second application of the continuity theorem, but on the obvious fact that(*) implies

$$\delta \sum_{r=1}^{k} q_r u_r \rightarrow \int_0^t q(x)f(x)\, dx,$$

where the step function $\{q_r\}$ converges uniformily to a continuous monotonic $q(x)$).

The following corrections apply to the paper, "On the normal approximation to the binomial distribution" (*Annals of Math. Stat.*, Vol. 16, (1945), pp. 319–329).

(1) Equation (27) gives two variants of an estimate for the error ρ. The second should simply restate the first one in terms of the variable x; in other words, the expression $(p^3 + q^3)$ in the second line of (27) should be replaced by $p^3(1 - px/\sigma)^{-3} + q^3(1 + qx/\sigma)^3$.

(2) The estimate $\rho < \sigma^{-6}/300$ given in (28) is not valid over the entire range for which it is claimed. However, the further theory depends only on the fact that $\rho = O(\sigma^{-4})$, and the estimate $\rho < \sigma^{-6}/30$ is both correct and sufficient for our purposes. (Actually, no changes whatever are required in the proofs, since (28) is used explicitly only for a range where it is correct as stated).

(3) On p. 324 it is stated that under the conditions of the main theorem (p. 325) $k \geq 4$, $n - k \geq 4$, whereas in reality the value 3 can occur in extreme cases. Fortunately, the assertion is not used anywhere in the proof, and the error ρ is negligible in all cases.

Accordingly, no changes are required either in the formulation or the proof of the theorems. I am indebted to Dr. W. Hoeffding for calling my attention to the slips.

(4) The first minus sign in footnote 5 should be an equality sign and the second minus in (70) a plus.

ABSTRACTS OF PAPERS

(Abstracts of papers presented at the Chapel Hill meeting of the Institute, March 17–18, 1950)

1. A Method of Estimating the Parameters of an Autoregressive Time Series.

S. G. Ghurye, University of North Carolina.

The general autoregressive process of the second order is defined by the equations

$$x_t = X_t + \eta_t,$$

$$X_t + \alpha_1 X_{t-1} + \alpha_2 X_{t-2} = \epsilon_t,$$

where x_t is the value actually observed at time t, X_t the corresponding theoretical value, ϵ_t the disturbance and η_t the superposed variation. The estimates of α_1, α_2 given by Yule's method are biased and inconsistent if η_t is not identically zero, the permanent bias being a function of the unknown variance of η_t. The present paper proposes a method of estimation which is unaffected by the presence of η_t, and seems to be better than any other known method; and this conjecture is supported by the results of application to observational and artificial series. In this method the estimates a_1, a_2 are obtained by minimizing

$$\sum_{k=3}^{n} \frac{1}{(N-k-2)} \left\{ \sum_{t=3}^{N-k} (x_t + a_1 x_{t-1} + a_2 x_{t-2})(x_{t+k} + a_1 x_{t+k-1} + a_2 x_{t+k-2}) \right\}^2,$$

where n is some number small in comparison with N (which is the number of observations). In the above expression the usual approximation of substituting $(N - k - 2)r_k$ for $\Sigma_{t=3}^{N-k} x_t x_{t+k}$ may be made for computational convenience. The method has been used for fitting autoregressive processes to the series of annual averages of Wolfer's sunspot numbers and that of Myrdal's Swedish cost of living index numbers. The method is applicable to higher order processes.

2. Most Powerful Rank Order Tests. (Preliminary Report). Wassily Hoeffding, University of North Carolina.

Let $X_{11}, \cdots, X_{1n_1}, \cdots, X_{k1}, \cdots, X_{kn_k}$ be random variables with a joint probability function $P(S)$ and let $P\{X_{ig} = X_{ih}\} = 0$ if $g \neq h$ $(i = 1, \cdots, k)$. Let H_0 be a hypothesis which implies that $P(S)$ is invariant under all permutations of $X_{i1}, \cdots, X_{in_i}$ $(i = 1, \cdots, k)$. Let r_{ij} $(j = 1, \cdots, n_i)$ be the ranks of $X_{i1}, \cdots, X_{in_i}$. Under H_0 the $M = \Pi n_i!$ rank permutations $R = (r_{11}, \cdots, r_{1n_1}, \cdots, r_{k1}, \cdots, r_{kn_k})$ have the same probability $P(R) = M^{-1}$. A test which depends only on the permutations R is called a rank order test (R.O.T.). A R.O.T. of size m/M which is most powerful (M.P.) against a simple alternative, $P_1(S)$, is determined by m permutations R for which $P_1(R)$ takes on its m largest values.

For example, let the pairs $(X_1, Y_1, \cdots, (X_n, Y_n)$ be independent and identically distributed. Let H_0 state that X_i, Y_i are independent, and let $H_1(\rho)$ be the hypothesis that X_i, Y_i have a bivariate normal distribution with correlation ρ. We may assume that $X_1 < \cdots < X_n$ and consider the ranks r_i of the Y's only. A R.O.T. which is uniformly M. P. against all $H_1(\rho)$ with $\rho > 0$ does not exist except for small n. The M.P.R.O.T. against small $\rho > 0$ is determined by the largest values of $\Sigma_{i=1}^{n} (EZ_i)(EZ_{r_i})$, where EZ_i is the expectation of the i-th order statistic in a sample of n from a standard normal distribuion. The M. P. unbiased R.O.T. against small values of $|\rho|$is based on the statistic $\Sigma_i \Sigma_j (EZ_iZ_j)(EZ_{r_i}Z_{r_j})$. The M.P. R.O.T. against ρ close to 1 is obtained by expanding the probability of $(r_1, \cdots, r_n)$ in powers of $\{(1 - \rho)/(1 + \rho)\}^{1/2}$.

