NOTES

This section is devoted to brief research and expository articles on methology and other short items.

BROWNIAN MOTION ON THE SURFACE OF THE 3-SPHERE

By Kôsaku Yosida

Mathematical Institute, Nagoya University

1. Introduction. Let S be a n-dimensional compact riemann space with the metric $ds^2 = g_{ij}(x)\,dx^i\,dx^j$ such that the totality G of the isometric transformations of S onto S constitutes a Lie group transitive on S. Consider a temporally homogeneous Markoff process by which $P(t, x, y)$, $t > 0$, is the transition probability that a point x is transferred to y after the elapse of t-unit time. We assume that $P(t, x, y)$ is a Baire function in (t, x, y) and continuous in t, then P satisfies Smoluchouski's equation

$$P(t + s, x, y) = \int_S P(t, x, z)P(s, z, y)\,dz \qquad (t, s > 0), \tag{1.1}$$

dz being the G-invariant measure $\sqrt{g(x)}dx^1\,dx^2 \cdots dx^n$, $g(x) = \det(g_{ij}(x))$, and

$$P(t, x, y) \geqq 0, \tag{1.2}$$

$$\int_S P(t, x, y)\,dy = 1. \tag{1.3}$$

The spatial homogeneity of the transition process may be defined by

$$P(t, Tx, Ty) = P(t, x, y) \qquad \text{for } T \in G. \tag{1.4}$$

The "continuity" of the transition process may be defined, following after A. Kolmogoroff and W. Feller,[1] as follows. Let $L_1(S)$ be the function space of integrable (with respect to dx) functions $f(x)$ on S, then, for those $f(x)$ which are dense in $L_1(S)$,

$$\begin{aligned} &\frac{\partial f(t, x)}{\partial t} = A \cdot f(t, x), \qquad (t \geqq 0); \\ &f(t, x) = \int_S f(y)P(t, y, x)\,dy, \qquad (t > 0), \qquad f(0, x) = f(x), \end{aligned} \tag{1.5}$$

where, with non-negative $b^{ii}(x)$

$$(Af)(x) = \frac{1}{\sqrt{g(x)}}\frac{\partial}{\partial x^i}(-\sqrt{g(x)}\,a^i(x)f(x)) + \frac{1}{\sqrt{g(x)}}\frac{\partial^2}{\partial x^i\,\partial x^j}(\sqrt{g(x)}\,b^{ij}(x)f(x)). \tag{1.6}$$

[1] A. Kolmogoroff, "Zur Theorie der stetigen zufälligen Prozesse," *Math. Annalen*, Vol. 108 (1933); W. Feller, "Zur Theorie der stochastischen Prozesse," *Math. Annalen*, Vol. 113 (1937).

The temporally and spatially homogeneous "continuous" Markoff process may, if it exists, be called a Brownian motion on the homogeneous space S. The purpose of the present note is to show that, under some derivability hypothesis concerning $a^i(x)$ and $b^{ij}(x)$, there exists one and (essentially) only one Brownian motion on the surface of the 3-sphere S^3.

I here express my hearty thanks to Dr. Kiyosi Itô who proposed to me the problem and discussed and much improved the manuscript.

2. The defining equation for the Brownian motion. The spatial homogeneity (1.4) is equivalent to the fact that A is commutative with every operator $\tilde{T}$ defined by

$$(\tilde{T}f)(x) = f(Tx), \qquad T \in G, \tag{2.1}$$

because we have

$$\int_S f(y)P(t, y, Tx)\,dy = \int_S f(Ty)P(t, Ty, Tx)\,dTy = \int_S f(Ty)P(t, y, x)\,dy.$$

The condition (2.1) is equivalent to

$$XA = AX \text{ for any infinitesimal operator } X = \xi^k(x)\frac{\partial}{\partial x^k} \tag{2.2}$$

induced on S by the infinitesimal operator of the Lie group G. Thus, assuming the derivability of $a^i(x)$ and $b^{ij}(x)$ of necessary orders, we obtain from (2.2) the conditions:

$$\xi^k(x)\frac{\partial}{\partial x^k}\left(\frac{1}{\sqrt{g(x)}}\frac{\partial G^i(x)}{\partial x^i}\right) = 0, \tag{2.3}$$

$$\left(G^i(x) = -\sqrt{g(x)}\,a^i(x) + \frac{\partial\sqrt{g(x)}b^{ij}(x)}{\partial x^j}\right),$$

$$\frac{1}{\sqrt{g(x)}}H^i(x)\frac{\partial\xi^k(x)}{\partial x^i} + b^{ij}(x)\frac{\partial^2\xi^k(x)}{\partial x^i\partial x^j} = \xi^i(x)\frac{\partial}{\partial x^i}\left(\frac{1}{\sqrt{g(x)}}H^k(x)\right), \tag{2.4}$$

$$(H^i(x) = G^i(x) + \frac{\partial}{\partial x^j}(\sqrt{g(x)}\,b^{ij}(x)),$$

$$b^{ij}(x)\frac{\partial\xi^k(x)}{\partial x^j} + b^{kj}(x)\frac{\partial\xi^i(x)}{\partial x^j} = \xi^j(x)\frac{\partial b^{ik}(x)}{\partial x^j}. \tag{2.5}$$

