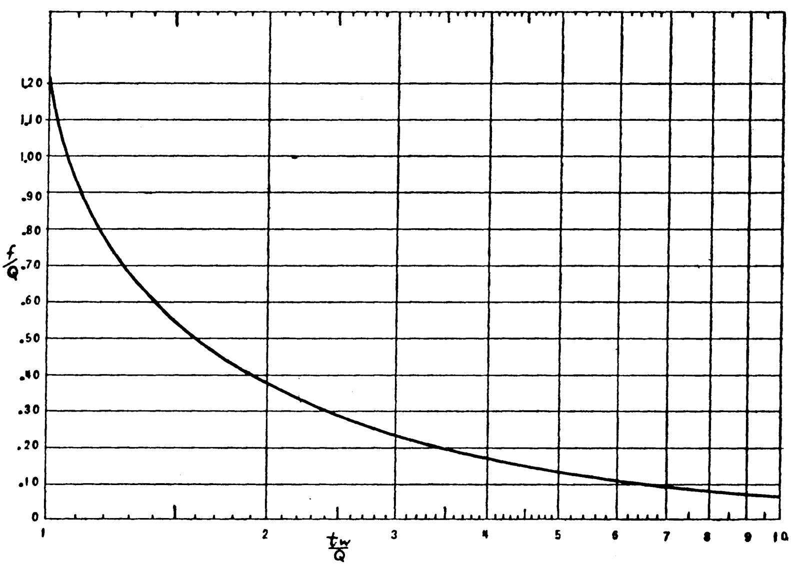

FIG. 3

As we allow the sub-interval (width t_m) to advance in time, this small rectangle will sweep out along a 45° line (Fig. 4); its horizontal displacement = vert. disp. is given by t as defined in (2). Since the phases must be measured modulo the periods, we must "switch back" the strip whenever it begins to leave the large rectangle: $0 \leq \phi_1 \leq T_1$, $0 \leq \phi_2 \leq T_2$; this is illustrated in Fig. 5.

The desired probability is then the area covered at least once by the strip divided by $(T_1 T_2)$, the total available area of the phase plane.

Using Fig. 4, one can easily show that, before the strip begins to overlap itself:

$$P = P_0 + wt, \tag{8}$$

where t, P_0, w are defined in (2).

A rectangle with opposite sides identified, as in Fig. 5, is topologically equivalent to a torus. This gives a good geometric picture of the overlap phenomena.

The strip winds diagonally about the torus until eventually (in general after several full circuits) it strikes sufficiently near its starting point to overlap itself on one edge. It then begins to fill the chinks between the previous circuits, and this single overlap continues until the chinks are almost filled. The strip then approaches its starting point from the side opposite to that on which single overlap occurred. Thereafter, only the center section of the strip is effective in increasing the area covered. This double overlap continues until the entire torus has been covered. A degenerate case is possible in which the strip, upon its first overlap, begins to retrace exactly its former path and the torus is never fully covered. This corresponds to interlocking of the original waves of Fig. 1.

A rigorous proof of the above statements may be constructed by using the fact that each change in behavior can occur only at the starting point. In this manner, it is easily shown that: (a) single and double overlap occur in that order,

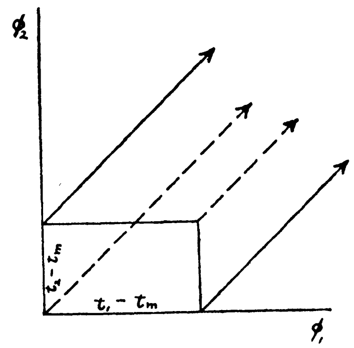

FIG. 4

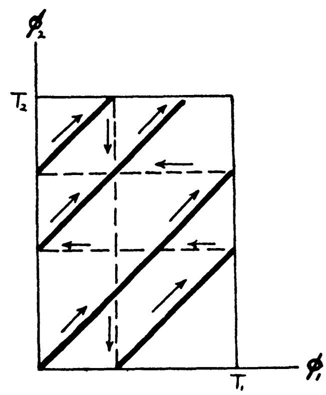

FIG. 5

(b) the strip area effective in covering changes only upon a change in the type of overlap, and (c) the two types of overlap must occur on opposite sides of the starting point.

