NOTES

This section is devoted to brief research and expository articles and other short items.

THE DISTRIBUTION OF STUDENT'S t WHEN THE POPULATION MEANS ARE UNEQUAL

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Let $x_1, \cdots, x_N$ be independent normal variates with the same variance σ^2 and with means $\mu_1, \cdots, \mu_N$ respectively. Set $n = N - 1$ and let

$$(1) \qquad \bar{x} = \sum_1^N x_i/N, \qquad s^2 = \sum_1^N (x_i - \bar{x})^2/n, \qquad t = N^{\frac{1}{2}}\bar{x}/s.$$

If all the μ_i are 0 then t has Student's distribution with n degrees of freedom; its frequency function will be denoted here by

$$(2) \qquad f_{n,0}(t) = n^{-\frac{1}{2}}\left[B\left(\frac{1}{2}, \frac{n}{2}\right)\right]^{-1} \cdot (1 + t^2/n)^{-\frac{1}{2}(n+1)}.$$

When dealing with situations involving mixtures of populations or in which the mean exhibits a secular trend, it is important to know the distribution of t when the μ_i are arbitrary; in the general case let

$$(3) \qquad \begin{aligned} \bar{\mu} &= \sum_1^N \mu_i/N, \qquad \beta^2 = \sum_1^N (\mu_i - \bar{\mu})^2/N, \\ \alpha &= N\bar{\mu}^2/2\sigma^2, \qquad \lambda = N\beta^2/2\sigma^2. \end{aligned}$$

The distribution of t will be shown to depend on the three parameters n, α, λ. If $\lambda = \beta^2 = 0$, so that all the μ_i are equal, then the distribution of t determines the power function of the ordinary t test. We shall here consider the case in which $\alpha = \bar{\mu} = 0$, although the μ_i are different. Denoting the frequency function of t in this case by $f_{n,\lambda}(t)$ we shall show that

$$(4) \qquad f_{n,\lambda}(t) = f_{n,0}(t) \cdot \exp\left\{\frac{-\lambda t^2/n}{1 + t^2/n}\right\} \cdot F(-\tfrac{1}{2}, n/2, -\lambda(1 + t^2/n)^{-1}),$$

where F denotes the confluent hypergeometric series, and where, since $\bar{\mu} = 0$,

$$(5) \qquad \lambda = \sum_1^N \mu_i^2/2\sigma^2.$$

In fact, the general distribution of t, of which (4) represents the case $\alpha = 0$, may be derived as follows. Using the standard orthogonal transformation [1, p. 387] let

$$(6) \qquad z_i = \sum_{j=1}^{N} c_{ij} x_j, \qquad x_i = \sum_{j=1}^{N} c_{ji} z_j \qquad (i = 1, \cdots, N),$$

where

$$(7) \qquad c_{1j} = N^{-\frac{1}{2}} \qquad (j = 1, \cdots, N);$$

then

$$(8) \qquad t = n^{\frac{1}{2}} z_1 \Big/ \left(\sum_2^N z_i^2 \right)^{\frac{1}{2}}.$$

The joint frequency function of the z_i is easily seen to be

$$(9) \qquad (2\pi)^{-N/2} \cdot \sigma^{-N} \cdot \exp\left\{ -\sum_1^N (z_i - a_i)^2 / 2\sigma^2 \right\},$$

where

$$(10) \qquad a_1 = N^{\frac{1}{2}} \bar{\mu}, \qquad \sum_2^N a_i^2 = N\beta^2.$$

Thus t is the ratio of a non-central normal variate to the square root of an independent non-central chi-square variate. It is known [2, p. 138] that the frequency function of $q^2 = \sum_2^N z_i^2/\sigma^2$ is

$$(11) \qquad \tfrac{1}{2} e^{-\lambda} \cdot (\tfrac{1}{2} q^2)^{\frac{1}{2}n - 1} \cdot e^{-q^2/2} \cdot \sum_{j=0}^{\infty} \frac{(\frac{1}{2}\lambda q^2)^j}{j!\,\Gamma(\frac{1}{2}n + j)},$$

where

$$(12) \qquad \lambda = \sum_2^N a_i^2/2\sigma^2 = N\beta^2/2\sigma^2.$$

The frequency function of $v = z_1/\sigma$ is

$$g(v) = \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{(\sigma v - a_1)^2}{2\sigma^2} \right\} = \frac{1}{\sqrt{2\pi}} e^{-\alpha} \cdot e^{-(v^2/2)} \cdot \sum_{k=0}^{\infty} \frac{(2\alpha)^{k/2}}{k!} x^k,$$

that of q is, by (11),

$$h(q) = 2^{1-(n/2)} e^{-\lambda} e^{-(q^2/2)} \sum_{j=0}^{\infty} \frac{\lambda^j q^{2j+n-1}}{2^j j!\,\Gamma((n/2) + j)}, \qquad (q > 0),$$

hence that of $u = v/q = n^{-\frac{1}{2}} t$ is

$$\int_0^\infty h(q) g(uq) q \, dq,$$

which, after integration, reduces to

$$(13) \qquad \pi^{-\frac{1}{2}} e^{-(\lambda+\alpha)} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{\lambda^j (2\alpha^{\frac{1}{2}} u)^k}{j!\,k!} \frac{\Gamma(N/2 + j + k/2)}{\Gamma(n/2 + j)} (1 + u^2)^{-(N+2j+k)/2}.$$