3. The Comparison of Percentages in Matched Samples. William G. Cochran, Johns Hopkins University.

In this paper the familiar χ^2 test for comparing the percentages of successes in a number of independent samples is extended to the situation in which each member of any sample is matched in some way with a member of every other sample. This problem has been encountered in the fields of psychology, pharmacology, bacteriology, and sample survey design. A solution has been given by McNemar (1949) when there are only two samples.

In the more general case, the data are arranged in a two-way table with r rows and t columns, in which each column represents a sample and each row a matched group. The test criterion proposed is

$$Q = \frac{c(c-1)\Sigma(T_j - \bar{T})^2}{c(\Sigma u_i) - (\Sigma u_i^2)},$$

where T_j is the total number of successes in the j^{th} sample and u_i the total number of successes in the i^{th} row. If the true probability of success is the same in all samples, the limiting distribution of Q, when the number of rows is large, is the χ^2 distribution with $(c - 1)$ degrees of freedom. The relation between this test and the ordinary χ^2 test, valid when samples are independent, is discussed.

In small samples the exact distribution of Q can be constructed by regarding the row totals as fixed, and by assuming that on the null hypothesis every column is equally likely to obtain one of the successes in a row. This exact distribution is worked out for eight examples in order to test the accuracy of the χ^2 approximation to the distribution of Q in small samples. The number of samples ranged from $c = 3$ to $c = 5$. The average error in the estimation of a significance probability was about 14 per cent in the neighborhood of the 5 per cent level and about 21 per cent in the neighborhood of the 1 per cent level. Correction for continuity did not improve the accuracy of the approximation, although it is recommended when there are only two samples. Another approximation, obtained by scoring each success as "1" and each failure as "0" and performing an analysis of variance on the data, was also investigated. The F-test, corrected for continuity, performed about as well as the χ^2 approximation (uncorrected), but is slightly more laborious.

The problem of subdividing χ^2 into components for more detailed tests is briefly discussed.

4. A Method of Estimating Components of Variance in Disproportionate Numbers. H. L. Lucas, North Carolina State College.

By including sufficient effects in the forward solution of the Abbreviated Doolittle method, components of variance may be estimated from disproportionate data. The procedure is very systematic, and thus, is adaptable to routine computational work. The computations will be described, and the utility of the method briefly discussed.

5. On the Theory of Unbiased Tests of Simple Statistical Hypotheses Specifying the Values of Two Parameters. (Preliminary Report). Stanley L. Isaacson, Columbia University.

In the Neyman-Pearson theory of testing simple hypotheses, in the one-parameter case, a locally best unbiased region is called "type A." It is obtained by maximizing the curvature of the power curve at the point $\theta = \theta_0$ specified by the hypothesis, subject to the conditions of size and unbiasedness. For the two-parameter case, Neyman and Pearson considered "type C" regions (*Stat. Res. Mem.*, vol. 2 (1938), p. 36). The definition of these regions requires one to choose in advance a family of ellipses of constant power in an infinitesimal neighborhood of the point $(\theta_1, \theta_2) = (\theta_1^0, \theta_2^0)$ specified by the hypothesis. The natural generalization of a "type A" region is a "type D" region, which maximizes the Gaussian curvature of the power surface at (θ_1^0, θ_2^0), subject to the conditions of size and unbiasedness. This definition does not require one to choose a family of ellipses in advance. This approach leads to a new problem in the calculus of variations. A sufficient condition is obtained which plays the role of the Neyman-Pearson fundamental lemma in the "type A" case. An illustrative example is given. (Prepared under sponsorship of the Office of Naval Research.)

6. A Note on Orthogonal Arrays. Raj Chandra Bose, University of North Carolina.

Consider a matrix $A = (a_{ij})$ with N rows and m columns, each element a_{ij} standing for one of the s integers $0, 1, 2, \cdots, s - 1$. Let us take the partial matrix obtained by choosing any $t \leq m$ columns of A. Each row now consists of an ordered t-plet of numbers, and each element has one of s possible values, there are s^t possible t-plets. The matrix A may be called an orthogonal array (N, m, s, t) of size N, m constraints, s levels and strength t, if by choosing any t columns whatsoever every possible t-plet occurs the same number of times. Clearly $N = \lambda s^t$ where λ is an integer. Such arrays have been considered by Rao and are useful for various experimental designs. The existence of an orthogonal array $(s^2\ M, s, 2)$ is equivalent to the existence of a set of orthogonal Latin squares of side s and m constraints (i.e., the number of Latin squares in the set is $m - 2$). The fundamental question that can be asked regarding orthogonal arrays is the following: What is the maximum number of constraints for an orthogonal array, given N, s and t? Denote this number by $f(N, s, t)$, then from known properties of Latin squares $f(s^2, s, 2) = s + 1$, if s is a prime or a prime power, and a theorem by Mann states that $f(s^2, s, 2) = r + 1$, if $s = p_1^{n_1} \cdots p_k^{n_k}$, where $p_1, \cdots, p_k$ are different primes, and r is the minimum of $p_0^{n_0}, p_1^{n_1} \cdots p_k^{n_k}$. The following generalisation of Mann's theorem is proved in this note:

$$f(N_1 N_2 \cdots N_k, s_1 s_2 \cdots s_k, t) = \mathrm{Min}\{f(N_1, s_1 t), f(N_2, s_2, t), \cdots, f(N_k, s_k, t)\}.$$

7. Transformations Related to the Angular and the Square Root. Murray F. Freeman and John W. Tukey, Princeton University.

The use of transformations to stabilize the variance of binomial or Poisson data is familiar (Anscombe, Bartlett, Curtiss, Eisenhart). The comparison of transformed binomial or Poisson data with percentage points of the normal distribution to make approximate significance tests or to set approximate confidence intervals is less familiar. Mosteller and Tukey have recently made a graphical application of a transformation related to the square-root transformation for such purposes, where the use of "binomial probability paper" avoids all computation. We report here on an empirical study of a number of approximations, some intended for significance and confidence work, and others for variance stabilization. (Prepared in connection with research sponsored by the Office of Naval Research).