The facts (a, b, c) may then be used to derive the probability function. For the analytic analysis, it is best to return to the (ϕ_1, ϕ_2) plane. Overlap of any type will first occur when the "unswitched-back" strip approaches sufficiently near a point (n_1T_1, n_2T_2) where n_1 and n_2 are non-negative integers not both zero. The analysis is greatly shortened by noticing that the behavior is completely determined by the distance of the line $\phi_1 = \phi_2$ from such points (even though the strip is not centered on this line), while the width of the strip is (Fig. 4) $wT_1T_2/\sqrt{2}$.

A slight fine-structure may arise in the probability function where it changes slope, depending on whether or not the leading corner of the moving rectangle strikes one of the sides of the original small rectangle. These effects are small if t_i/T_i are small and will be neglected below by supposing the strip to be generated by a line segment oriented perpendicularly to its path. The error arising from this procedure consists essentially in a delay or advance in the time at which P changes slope. It may be seen that the maximum effect represents a delay of $\Delta t = wT_1T_2/2$. The error introduced is then less than $\Delta t\sqrt{2}$ multiplied by that portion of the total width of the strip which becomes ineffective due to the overlap considered. The sum of these effects must be less than that given by using the total width of the strip; this gives the maximum error $w^2T_1T_2/2$.

The results of the method outlined are then as follows. Single overlap occurs at $t = s$ where

$$s = \tfrac{1}{2}(m_1T_1 + m_2T_2), \tag{9}$$

and (m_1, m_2) is that pair of non-negative integers not both zero such that s is a minimum and

$$p_1 = \left| \frac{m_1}{T_2} - \frac{m_2}{T_1} \right| < w. \tag{10}$$

Double overlap occurs at $t = d$, where

$$d = \tfrac{1}{2}(n_1T_1 + n_2T_2), \tag{11}$$

and (n_1, n_2) is that pair of non-negative integers not both zero such that d is a minimum and the conditions

$$\begin{gathered} \left| \frac{n_1}{T_2} - \frac{n_2}{T_1} \right| < w, \\ \left(\frac{n_1}{T_2} - \frac{n_2}{T_1}\right)\left(\frac{m_1}{T_2} - \frac{m_2}{T_1}\right) < 0, \end{gathered} \tag{12}$$

are satisfied. If we set

$$p_2 = p_1 + \left| \frac{n_1}{T_2} - \frac{n_2}{T_1} \right| - w, \tag{13}$$

the probability function is then

$$P \begin{aligned} &= P_0 + wt && \text{for } t \leq s, \\ &= P_0 + sw + (t - s)p_1 && \text{for } s \leq t \leq d, \\ &= P_0 + sw + (d - s)p_1 + (t - d)p_2 && \text{for } d \leq t, \end{aligned} \tag{14}$$

where it is understood that $P = 1$ if (14) gives $P > 1$.

The degenerate case where the waves interlock is given correctly by this formalism. Namely, if the strip starts to retrace its path exactly, then $p_1 = 0$ and the second part of (12) shows that d does not exist. Equation (14) then gives the correct result: P rises to the value $P_0 + sw$ and never increases further.

4. The method of smoothing. We have already discussed in section 1 the inadequacy of the formal mathematical solution (14) for purposes of practical application. Either mathematical analysis or intuitive consideration of interlock shows that the erratic behavior of P is due almost entirely to small changes in the *ratio* T_1/T_2. As this ratio passes through certain rational values, possibilities of interlock appear and disappear. Consequently, we next alter (14) to a form in which the dependence on this ratio is more evident.

We may, without loss of generality, assume:

$$(15) \qquad T_1 = 1, \qquad T_2 < 1.$$

Also introduce the standard notation:

$$(16) \qquad [x] = (\text{largest integer} \leq x).$$

It will then be seen that (10) and (12) may be thrown into the form:[2]

(17) k = smallest positive integer such that $p_1 = | ke - i | < w$ (i = integer);

(18) K = smallest positive integer such that $| Ke - I | < w$ and also

$$(ke - i)(Ke - I) < 0 \; (I = \text{integer});$$

where either

$$(19) \qquad e = \frac{1}{T_2} - \left[\frac{1}{T_2}\right], \qquad \text{or} \qquad e = 1 + \left[\frac{1}{T_2}\right] - \frac{1}{T_2}.$$

Now from (9) and (10), we note that s differs from m_1T_1 by at most $wT_1T_2/2$, while from (11) and (12), d differs from n_1T_1 by less than the same amount. Moreover, by the second half of (12), d is thereby made too small if s has been made too large and vice versa. Hence the use of these approximations in (14) will contribute an error certainly less than $w^2T_1T_2/2$. Adding the error discussed in section 3, the total introduced thus far cannot exceed $w^2T_1T_2$.