In particular, if $\alpha = \bar{\mu} = 0$ then (13) reduces by means of the relation $F(\alpha, \gamma, x) = e^x F(\gamma - \alpha, \gamma, -x)$ to

$$(14)\quad \left[B\left(\frac{1}{2}, \frac{n}{2}\right)\right]^{-1} \cdot e^{-\lambda u^2/(1+u^2)} \cdot (1 + u^2)^{-\frac{1}{2}N} \cdot F\left(-\frac{1}{2}, \frac{n}{2}, -\lambda(1 + u^2)^{-1}\right),$$

from which it follows that the frequency function of t is given by (4).

Again, let $x_1, \cdots, x_{N_1+N_2}$ be independent normal variates with the same variance σ^2 and with means $\mu_1, \cdots, \mu_{N_1+N_2}$ respectively. Set $n_1 = N_1 - 1$, $n_2 = N_2 - 1$, $n = n_1 + n_2$, and let

$$(15)\quad \begin{gathered} \bar{x}_1 = \sum_1^{N_1} x_i/N_1, \qquad \bar{x}_2 = \sum_{N_1+1}^{N_1+N_2} x_i/N_2 \\ s_1^2 = \sum_1^{N_1} (x_i - \bar{x}_1)^2/n_1, \qquad s_2^2 = \sum_{N_1+1}^{N_1+N_2} (x_i - \bar{x}_2)^2/n_2 \\ s^2 = (n_1 s_1^2 + n_2 s_2^2)/(n_1 + n_2), \quad t = [N_1 N_2/(N_1 + N_2)]^{\frac{1}{2}}(\bar{x}_1 - \bar{x}_2)/s. \end{gathered}$$

If all the μ_i are equal then t again has Student's distribution with n degrees of freedom. In the general case let

$$(16)\quad \begin{gathered} \bar{\mu}_1 = \sum_1^{N_1} \mu_i/N_1, \qquad \bar{\mu}_2 = \sum_{N_1+1}^{N_1+N_2} \mu_i/N_2, \\ \beta_1^2 = \sum_1^{N_1} (\mu_i - \bar{\mu}_1)^2/N_1, \qquad \beta_2^2 = \sum_{N_1+1}^{N_1+N_2} (\mu_i - \bar{\mu}_2)^2/N_2. \end{gathered}$$

Then we may show as before [1, p. 388] that in this case $u = n^{-\frac{1}{2}}t$ has the frequency function (13), where now

$$(17)\quad \begin{gathered} N = N_1 + N_2 - 1, \qquad \lambda = (N_1\beta_1^2 + N_2\beta_2^2)/2\sigma^2, \\ \alpha = [N_1 N_2/(N_1 + N_2)](\bar{\mu}_1 - \bar{\mu}_2)^2/\sigma^2. \end{gathered}$$

In particular, when $\alpha = \bar{\mu}_1 - \bar{\mu}_2 = 0$, so that $\bar{\mu}_1 = \bar{\mu}_2 = \bar{\mu}$, say, the frequency function $f_{n,\lambda}(t)$ of t is again given by (4), where now

$$(18)\quad \lambda = \sum_1^{N_1+N_2} (\mu_i - \bar{\mu})^2/2\sigma^2.$$

Extensions in this direction to the general linear hypothesis in the analysis of variance will not be treated here

If we set

$$(19)\quad w = (1 + t^2/n)^{-1}$$

where t has the frequency function (4), then w will have the frequency function

$$(20)\quad g_{n,\lambda}(w) = \left[B\left(\frac{1}{2}, \frac{n}{2}\right)\right]^{-1} \cdot e^{-\lambda(1-w)} \cdot w^{\frac{1}{2}n-1} \cdot (1 - w)^{-\frac{1}{2}} \cdot F\left(-\frac{1}{2}, \frac{n}{2}, -\lambda w\right),$$

for $0 < w \leq 1$. Thus for every t,

$$(21) \qquad 1 - \int_{-t}^{t} f_{n,\lambda}(x)\, dx = \int_0^{(1+t^2/n)^{-1}} g_{n,\lambda}(w)\, dw.$$

It would be interesting to have numerical values of the integral on the left side of (21) for that value of t for which

$$(22) \qquad 1 - \int_{-t}^{t} f_{n,0}(x)\, dx = 0.01 \text{ or } 0.05 \quad \text{(say)},$$

but existing tables (e.g. those in [2] and [3]) of the integral of (20) were compiled for a different purpose and do not supply this information. The following remarks throw some light on this subject.