8. Standard Inverse Matrices for Fitting Polynomials. F. J. Verlinden, North Carolina State College.

For fitting polynomials of the type, $y = b_0 x^0 + b_1 x + b_2 x^2 + \cdots + b_m x^m$, with the x's equally spaced, published tables of orthogonal polynomials may be used. This procedure does not yield the b's directly, nor their variances or covariances, although such may be obtained by proper computations which are moderately tedious. In some types of statistical work, the b's and their variances and covariances may be desired. These may of course be obtained directly by the method of least squares but the computational work is prodigious relative to that for the orthogonal polynomial approach. When the x's are equally spaced the elements of the variance-covariance matrix may be put in the simple form of sums of powers (including the zero power) of successive integers from zero to n (n equals one less than the number of observations). The elements of the inverses of matrices of this type have been worked out algebraically in terms of n for polynomials up to and including the quintic ($m = 5$). With these standard inverse matrices, the b's and their variances and covariances may quickly be obtained once the elements are evaluated numerically. These elements have been evaluated numerically up to $n = 20$.

9. Mathematical Models in Biology. J. A. Rafferty, Department of Biometrics, School of Aviation Medicine, Randolph Field, Texas.

From the point of view of a bio-medical research administrator, mathematical models will assume a greater role in biological research than heretofore. In anticipation of this trend, certain philosophical implications of models in biological theory and scientific theory in history are examined. A hierarchy of abstraction-levels in biology is delineated, and the role of mathematical models at these levels is illustrated by examples from the literature. Proposals are made for a concentration of mathematical effort on certain important biological problems. Remarks are made on the capabilities and limitations of models in biology.

10. Small Sample Performance of Biological Statistics. IRWIN BROSS, Johns Hopkins University.

In this paper the dilution method for estimating bacterial density is investigated by an exact small sample method and also by an approximate one. Methodologies and design of experiments are compared for various small sample cases.

11. Methodology in the Study of Physical Measurements of School Children. B. G. GREENBERG AND A. HUGHES BRYAN, University of North Carolina.

In a series of investigations to determine by small-sampling technique what physical differences, if any, occur between children of differing socio-economic backgrounds, several problems of methodology arose. A pilot study was undertaken to assure maximum efficiency at each step. This paper reports some of these results. It was found that the children could remain dressed (with the exception of boys' bi-iliac measurement) without changing the magnitude of the differences. The pilot study enabled us to decide how many observers to use, and how much duplication of measurements by them was necessary. Minimum sample sizes were estimated to indicate physical differences of predetermined magnitudes. It was found that the age grouping 96–143 months was optimal from the standpoint of indicating physical differences between children of differing socio-economic levels. Boys and girls in the upper socio-economic levels were both taller and heavier for their age in this age group. There were no weight differences, however, when weight was adjusted for age and height. Measurement of the bi-iliac and transverse chest diameter provided little additional information on physical differences. The calf circumference, an indicator of muscle mass and subcutaneous fat, is suggested as being a sensitive supplementary index to indicate physical differences when age and height are adjusted.

12. Tetrad Analysis in Yeast. A. S. HOUSEHOLDER, Oak Ridge National Laboratory, Oak Ridge, Tennessee.

In neurospora all four products of meiosis are recovered in the four spores of an ascus. In crosses $AB \times ab$ the asci are of three types, designated I, II or III according as all four, none, or two spores resemble parents. Frequencies of these types, P, P′ and P″ are the observables. If there were no exchange P'' would be zero; and one should have $P' = 0$ or $\frac{1}{2}$ according to whether the loci were on the same or different chromosomes.

Assuming only that no exchange occurs between sister chromatids and neglecting chromatid interference, one can calculate without further assumptions a frequency P'' of exchanges between a single locus and its centromere from data on three or more genes taken in pairs by equations

$$s_{ij} = s_{0i}s_{0j}\,, \qquad P'' = 2(1 - s)/3,$$

where the subscript 0 refers to a centromere. Lindegren makes such calculations from his own data, by taking groups of three, but makes no effort to reconcile discrepancies. Neyman's modified chi-square, however, permits combining all observations in a set of equations that yields easily to rapidly converging iterative solution. The equations are

$$2s_i \sum_{j \neq i} s_j^2 (n_{ij} + n'_{ij})^2 (n_{ij}^{-1} + n'^{-1}_{ij}) = \sum_{j \neq i} s_j (n_{ij} + n'_{ij})^2 (2n_{ij}^{-1} - n'^{-1}_{ij}),$$

where n_{ij} is the number in class I and II combined for the loci i and j, n'_{ij} the number class in III, and only those pairs (i, j) are included which are found to be independent.

The argument of A. R. G. Owen (*Proc. Roy. Soc., Ser. B*, Vol. 136 (1949) pp. 67–94.) can be paraphrased for the present case and a suitable generating function $P(\lambda, u)$ is being sought providing a metric. The specific one proposed by Owen is ruled out since $s = P(-\frac{1}{2}, u)$ takes on a negative value for one locus, which is not possible with Owen's function.

13. Contribution to the Probabilistic Theory of Neural Nets. I. Randomization of Refractory Periods and of Stimulus Intervals. Anatol Rapoport, University of Chicago.

Aggregates of neurons are considered in which the frequency of occurrence of neurons with a specified value of the refractory period follows certain probability distributions. Input-output functions are derived from such aggregates. In particular, if input and output intensities are defined in terms of stimulus frequencies and firing frequencies per neuron respectively, it is shown that a rectangular distribution of refractory periods leads to a logarithmic input-output curve. If input and output are defined in terms of the total number of stimuli and firings in the aggregate, it is shown how the "mobilization" picture leads to the logarithmic input-output curve.