We thus use in the present notation $s = k, d = K$; (13) and (14) then become:

$$(20) \qquad p_2 = p_1 + | Ke - I | - w$$

$$(21) \qquad \begin{aligned} &(a)\; P = P_0 + wt, && \text{for } t \leq k \\ &(b)\; P = P_0 + kw + (t - k)p_1, && \text{for } k \leq t \leq K \\ &(c)\; P = P_0 + kw + (K - k)p_1 + (t - K)p_2, && \text{for } K \leq t \end{aligned}$$

where, as before, $P = 1$ if (21) gives a value greater than unity. Equations (17)–(21) are the formulation which will be used, with conditions (15), henceforth.

We wish now to smooth P with respect to variations in e. The number-theoretic requirement (17) is extremely difficult to work with. For reasons of simplicity, then, we shall assume that e is the only parameter which changes as

[2] Note that, even though the periods appear explicitly only in (19) hereafter, all the following equations are true only for $T_1 = 1$. (This is evident if we recall that w has the dimensions of inverse time.) Thus we are definitely assuming that T_1 = constant.

T_2 is varied. The errors which may arise from this assumption are treated at the end of section 5.

From (19)—or from the absolute value signs in (17), (18)—it will be seen that all possible situations arise if e varies merely from zero to one-half. In order that this should entail as little variation in T_2 as possible, our conventions should be chosen as already stated in (15). Even under these circumstances, a maximum variation of 33% in T_2 may be required to cover the range $e = 0$ to $\frac{1}{2}$.

Equation (21) cannot be used directly without the interpretational convention there noted. This leads to difficulties of treatment which the author was unable to solve. The difficulties may be avoided by the following device, which admittedly has less direct significance than an averaged value for P.

We enquire after the fraction f of the range of e over which P has a value (at fixed t) less than some given value $Q + P_0$. We may then say that, if a large number of trials each of length t is made, then in f of them, the probability of coincidence will be less than $Q + P_0$.

5. Calculation of f. The exceptional behavior of P is that caused by interlock possibilities. This corresponds to $p_1 = 0$ in (17). Thus the exceptional values of P center about the points $e = i/k$, where i and k are relatively prime (otherwise, k would not be the smallest integer satisfying (17)). Moreover, by a standard theorem [1], $k \leq 1/w$. Thus the critical points form the Farey series of order $1/w$ in the range $(0, \frac{1}{2})$. About each Farey point, we may suspect that there will be an interval over which k is constant, and that the entire range may thereby be divided up into ranges of constant k.

In thinking about the use of (17) in a typical calculation, it is convenient to eliminate the integer i by representing multiples of e as a series of points progressing around and around a circle of unit circumference. When $e = i/k$, the kth multiple will (after i revolutions) coincide with the origin; this and the earlier points, it is easily shown, will be distributed uniformly about the circle with a separation $1/k$.

As e moves away from the Farey point, k will, by definition (17), remain constant until either (a) the point ke moves a distance greater than w from the origin or (b) an *earlier* point moves to a distance less than w from the origin (Fig. 6).

Let (me) be that *earlier* point nearest (initially $1/k$ from) the origin and moving toward it as e varies in a particular direction. Of course,

$$(22) \qquad m < k.$$

For each Farey point, there will be two values of m; one for decreasing e and one for increasing e. If we introduce the new variable: h = the absolute value of the change in e from the Farey point i/k, then each point, ne, on the reference circle will move a distance nh, and (17) gives as the conditions for constant k (Fig. 7):

$$(23) \qquad \begin{aligned} &\text{(a)}\ w > kh = p_1, \\ &\text{(b)}\ mh < (1/k) - w. \end{aligned}$$

Thus we have divided the range $(0, \frac{1}{2})$ into small ranges where k (and m) are fixed. The number of small ranges is roughly twice the number of Farey points in $(0, \frac{1}{2})$.