Now for the surface of the 3-sphere S^3,

$$ds^2 = d\theta^2 + \sin^2\theta\cdot d\varphi^2, \qquad g(\theta, \varphi) = \sin^2\theta,$$

and the infinitesimal operators

$$X_x = \sin\varphi\frac{\partial}{\partial\theta} + \frac{\cos\theta\cos\varphi}{\sin\theta}\frac{\partial}{\partial\varphi},$$

$$X_y = \cos\varphi\frac{\partial}{\partial\theta} - \frac{\cos\theta\sin\varphi}{\sin\theta}\frac{\partial}{\partial\varphi},$$

$$X_z = \frac{\partial}{\partial \varphi}$$

respectively correspond to the rotations about the x-, y- and z-axis.

From (2.5) we see that, by taking $X = X_z$,

(2.6) $b^{ij}(\theta, \varphi)$ is independent of φ.

By taking $X = X_z$ in (2.4) we see that H^k is independent of φ. Hence, by (2.6),

(2.7) $a^i(\theta, \varphi)$ is independent of φ.

Thus, by taking $k = 1$, $X = X_x$ we obtain from (2.4),

$$\frac{1}{\sin\theta} H^2(\theta) \cos\varphi - b^{22}(\theta) \sin\varphi = \sin\varphi \frac{d}{d\theta}\left(\frac{1}{\sin\theta} H^1(\theta)\right)$$

and thus

$$H^2(\theta) = 0, \qquad b^{22}(\theta) + \frac{d}{d\theta}\left(\frac{1}{\sin\theta} H^1(\theta)\right) = 0. \tag{2.8}$$

Hence, by taking $k = 2$, $X = X_x$ or $X = X_y$, we obtain from (2.4)

$$\frac{-H^1(\theta)\cos\varphi}{\sin^3\theta} + 2b^{11}(\theta)\frac{\cos\theta\cos\varphi}{\sin^3\theta} + 2b^{12}(\theta)\frac{\sin\varphi}{\sin\theta} - b^{22}(\theta)\frac{\cos\theta\cos\varphi}{\sin\theta} = 0,$$

$$\frac{H^1(\theta)\sin\varphi}{\sin^3\theta} - 2b^{11}(\theta)\frac{\cos\theta\sin\varphi}{\sin^3\theta} + 2b^{12}(\theta)\frac{\cos\varphi}{\sin^2\theta} + b^{22}(\theta)\frac{\cos\theta\sin\varphi}{\sin\theta} = 0.$$

From these two equations we obtain

$$b^{12}(\theta) = 0, \qquad \frac{H^1(\theta)}{\sin^3\theta} - 2b^{11}(\theta)\frac{\cos\theta}{\sin^3\theta} + b^{22}(\theta)\frac{\cos\theta}{\sin\theta} = 0. \tag{2.9}$$

By taking $i = 2$, $k = 1$, $X = X_x$, we obtain from (2.5), (2.9)

$$b^{22}(\theta)\cos\varphi + b^{11}(\theta)\frac{d}{d\theta}\left(\frac{\cos\theta\cos\varphi}{\sin\theta}\right) = 0$$

and hence

$$b^{22}(\theta) = \frac{b^{11}(\theta)}{\sin^2\theta}. \tag{2.10}$$

Similarly by taking $i = 1$, $k = 1$, $X = X_x$ we obtain from (2.5)

$$b^{12}(\theta)\cos\varphi + b^{12}(\theta)\cos\varphi = \sin\varphi\frac{db^{11}(\theta)}{d\theta}$$

and hence by (2.9), (2.10)

$$b^{11}(\theta) = \text{constant } C, \qquad b^{22}(\theta) = \frac{C}{\sin^2\theta}. \tag{2.11}$$

Thus we obtain from (2.4)

$$H^1(\theta) = -a^1(\theta)\sin\theta + 2C\cos\theta, \qquad H^2(\theta) = -\sin\theta\cdot a^2(\theta)$$

and thus, by (2.8),

$$a^2(\theta) = 0. \tag{2.12}$$

Substituting (2.11) in (2.9) we obtain

$$a^1(\theta) = \frac{C\cos\theta}{\sin\theta}. \tag{2.13}$$

Therefore since $b^{11}(\theta)$ and $b^{22}(\theta)$ are non-negative, A is (essentially) equal to the Laplace operator

$$\Lambda = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}. \tag{2.14}$$

Thus we may obtain $P(t, x, y)$ by integrating the equation

$$\frac{\partial f(t;\theta,\varphi)}{\partial t} = \Lambda\cdot f(t;\theta,\varphi), \qquad (t \geqq 0), \tag{2.15}$$

and by putting

$$f(t;\theta,\varphi) = f(t,x) = \int_{S^3} f(y)P(t,y,x)\,dy. \tag{2.16}$$

3. Integration of the equation (2.15)–(2.16). Consider the Laplacian (real) spherical harmonics

$$Y_k^{(m)}(\theta,\varphi) = Y_k^{(m)}(x), \qquad (-k \leqq m \leqq k;\ k = 0, 1, \cdots). \tag{3.1}$$