By randomizing the intervals between stimuli received by a single neuron and by introducing an inhibitory neuron a very simple "filter net" can be constructed whose output will be sensitive to a particular range of the input, and this range can be made arbitrarily small.

14. Theoretical and Experimental Aspects in the Removal of Air-Borne Matter by the Human Respiratory Tract. H. D. Landahl, University of Chicago.

The principal factors governing the fate of a particle in the respiratory tract are impaction due to inertia, settling due to gravity and Brownian movements. For a given respiratory pattern, it is possible to calculate the probable fate of a particle from a knowledge of the geometry of the passages. These calculations have been carried out in such a manner as to obtain the theoretical amounts of material deposited in various regions of the lungs as well as the relative amounts in various fractions of the expired air. Similarly, it is possible to estimate the probable fate of a particle which passes through the nasal passages. Experiments have been carried out to verify a number of these predictions. On the whole, the agreement, as illustrated in the slides, is fairly satisfactory when one considers the complexity of the calculations.

15. An Application of Biometrics to Zoological Classification. F. M. Wadley, Navy Department, Washington, D. C.

Statistical problems in taxonomy are discussed; attention must be paid to variation of individuals as well as of group means. Covariance analysis and the discriminant function technique are applied to multiple measurements in groups of molluscan fossils.

16. The Analysis of Hemotological Effects of Chronic Low-Level Radiation. Jack Moshman, United States Atomic Energy Commission, Oak Ridge, Tennessee.

Several methods are investigated for analyzing the possible effects of chronic low-level irradiation upon the employees of the operating contractors of the US AEC. The effects investigated are those on the red blood count, hemoglobin, white blood count, lymphocytes and neutrophils. The analysis includes measurements of significant differences among individuals, geographic sites and the exploration of various indices of exposure to radiation. A non-parametric determination of trend values for individuals which may be applied to mass data is considered.

17. Statistical Problems in Psychological Testing. Edward E. Cureton, University of Tennessee.

Though great progress has been made in mathematical statistics in recent years, a number of the major statistical problems encountered in the development and use of psychological tests remain unsolved. Some of these problems are outlined, with particular reference to the mathematical models and assumptions implied by psychological theory, by the nature of the experimental data, and by the conditions under which the results and findings are to be applied.

18. Accuracy of a Linear Prediction Equation in a New Sample. George E. Nicholson, Jr., University of North Carolina.

The problem considered is as follows. Given two samples S_1 and S_2 of N_1 and N_2 observations on a $p + 1$ character random variable $(y, x_1 \cdots x_p)$. Let Y_1 and Y_2 be the linear regression equation computed by the method of least squares from each sample. The effect of using Y_1 to predict the y's in S_2 is considered. The ratio $k \cdot \dfrac{S(y_2 - Y_1)^2}{S(y_2 - Y_2)}$ s used as a measure of the predicting efficiency of Y_1 in S_2 relative to Y_2 when the X_i are fixed for the usual regression model. The general multivariate case is also considered.

19. Independence of Quadratic Forms in Normally Correlated Variables. Yukiyosi Kawada, Tokyo University of Literature and Science, Tokyo, Japan.

An extension is given of theorems of Craig, Hotelling and Matérn which includes the following theorem, proved by a new method: If two quadratic forms Q_1, Q_2 in normally and independently distributed variates with zero means and unit variances satisfy the four conditions $E(Q_1^i Q_2^j) = E(Q_1^i)E(Q_2^j)$, for $i, j = 1, 2$, then the product of the matrices of the two forms in either order is zero.

20. Bounds on the Distribution of Chi-square. S. A. Vora, University of North Carolina.

Let

$$\chi^2 = \sum_{j=1}^{k} (v_j - np_j)^2/np_j, \qquad \chi'^2 = \sum_{j=1}^{k} (v_j + \tfrac{1}{2} - Np_j)^2/Np_j,$$

where $v_j \geq 0, \sum_1^k v_j = n, p_j > 0, \sum_1^k p_j = 1$ and $N = n + k/2$. Bounds on the multinomial probability T in terms of χ'^2 are obtained. A triangular transformation of

$$x_i = (v_i + \tfrac{1}{2} - Np_i)/\{Np_i(1 - p_i)\}^{1/2} \qquad (i = 1, \cdots, k - 1),$$

to y_i is applied so that

$$d \cdot \chi'^2 = \sum_{i=1}^{k-1} y_i^2,$$

where d is determined later by equating the coefficients of χ'^2. Certain rectangles $r(v)$ with $(y_1, \cdots, y_{k-1})$ as a mid-point are non-overlapping and cover the entire space R_{k-1} for $v_i = 0, \pm 1, \pm 2, \cdots$. If $\chi'^2 \leq c$, then bounds on T in terms of the integral of the $(k-1)$ dimensional normal frequency function over the rectangle $r(v)$ are obtained. Prob. $\{\chi'^2 \leq c\}$ is the sum of T over $\chi'^2 \leq c$, so the integral over the sum of rectangles whose mid-points lie within the hypersphere $\chi'^2 \leq c$ is considered. Two hyperspheres, one which contains the sum of those rectangles, and one which is contained in it are used for the bounds, giving

$$\lambda_2 \cdot F_{k-1}(c_2) \leq \text{Prob. } \{\chi'^2 \leq c\} \leq \lambda_1 \cdot F_{k-1}(c_1),$$

where $F_{k-1}(x)$ is a chi-square distribution function with $(k-1)$ degrees of freedom and $\lambda_1, \lambda_2, c_1, c_2$ are functions of c, n, k and $p_1, \cdots, p_k$. As $n \to \infty$, both bounds tend to $F_{k-1}(c)$. Bounds of the same form are obtained for Prob.$\{\chi^2 \leq C\}$. Closer bounds for Prob.$\{\chi^2 \leq C\}$ are given in terms of a non-central chi-square distribution.