Within each small range p_1, K, p_2 still vary with e. The behavior of p_1 is

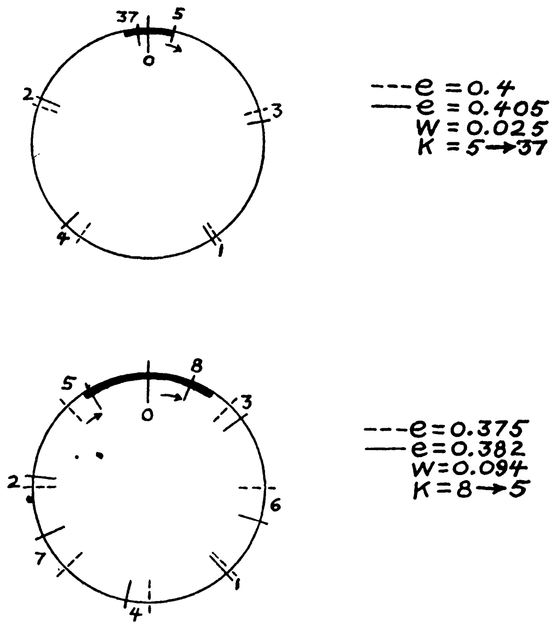

FIG. 6

already given in (23a); we shall find that we do not need p_2. Using (18) and Fig. 7, it may easily be shown that:

(24) $$K = m + jk + k,$$

where

(25) $$j + a = (1 - mkh - kw)/k^2h, \qquad j = [j], \qquad 0 \leq a < 1.$$

From (23a), (24), (25), we obtain:

(26) $$(K - k)p_1 = 1 - kw - ak^2h \qquad (0 \leq a < 1).$$

Having thus divided the range of e into small regions within each of which the number-theoretic requirements (17, 18) take a relatively simple form, we must now turn to the calculation of f = that fraction of the range $e = (0, \frac{1}{2})$ over which $P < P_0 + Q$ at fixed t. We shall specialize the further analysis to the case $Q \leq \frac{1}{2}$. This considerably shortens the discussion and yields essentially all the useful results of the more general inquiry.

We first note from (21) that, since $p_2 < p_1 < w$ (i.e. because of (4)), we have $P < P_0 + Q$ independently of e if $t < Q/w$

(27) $$f = 1, \quad \text{for} \quad t < Q/w.$$

Similar reasoning shows on the other hand that, when $t > Q/w$, those regions with $k \geq Q/w$ do not contribute to f. In the following, we shall therefore employ:

(28) $$k \lesssim Q/w < t, \qquad Q \leq \tfrac{1}{2}.$$

Equation (28) implies that we must use either (21b) or (21c); we shall next show that we do not need (21c). The value of P whenever (21c) is applicable is certainly greater than $(P_0 + kw + (K - k)p_1)$. From (26), this value is equal to $(P_0 + 1 - ak^2h)$. Now from (28), $w < 1/2k$, whence by (23a) $h < 1/2k^2 < 1/2ak^2$ (since $a < 1$). Thus $(P_0 + 1 - ak^2h) > P_0 + \frac{1}{2} \geq P_0 + Q$, and consequently (21c) never applies until $P \geq P_0 + Q$. (This means merely that the double overlap discussed in section 3 cannot occur until at least half the torus is covered.) Accordingly, we can confine our attention entirely to (21b) in any further discussion of f.

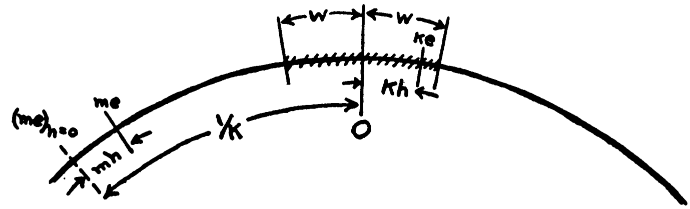

FIG. 7

Substituting for p_1 from (23) and recalling that $(t - k)$ is positive (by (28)), we find from (21b) that the condition $P < P_0 + Q$ becomes:

(29) $$h < \frac{Q - kw}{k(t - k)}.$$

However, h is subject also to the restrictions (23), which insure that we do not stray from the small region where k is constant. We assert that (29) implies (23) and may therefore be used as the final expression of the requirement $P < P_0 + Q$.