Let us set

$$(23) \qquad \begin{aligned} R(t) = f_{n,\lambda}(t)/f_{n,0}(t) &= \exp\left\{\frac{-\lambda t^2/n}{1 + t^2/n}\right\} \cdot F\left(-\frac{1}{2}, \frac{n}{2}, -\lambda(1 + t^2/n)^{-1}\right) \\ &= \{1 - \lambda(t^2/n)/(1 + t^2/n) + o(\lambda)\} \\ &\qquad \cdot \{1 + \lambda/(n + t^2) + o(\lambda)\} \\ &= 1 + \lambda(n + t^2)^{-1}(1 - t^2) + o(\lambda). \end{aligned}$$

Then as $\lambda \to 0$ we have ultimately

$$(24) \qquad \begin{aligned} &R(t) > 1 \text{ if } |t| < 1, \\ &R(t) < 1 \text{ if } |t| > 1. \end{aligned}$$

Hence for any $t > 1$ and for sufficiently small λ,

$$(25) \qquad 1 - \int_{-t}^{t} f_{n,\lambda}(x)\, dx < 1 - \int_{-t}^{t} f_{n,0}(x)\, dx.$$

The exact range of values of t for which $R(t) < 1$ depends of course on n and λ. However we shall show that always

$$(26) \qquad R(t) < 1 \text{ if } |t| > 1,$$

so that (25) holds for all n and $\lambda > 0$, provided $t > 1$. The proof is as follows. In terms of w we have

$$(27) \qquad R(t) = e^{-\lambda(1-w)} \cdot F(-\tfrac{1}{2}, n/2, -\lambda w) = e^{-\lambda} F((n + 1)/2, n/2, \lambda w).$$

Now

$$(28) \qquad \begin{aligned} F((n + 1)/2, n/2, \lambda w) &= 1 \\ &\quad + \sum_{k=1}^{\infty} \frac{(n + 1)(n + 3)\cdots(n + 2k - 1)}{n(n + 2)\cdots(n + 2k - 2)} (\lambda w)^k/k!, \end{aligned}$$

and by induction on k we may show that for all $k = 1, 2, \cdots$,

$$\frac{(n+1)(n+3)\cdots(n+2k-1)}{n(n+2)\cdots(n+2k-2)} \leq 1 + k/n, \tag{29}$$

where the equality holds only for $k = 1$. Hence

$$F((n+1)/2, n/2, \lambda w) < 1 + \sum_{k=1}^{\infty} (1 + k/n)\cdot(\lambda w)^k/k! = e^{\lambda w}(1 + \lambda w/n), \tag{30}$$

$$R(t) < e^{-\lambda(1-w)}\cdot(1 + \lambda w/n) < e^{-\lambda(1-w)}\cdot e^{\lambda w/n} = e^{-\lambda[1-w(1+1/n)]}. \tag{31}$$

Hence $R(t) < 1$ if $w < n/(n+1)$, which is equivalent to (26).

REFERENCES

[1] H. CRAMÉR, *Mathematical Methods of Statistics*, Princeton University Press, Princeton, 1946.

[2] P. C. TANG, "The power function of the analysis of variance tests with tables and illustrations of their use," *Stat. Res. Memoirs*, Vol. 2 (1938), pp. 127–149.

[3] EMMA LEHMER, "Inverse tables of probabilities of errors of the second kind," *Annals of Math. Stat.*, Vol. 15 (1944), pp. 388–398.

A DISTRIBUTION-FREE CONFIDENCE INTERVAL FOR THE MEAN

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1. Summary. Consider a random sample of N observations $x_1, x_2, \cdots, x_N$, from a universe of mean μ and variance σ^2. Let m and s^2 be the sample mean and variance respectively:

$$m = \frac{1}{N}\sum_{i=1}^{N} x_i, \qquad s^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - m)^2. \tag{1}$$

It is shown that the following conservative confidence interval holds for μ:

$$\text{Prob}\,\{(m-\mu)^2 \leqq s^2/(N-1) + \lambda\sigma^2\sqrt{2/N(N-1)}\} > 1 - \lambda^{-2}, \tag{2}$$

where λ is any positive constant. Inequality (2) also holds if, in the braces, λ is replaced by $\sqrt{\lambda^2 - 1}$, with $\lambda \geqq 1$.

Inequality (2) is much more efficient on the average than Tchebychef's inequality for the mean, namely,

$$\text{Prob}\,\{(m-\mu)^2 \leqq \lambda^2\sigma^2/N\} > 1 - \lambda^{-2}, \tag{3}$$

yet (2) and (3) are both distribution-free, requiring only knowledge about σ^2. At the $1 - \lambda^{-2} = .99$ level of confidence, the expected value of the right member in the braces of (2) is only about 1/6 the corresponding member of (3); at the .999 level of confidence the ratio is about 1/20.