21. Estimation of Genetic Parameters. C. R. Henderson, Cornell University.

Many applications of genetics and statistics to the improvement of plants and animals deal with experimental data for which the underlying model is assumed to be

$$y_\alpha = \sum_{i=1}^{p} b_i x_{i\alpha} + \sum_{i=1}^{q} u_i z_{i\alpha} + e_\alpha,$$

where b_i are unknown fixed parameters, $x_{i\alpha}$ and $z_{i\alpha}$ are observable parameters, the u_i are a random sample from a multivariate normal distribution with means zero and covariance matrix $\| \sigma_{ij} \|$, and the e_α are normally and independently distributed with means zero and variances σ_α^2. If $\sigma_{ij} = 0$ when $i \neq j$ and if $\sigma_\alpha^2 = \sigma_e^2$, the model is the one usually assumed when components of variance are estimated.

Three different estimation problems are involved, (1) estimation of b_i under the assumptions of the model, (2) estimation of u_i and (3) estimation of σ_{ij}. The first two problems are not solved satisfactorily by the least squares procedure in which the u_i are regarded as fixed, but the maximum likelihood solution does lead to a satisfactory estimation procedure.

Assuming that the σ_{ij} and σ_α^2 are known, the joint maximum likelihood estimates of b_i and u_i are the solution to the set of linear equations

$$\sum_{i=1}^{p} b_i \left(\sum_\alpha x_{h\alpha} x_{i\alpha}/\sigma_\alpha^2\right) + \sum_{i=1}^{q} u_i \left(\sum_\alpha x_{h\alpha} z_{i\alpha}/\sigma_\alpha^2\right) = \sum_\alpha x_{h\alpha} y_\alpha/\sigma_\alpha^2, \qquad h = 1, \cdots, p,$$

$$\sum_{i=1}^{p} b_i \left(\sum_\alpha x_{i\alpha} z_{h\alpha}/\sigma_\alpha^2\right) + \sum_{i=1}^{q} u_i \left(\sigma^{ih} + \sum_\alpha z_{i\alpha} z_{h\alpha}/\sigma_\alpha^2\right) = \sum_\alpha z_{h\alpha} y_\alpha/\sigma_\alpha^2, \qquad h = 1, \cdots, q.$$

Some important applications of this estimation procedure to genetic studies are described and certain computational short-cuts are suggested.

The problem of estimating σ_{ij} has not been solved satisfactory although under certain quite general assumptions the equations for the joint estimation of b_i, u_i, σ_{ij}, and σ_α^2 can easily be written. The solution to the equations, however, is too difficult to make the procedure practical. Nevertheless unbiased estimates of σ_{ij} can be obtained by equating to their expected values the differences between certain reductions in sums of squares computed by least squares and solving for the σ_{ij}. In general, the expectation of the reduction due to $b_1, \cdots, b_p, u_1, \cdots, u_k (k \leq q)$ is $\sum \sum_{gh} d^{gh} E(Y_g Y_h)$, where d^{gh} are the elements of the matrix which is the inverse of the $(p + k)^2$ matrix of coefficients and the Y_g are the right members of the least squares equations.

22. Estimating the Mean and Standard Deviation of Normal Populations from Double Truncated Samples. A. C. Cohen, Jr., University of Georgia.

The method of maximum likelihood is employed to obtain estimates of the mean and standard deviation of a normally distributed population from double truncated random samples. Two cases are considered. In the first, the number of missing variates is assumed to be unknown. In the second, the number of missing (unmeasured) variates in each tail is known. Variances for the estimates involved in each case are obtained from the maximum likelihood information matrices. A numerical example is given to illustrate the practical application of the estimating equations obtained for each of the two cases considered.

23. Minimax Estimates of Location and Scale Parameters. Gopinath Kallianpur, University of North Carolina.

If the joint fr. f. of the random variables $X_1, \cdots, X_N$ contains only a scale parameter and is of the form

$$\frac{1}{\alpha^N} p\left(\frac{x_1}{\alpha}, \cdots, \frac{x_N}{\alpha}\right),$$

then under mild restrictions the following theorem is proved:

Theorem 1: *If the loss function is of the form* $W\left(\frac{\alpha - \bar{\alpha}}{\alpha}\right)$, *the best or minimax estimate* $\bar{\alpha}_0(x)$ *of* α *minimizes*

$$\int_0^\infty W\left(\frac{\alpha - \bar{\alpha}}{\alpha}\right)\frac{1}{\alpha^N} p\left(\frac{x_1}{\alpha}, \cdots, \frac{x_N}{\alpha}\right) d\alpha$$

w.r.t. $\bar{\alpha}$ *and further,*

$$\bar{\alpha}_0(\mu x_1, \cdots, \mu x_N) = \mu\bar{\alpha}_0(x_1, \cdots, x_N), \qquad \mu > 0.$$

When both location and scale parameters are present and the joint fr. f. is of the form

$$\frac{1}{\alpha^N} p\left(\frac{x_1 - \theta}{\alpha}, \cdots, \frac{x_N - \theta}{\alpha}\right),$$

(under conditions similar to those in Theorem 1) we obtain two results for the estimation of θ and α, respectively, one of which is:

Theorem 2: *If the loss function is of the form* $W\left(\frac{\theta - \bar{\theta}}{\alpha}\right)$, *the best estimate* $\bar{\theta}_0(x)$ *of* θ *minimizes*

$$\int_{-\infty}^\infty \int_0^\infty W\left(\frac{\theta - \bar{\theta}}{\alpha}\right)\frac{1}{\alpha_N} p\left(\frac{x_1 - \theta}{\alpha}, \cdots, \frac{x_N - \theta}{\alpha}\right) d\theta\, d\alpha$$

$$\text{and} \quad \bar{\theta}_0\left(\frac{x_1 + \lambda}{\mu}, \cdots, \frac{x_N + \lambda}{\mu}\right) = \frac{\bar{\theta}_0(x_1, \cdots, x_N) + \lambda}{\mu}.$$

These theorems have been applied to derive minimax estimates in the case of standard distributions. Finally, the problem of estimating the difference between the location parameters of two populations is briefly considered. The results obtained in this paper are a continuation of the line of approach suggested in Theorem 5 of Wald's, "Contributions to the Theory of Statistical Estimation and Testing Hypotheses." (*Annals of Math. Stat.*, Vol. 10 (1939), pp. 299–225). (The present work was carried out under Office of Naval Research contract.)