To prove this, note first that (29) and (28) immediately give $h < w/k$, which is (23a). Secondly, (28) implies $1/k > 2w$ so that, using (23a) and (22): $(1/k) - w > w > kh > mh$, which is (23b).

Thus we arrive at the result that f receives contributions only from those elementary regions where k satisfies (28) and that the contribution of each such region is governed by (29).

Since the variable h was defined as the absolute value of the change of e from the Farey point i/k, each Farey point (satisfying (28)) contributes an amount equal to twice[3] the right-hand side of (29). Since this amount is independent

[3] This is not true of the Farey points 0 and $\frac{1}{2}$, the ends of the range of e, but the terms $k = 1, 2$ in (31) correctly account for these contributions since $\phi(1) = \phi(2) = 1$.

of i, we may immediately sum over all Farey points i/k with fixed k. There are $\frac{1}{2}\phi(k)$ such points[3] in the range $(0, \frac{1}{2})$, where Euler's function ϕ is defined by:

$$\phi(k) = \text{the number of integers} \leq k \text{ and relatively prime to } k. \tag{30}$$

(Note that $\phi(k)$ is even for $k \geq 3$ since if k and i have no common divisor > 1, neither do k and $k - i$.)

Thus, summing over all these contributions and dividing by the length of the total range:

$$f = 2 \sum_{1 \leq k < Q/w} \phi(k) \frac{Q - kw}{k(t - k)}, \qquad \text{for } t > Q/w. \tag{31}$$

Regarding error in (31) due to the inaccuracy of (21), note that this can enter only when we set $P = P_0 + Q$ in deriving (29). Actually the difference between (21b) and the correct value of P will change as e is changed so that there is considerable possibility that these effects will cancel out in (31). (In fact, a detailed study shows that the error in (21b) assumes opposite signs as e varies in opposite directions from any given Farey point.) In any case, because (31) is monotone in Q, the error in (31) can be no greater than that found by substituting $Q \pm w^2 T_1 T_2$ for Q. Taking account also of the variation of P_0 with T_2, the same argument establishes the "Q-dependence" of (6) given in section 2.

Finally, we investigate the error due to change in w with T_2. If $\overline{w}$ is the maximum value of w, Farey points with $k < Q/\overline{w}$ are certain to contribute to f, and this contribution will be at least as great as $(Q - k\overline{w})/k(t - k)$ so that $f > f(\overline{w})$. On the other hand, if $\underline{w}$ is the minimum value of w, Farey points with $k \geq Q/\underline{w}$ cannot possibly contribute to f, and the remaining points can contribute no more than $(Q - k\underline{w})/k(t - k)$ so that $f < f(\underline{w})$. Hence we arrive at the final statement (6) in section 2.

6. Approximations for f. Computational difficulties in the use of (31) suggested approximating it by a more readily computed expression. By a standard theorem [1, p. 266]:

$$\phi(k) \approx 6k/\pi^2. \tag{32}$$

We may then approximate (31) by:

$$f = 1.216 \int_{\frac{1}{2}}^{(Q/w)+\frac{1}{2}} \frac{Q - kw}{t - k}\, dk$$

$$= 1.216\, Q \left(1 + \frac{tw - Q}{Q} \log \frac{tw - Q - \frac{1}{2}w}{tw - \frac{1}{2}w}\right).$$

If Q/w is large compared to $\frac{1}{2}$ (recall $t > Q/w$), this becomes very nearly:

$$f = 1.216\, Q \left(1 + \left(\frac{tw}{Q} - 1\right) \log \left(1 - \frac{Q}{tw}\right)\right) \qquad \text{for } t > Q/\underline{w}. \tag{33}$$

Despite the cavalier derivation of (33), its agreement with (31) is remarkably close. Fig. 2 shows a perfectly general comparison of (31) and (33), where the agreement will be seen to be fairly good even for t and Q/w of the order of 4 or 5. Note also that (33) nearly always gives a value of f that is too large.

For completeness, we may repeat (27).

$$f = 1 \quad \text{for} \quad t < Q/w. \tag{34}$$

Note that only the dimensionless quantities tw, Q enter into (33, 34) which are therefore independent of the normalization (15).

REFERENCE

[1] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Clarendon Press, Oxford, 1938, p. 30.