They constitute an orthonormal function system complete for continuous functions on S^3, and we have

$$\Lambda\cdot Y_k^{(m)}(\theta,\varphi) = -k(k+1)Y_k^{(m)}(\theta,\varphi). \tag{3.2}$$

Since, as is well-known,

$$Y_k^{(m)}(T^{-1}x) = \sum_{n=-k}^{k} u_{nm}^{(k)}(T)Y_k^{(n)}(x) \tag{3.3}$$

by an irreducible orthogonal representation $(u_{nm}^{(k)}(T))$ of the rotation group G, we have

$$\max_x |Y_k^{(m)}(x)|^2 \leqq (2k+1)\min_x \sum_{n=-k}^{k} |Y_k^{(n)}(x)|^2, \tag{3.4}$$

by applying the Schwarz inequality and the transitivity of the group G on S^3. The right hand member satisfies, by the orthonormality

$$(2k+1)^2/(\text{area of } S^3). \tag{3.5}$$

Therefore the double series (for $t > 0$)

$$(3.6) \quad P(t; \theta, \varphi; \theta', \varphi') = \sum_{k=0}^{\infty} \sum_{m=-k}^{k} \exp\left(-k(k+1)t\right) Y_k^{(m)}(\theta, \varphi) Y_k^{(m)}(\theta', \varphi')$$

is absolutely and uniformly convergent on S^3. We will show that this P is the required (unique) Brownian motion on S^3.

The proof may be given in three steps. i) We see by (3.2) and (3.6), that $\int_{S_3} f(y)P(t, y, x)\, dx$ satisfies (2.15) if

$$f(x) \sim \sum_{k=0}^{\infty} \sum_{m=-k}^{k} d_k^{(m)} Y_k^{(m)}(x), \qquad \sum_{k=0}^{\infty} \sum_{m=-k}^{k} \exp\left(-k(k+1)t\right)k(k+1)\, d_k^{(m)} Y_k^{(m)}(x)$$

are both absolutely and uniformly convergent. By the completeness of $\{Y_k^{(m)}(x)\}$, such $f(x)$ are dense in $L_1(S)$.

ii) Because of (3.3) we see that (3.6) satisfies the spacial homogeneity (1.4).

iii) (1.3) is obvious by the orthonormality of $\{Y_k^{(m)}(x)\}$ and the constancy on S^3 of $Y_0^{(0)}(x)$. Next, for the solution $f(t, x)$ of (2.15)–(2.16), let $f(x) = f(0, x)$ be non-negative on S^3, then $g_\epsilon(t, x) = \exp(-\epsilon t)f(t, x)$, $(\epsilon > 0)$, satisfies

$$\frac{\partial g_\epsilon(t, x)}{\partial t} = \Lambda \cdot g_\epsilon(t, x) - \epsilon g_\epsilon(t, x), \qquad (t > 0),$$

$$g_\epsilon(0, x) = f(x) \geqq 0 \qquad \text{(on } S^3\text{)}.$$

Thus $g_\epsilon(t, x) \geqq 0$ on S^3, since $g_\epsilon(t, x)$ cannot have a negative minimum on the product space $[t_1, t_2] \times S^3$, for any $t_2 > t_1 > 0$. For at such minimizing point we must have

$$\frac{\partial g_\epsilon}{\partial t} = 0, \qquad \frac{\partial g_\epsilon}{\partial \theta} = 0, \qquad \frac{\partial g_\epsilon}{\partial \varphi} = 0, \qquad \frac{\partial^2 g_\epsilon}{\partial \theta^2} \geqq 0, \qquad \frac{\partial^2 g_\epsilon}{\partial \varphi^2} \geqq 0.$$

Therefore, since $\epsilon > 0$, $t_2 > t_1 > 0$ were arbitrary, we conclude that $f(t, x) \geqq 0$ on S^3 for $t > 0$ if $f(x) = 0$ on S^3. This proves (1.2). The same argument simultaneously shows us that the solution P of (2.15)–(2.16) and (1.2)–(1.3) is unique.

ON THE STRONG STABILITY OF A SEQUENCE OF EVENTS

By Aryeh Dvoretzky

Hebrew University, Jerusalem, and Institute for Advanced Study

1. Summary. M. Loève [3] has found conditions under which a sequence of events which may be interdependent in an arbitrary manner is strongly stable. In this note it is established that considerably weaker conditions imply the strong stability.

2. Introduction. Let

$$(1) \qquad A_1, A_2, \cdots, A_n, \cdots$$