24. On Some Features of the Neyman-Pearson and the Wald Theories of Statistical Inference, Their Interrelations and Their Bearing on Some Usual Problems of Statistical Inference. S. N. Roy, University of North Carolina.

With two alternative hypotheses H_1 and H_2 it is shown that (i) the most powerful test of H_1 with respect to H_2 is automatically an unbiased test in the sense that its power is never less than (and usually greater than) the level of significance α and (ii) there is also a least powerful test with its power not greater (usually less) than α. This means that all tests have powers lying in between, which gives a complete picture of the possible family of tests and provides a basis for defining efficiency of tests.

With the first kind of error α is tied up a minimum second kind of error β (complementary to the maximum power P), and the level at which α is fixed depends upon some compromise between α and β. This intuitive approach is formalised by the introduction of loss functions related to and apriori probability weights for H_1 and H_2, thus leading to the first stage in the Wald treatment of dichotomy with two solutions in the observation space corresponding respectively to minimum and maximum total risks. This is immediately generalised to the first stage in the Wald treatment of multichotomy with minimum and maximum total risk solutions. An important special case is discussed in which all the possible alternatives to a particular hypothesis are, by our test procedure, indistinguishable among themselves, thus effectively forming only one alternative to the hypothesis, which means a degenerate multichotomy. The bearing of this on most powerful tests on an average under the Neyman-Pearson theory is also discussed.

The problem of testing a composite hypothesis which is usually treated in terms of the Neyman-Pearson theory is posed and treated in terms of the (first stage) Wald theory and an indication is given of how these notions could be applied to the usual problems of univariate and multivariate analysis.

25. Note on Uniformly Best Unbiased Estimates. R. C. Davis, Naval Ordnance Test Station, Inyokern, California.

For the estimation in an absolutely continuous probability distribution of an unknown parameter which does not possess a sufficient statistic, it is shown that no unbiased estimate for the unknown parameter exists which attains minimum variance uniformly over a parameter set of arbitrary nature. This result demonstrates the impossibility of obtaining a generalized sufficient statistic first proposed by Bhattacharyya. Although not used in this note it is surmised that Barankin's powerful results on locally best unbiased estimates can be applied to yield further results in this direction.

26. Competitive Estimation. Herbert Robbins, University of North Carolina.

Let θ be a vector random variable with distribution function $G(\theta)$ and let x be a vector random variable whose frequency function $f(x; \theta)$ depends on θ. Two statisticians, A and B, are required to estimate θ from the value of x. If A's estimate is closer to θ he wins one dollar from B, and *vice versa*; in case of a tie no money changes hands. It is shown that A should estimate θ by the function $a(x)$ = median of posterior distribution of θ given x; his expected gain will then be ≥ 0 whatever estimate B may use. If $G(\theta)$ is not known to A he should estimate it from the series of values of θ which have been observed in previous trials. If these are not known, A should estimate $G(\theta)$ from the values of x which have previously occurred; how this may be done is discussed elsewhere (see Abstract 35).

From the point of view of the theory of games, when $G(\theta)$ is unknown we have a game in which the "rules" are unknown and must be successively estimated from past experience. Other examples arise whenever a game involves random devices whose probability distributions are not known to the players but must be inferred by statistical methods, in general from secondary variables which contain only part of the total information. The role of statistical inference in such "long term" games is fundamental.

27. The Effect of an Unknown 'Location Disturbance' on "Student's" t based on a Linear Regression Model. Uttam Chand, Boston University.

Consider $y_1, \cdots, y_{N_1}, y_{N_1+1}, \cdots y_N$, a set of observations ordered in time. If the y's are normally and independently distributed according to $N(\alpha + \beta(t - \bar{t}), \sigma^2)$ and we want to find out if the y's have changed with time, we usually employ a "Student's" t type of statistic with $N - 2$ degrees of freedom. If, as a consequence of the impact of a certain unknown political or economic change in the past on the y's, the y's actually constitute two independent, normal samples $y_1, \cdots y_{N_1}, y_{N_1+1}, \cdots y_N$ distributed according to $N(m_1, \sigma^2)$, $N(m_2, \sigma^2)$ respectively, a two-sample "Student's" t also based on $N - 2$ degrees of freedom would be the appropriate statistic to use for the hypothesis $m_1 = m_2$. If, in fact, the latter situation describes the correct state of affairs, and the statistician employs the "Student's" t based on the regression model, he commits an error. The present paper investigates the nature of such an error in the light of the point of impact as determined by the magnitude of N_1 and the intensity of the impact as determined by the standardized 'distance' $\dfrac{m_2 - m_1}{\sigma\sqrt{\dfrac{1}{N_1} + \dfrac{1}{N - N_1}}}$ of this extraneous 'shock' on the ordered set of observations y.

28. Corrections for Non-normality for the Two-sample t and the F Distributions Valid for High Significance Levels. Ralph A. Bradley, McGill University.

The effects of non-normality of the parent population on common tests of significance have long been of concern in the application of statistical methods to experimental data. In this paper, the two-sample t-statistic is expressed as a simple multiple of the cotangent of an angle between two lines in a space of dimensionality one less than the total of the sample sizes; the F-statistic for k samples is expressed as a multiple of the cotangent of an angle between a line and a plane of $(k - 1)$ dimensions in a space, again, of dimensionality one less than the total of the sample sizes. The geometrical formulation is such as to suggest approximations to the distributions of these statistics valid for large values of the statistics, and these approximations are obtained. The approximations are shown to be exact in the special cases where the parent population is normal, and a method of evaluation of correction factors is given for a wide class of parent populations. The approximation procedures are valid for the distributions under both null and non-null hypotheses.

29. Some Tests Based on the Empirical Distribution Function. (Preliminary Report). James F. Hannan, University of North Carolina.

Let $X = (X_1, X_2, \cdots, X_n)$ be an independent sample of n where X_i has the continuous c.d.f. $F(x)$. Let $S_n(x)$ be the empirical distribution function. Acceptance regions of the type $\{X: S_n(x) \leq \phi(x)$ for all $x\}$ are considered for different specifications of ϕ and their probabilities evaluated. The method of evaluation consists in identifying the regions with regions defined in terms of the order statistics of a sample of n from the uniform distribution on the interval (0, 1). The result obtained for $\phi(x) = F(x) + c/n, 0 \leq c$, integral $\leq n$ is used to provide a direct proof of the Kolmogoroff result

$$\lim_{n\to\infty} P[n^{1/2} \sup_x (S_n(x) - F(x)) \leq z] = 1 - e^{-2z^2},$$

while that obtained for $\phi(x) = F(x) + t, 0 \leq t \leq 1$, gives the exact c.d.f. of the statistic $\sup_x (S_n(x) - F(x))$.

30. On a Generalization of the Behrens-Fisher Problem. (By Title). JOHN E. WALSH, Rand Corporation, Santa Monica, California.

Let $m + n$ independent observations be available where it is only known that a specified m of them are from continuous symmetrical populations with common median μ while the remaining n are from continuous symmetrical populations with common median ν. This is the generalization of the Behrens-Fisher problem investigated; some tests and confidence intervals for $\mu - \nu$ which are valid for the generalized situation are presented. For definiteness, suppose that $n \leq m$. The procedure used is to subdivide the m observations (common median μ) into n groups of nearly equal size and form the mean of the observations for each group. Pair the n means with remaining n observations and subtract the value of each observation from the value of the mean with which it is paired. The resulting n values represent independent observations from populations with common median $\mu - \nu$. Tests and confidence intervals for $\mu - \nu$ are obtained by applying the results of "Applications of Some Significance Tests for the Median Which are Valid Under Very General Conditions" (*Jour. Amer. Stat. Assn.*, Vol. 44 (1949), pp. 342–55) to these n values. To measure the "information" lost by using the generalized tests when one actually has two independent samples from normal populations, power efficiencies are computed with respect to: (a) Scheffé's "best" t-test solution and (b) most powerful solution when ratio of variances is known. Case (a) yields an upper bound while case (b) furnishes a lower bound for the actual efficiency.

31. Construction of Partially Balanced Designs with two Accuracies. (By Title). S. S. SHRIKHANDE, University of North Carolina and Nagpur College, Nagpur, India.

Various methods of construction of partially balanced designs first introduced by Bose and Nair (*Sankhyā*, Vol. 4 (1939), pp. 337–373) have been considered. Two of the methods given are generalisations of a difference theorem given by them. Another method is the inversion of an unreduced balanced incomplete block design with $k = 2$. Use has also been made of the existing balanced incomplete block design in another direction. A number of designs can also be obtained by methods of finite geometries and especially by omitting a number of treatments and certain blocks from the complete lattice designs. Use of curves and surfaces in finite geometries and the use of multifactorial designs given by Plackett and Burman (*Biometrika*, Vol. 33 (1946), pp. 305–325) are also indicated.

32. Designs for Two-way Elimination of Heterogeneity. (By Title). S. S. SHRIKHANDE, University of North Carolina and Nagpur College, Nagpur, India.

Use has been made of the existing balanced and some partially balanced designs for two-way elimination of heterogeneity with at most two accuracies. Particular cases of these designs were given by Youden (*Contributions from Boyce Thompson Institute*, Vol. 9 (1937), pp. 317–326) and Bose and Kishen (*Science and Culture* (1939), pp. 136–137). The method depends upon interchanging the positions of various treatments in the different columns (blocks), if necessary, so as to satisfy certain conditions.

33. Designs for Animal Feeding Experiments. (By Title). S. S. SHRIKHANDE, University of North Carolina and Nagpur College, Nagpur, India.

In animal-feeding experiments change-over designs are generally preferable to continuous feeding experiments. In change-over designs both the direct and carry-over treatment effects are important. Use of balanced and partially balanced incomplete block designs toward this end has been considered.

34. A Truncated Sequential Procedure for Interval Estimation, with Applications to the Poisson and Negative Binomial Distributions. (Preliminary Report). (By Title). D. MARTIN SANDELIUS, University of Uppsala, Sweden, and University of Washington.

Let $x, y_1, y_2, \cdots$ be a sequence of random variables defined in $(0, \infty)$, and let n be the smallest integer satisfying $\sum_{i=1}^{n+1} y_i > tx$, where $t > 0$ is a non-random quantity. Define u_k either as $\sum_{i=1}^{k} y_i/x$ or as the smallest integer exceeding $\sum_{i=1}^{k} y_i/x$, $k = 1, 2, \cdots$. Given the distribution function $F(x, \theta)$ of x and, for any t, the conditional distribution of n with respect to x, the distribution of u_k is obtained. The problem is to determine a confidence interval for θ with confidence coefficient $1 - \alpha$ on the basis of either an observation on u_k, if $u_k \leq t$, or an observation on n, if $n \leq k - 1$. The following procedure is proposed: If $u_k \leq t$, choose θ_{10} and θ_{11} according to a rule satisfying Prob $(\theta_{10} \leq \theta \leq \theta_{11}) \mid u_k \leq t) \geq 1 - \alpha$. If $n \leq k - 1$, choose θ_{20} and θ_{21} such that Prob $(\theta_{20} \leq \theta \leq \theta_{21} \mid n \leq k - 1) \geq 1 - \alpha$. For continuous u_k the following cases are discussed: a) $x = \theta$ with probability 1, and n has, for any t, a Poisson distribution with mean $t\theta$, b) x has a Gamma distribution with mean θ, and the conditional distribution of n with respect to x is, for any t, a Poisson distribution. Both cases may, for instance, be applied to bacterial counting.

35. A Generalization of the Method of Maximum Likelihood: Estimating a Mixing Distribution. (Preliminary Report). (By Title). HERBERT ROBBINS, University of North Carolina.

Let θ be a vector random variable with distribution function $G(\theta)$ belonging to some class $\mathfrak{G}$, let x be a vector random variable whose frequency function $f(x; \theta)$ depends on θ, and let $g^*(x) = \int f(x; \theta)\, dG(\theta)$ be the resulting frequency function of x. From a sample $x_1, x_2, \cdots$ it is required to estimate $G(\theta)$. The generalized method of maximum likelihood consists in using the estimates $G_n(\theta; x_1, \cdots, x_n)$ in $\mathfrak{G}$ for which $\Pi\, g^*(x_i)$ is a maximum. Under certain restrictions this method is consistent as $n \to \infty$.

Any consistent method of estimating the mixing distribution $G(\theta)$ from the sequence $x_1, x_2, \cdots$ yields a solution of parametric statistical decision problems in the following manner: from *past* values $x_1, \cdots, x_{n-1}$ we estimate $G(\theta)$, and then use the corresponding Bayes solution of the decision problem to reach our decision for x_n, even though the value θ_n which produced x_n is different from those which produced $x_1, \cdots, x_{n-1}$. In certain cases of long-term experimentation this approach seems more reasonable than the minimax method which decides on the course of action appropriate to θ_n on the basis of x_n only, and ignores the information about the prior distribution of θ which is contained in $x_1, \cdots, x_{n-1}$.

36. Smallest Average Confidence Sets for the Simultaneous Estimation of k Normal Means. (By Title). RAGHU RAJ BAHADUR, University of North Carolina.

Let $v = (x_{11}, \cdots, x_{1n_1}; \cdots; x_{k1}, \cdots, x_{kn_k})$ denote the combined sample point in samples of sizes $n_1, n_2, \cdots, n_k$ from normal populations $\pi_1, \pi_2 \cdots, \pi_k$, respectively, π_i having mean μ_i and variance σ_i^2. Writing $\mu = (\mu_1, \mu_2, \cdots, \mu_k)$, denote the k dimensional Euclidean space of all points μ by R. Given any parameter point (μ, σ), where $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_k)$, and any set-valued function $f(v)$ defined for all sample points v and having subsets of R as its values (which satisfies certain measurability hypotheses), let $\alpha(f \mid \mu, \sigma)$ = probability of the statement "$\mu \epsilon f(v)$" being false, and $\beta(f \mid \mu, \sigma)$ = expected Lebesgue measure of $f(v)$. We consider the problem of constructing $f(v)$ so as to make both α and β "as small as possible." One of the results obtained is as follows: Given p, $0 < p < 1$, let $f^0_{\lambda:\zeta(p)}(v) = \{\mu: \Sigma_1^k n_i[(\bar{x}_i - \mu_i)/l_i]^2 < \zeta(p)\cdot\Sigma_1^k n_i[s_i/l_i]^2\}$, where $\bar{x}_i = n_i^{-1}\Sigma_1^{n_i} x_{ij}$, $s_i^2 = n_i^{-1}\Sigma_1^{n_i} (x_{ij} - \bar{x}_i)^2$, $\lambda = (l_1, l_2, \cdots, l_k)$, the l_i's being given positive constants, and $\zeta(p)$ being determined by $P(\chi_k^2 > \zeta(p)\cdot\chi_{N-k}^2) = p$, where χ_k^2, χ_{N-k}^2 are independent chi-square variables with k, $N - k$ degrees of freedom $(k < N = \Sigma_1^k n_i)$. Then (a) obviously $\alpha(f^0_{\lambda:\zeta(p)} \mid \mu, c\lambda) = p$ for all μ and all c, $0 < c < \infty$, and (b) if $f(v)$ is any other function such that $\alpha(f \mid \mu, c\lambda) \leq p$ for all μ and all c, either (i) $f(v)$ and $f^0_{\lambda:\zeta(p)}(v)$ differ by a set of measure zero for almost every v, or (ii) $\sup_{\mu \epsilon R} \{\beta(f \mid \mu, c\lambda)\} > \sup_{\mu \epsilon R} \{\beta(f_{\lambda:\zeta(p)} \mid \mu, c\lambda\}$ for every c.

NEWS AND NOTICES

Readers are invited to submit to the Secretary of the Institute news items of general interest

Personal Items

Mr. Harry H. Goode, formerly head of the Special Projects Branch, Special Device Center, Office of Naval Research 1, New York, is now Superviser of the Aero-Physics Group, Aeronautical Research Center, University of Michigan, Ann Arbor, Michigan.

Mr. William G. Howard, who was previously employed by the Johns Hopkins University, Institute for Cooperative Research, is presently employed as Mathematical Statistician in the Air Studies Division of the Library of Congress.

Miss Margaret Kampschaefer has accepted a position as Statistician in the U. S. Bureau of Labor Statistics, Minnesota Payroll Project, Minnesota Division of Employment and Security. She was formerly employed as Junior Mathematician at the Argonne National Laboratory, Naval Reactor Division, Chicago, Illinois.

Dr. Albert Noack has recently been appointed Professor of Actuarial Mathematics at the University of Koeln, Germany.

Second Berkeley Symposium on Mathematical Statistics and Probability

The Second Berkeley Symposium will be held at the Statistical Laboratory, University of California, Berkeley, from July 31 to August 12, 1950, with the