# ASYMPTOTIC PROPERTIES OF THE WALD-WOLFOWITZ TEST OF RANDOMNESS

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**1. Summary.** The paper investigates certain asymptotic properties of the test of randomness based on the statistic $R_h = \sum_{i=1}^{n} x_i x_{i+h}$ proposed by Wald and Wolfowitz. It is shown that the conditions given in the original paper for asymptotic normality of $R_h$ when the null hypothesis of randomness is true can be weakened considerably. Conditions are given for the consistency of the test when under the alternative hypothesis consecutive observations are drawn independently from changing populations with continuous cumulative distribution functions. In particular a downward (upward) trend and a regular cyclical movement are considered. For the special case of a regular cyclical movement of known length the asymptotic relative efficiency of the test based on ranks with respect to the test based on original observations is found. A simple condition for the asymptotic normality of $R_h$ for ranks under the alternative hypothesis is given. This asymptotic normality is used to compare the asymptotic power of the $R_h$-test with that of the Mann $T$-test in the case of a downward trend.

**2. Introduction.** The hypothesis of randomness, i.e., the assumption that the chance variables $X_1, \cdots, X_n$ have the joint cumulative distribution function (*cdf*) $F(x_1, \cdots, x_n) = F(x_1) \cdots F(x_n)$ where $F(x)$ may be any *cdf*, is basic in many statistical problems. Several tests of randomness designed to detect changes in the underlying population have been suggested, however mostly on intuitive grounds. Very seldom has the actual performance of a test with respect to a given class of alternatives been investigated. It is the intention of this paper to carry out such an investigation for the particular test based on the statistic

$$R_h = \sum_{i=1}^{n} x_i x_{i+h}, \qquad x_{n+j} = x_j,$$

proposed by Wald and Wolfowitz [1]. It is suggested in [1] that this test is suitable if the alternative to randomness is the existence of a trend or a regular cyclical movement. Both these cases will be treated.

Let $a_1, \cdots, a_n$ be observations on the chance variables $X_1, \cdots, X_n$ and assume that the hypothesis of randomness is true. (Henceforth this hypothesis will be denoted by $H_0$ while the hypothesis that an alternative to randomness is true will be denoted by $H_1$.) Restricting then $X_1, \cdots, X_n$ to the subpopulation of permutations of $a_1, \cdots, a_n$, any one of the $n!$ possible permutations is equally likely, and the distribution of $R_h$ in this subpopulation can be found. If the level of significance $\alpha$ is chosen in such a way that $\alpha = m/n!$ where $m$ is a positive integer, the test is performed by selecting $m$ of the $n!$ possible values of $R_h$ and rejecting $H_0$ when the actually obtained value of $R_h$ is one of these $m$ values. The particular choice of the critical values should be such as to maximize the power of the test with respect to the class of alternatives under consideration.

Denote the expected value and variance of $R_h$ in the subpopulation of equally likely permutations of $n$ observations $a_1, \cdots, a_n$ by $E^0R_h$ and $V^0R_h$, respectively. Then it is shown in [1] that if $h$ is prime to $n$

$$E^0 R_h = \frac{1}{n-1}(A_1^2 - A_2) \tag{2.1}$$

and

$$\begin{aligned} V^0 R_h = {} & \frac{1}{n-1}(A_2^2 - A_4) \\ & + \frac{1}{(n-1)(n-2)}(A_1^4 - 4A_1^2A_2 + 4A_1A_3 + A_2^2 - 2A_4) \\ & - \frac{1}{(n-1)^2}(A_1^2 - A_2)^2, \end{aligned} \tag{2.2}$$

where $A_r = a_1^r + \cdots + a_n^r$, $(r = 1, 2, 3, 4)$. Actually (2.1) and (2.2) are valid as soon as $n > 2h$.

Let $R_h^0 = (R_h - E^0R_h)/\sqrt{V^0R_h}$. Then it is also shown in [1] that if $h$ is prime to $n$, $R_h^0$ is asymptotically normally distributed with mean 0 and variance 1 provided the $a_i$, $(i = 1, \cdots, n)$, satisfy condition $W$:

$$\frac{\frac{1}{n}\sum_{i=1}^{n}(a_i - \bar{a})^r}{\left[\frac{1}{n}\sum_{i=1}^{n}(a_i - \bar{a})^2\right]^{r/2}} = O(1),^1 \qquad (r = 3, 4, \cdots),$$

where $\bar{a} = n^{-1}\sum_{i=1}^{n} a_i$.

It is easily seen that condition $W$ is satisfied when the original observations are replaced by ranks. When the $a_1, \cdots, a_n$ are independent observations on the same chance variable $X$, condition $W$ is satisfied with probability 1 provided $X$ has positive variance and finite moments of all orders. It is interesting to compare this condition for asymptotic normality of $R_h$ in the population of permutations of observations on the chance variable $X$ with the condition for asymptotic normality of $R_h$ under random sampling. For this case Hoeffding and Robbins [3] have shown that it is sufficient to assume that $X$ has a finite absolute moment of order 3. Thus it is desirable to weaken condition $W$. This will be done in Section 3.

In further sections the consistency and efficiency of the test based on $R_h$ will

[1] The symbol $O$, as well as the symbols $o$ and $\sim$ to be used later, have their usual meaning. See, for example, Cramér [2], p. 122.

be examined assuming that under the alternative hypothesis observations, though still independent, are drawn from changing populations. Throughout the paper the circularly defined statistic $R_h$ is used. However, if with probability 1

$$x_{n-h+1}x_1 + \cdots + x_n x_h = o(R_h),$$

it is seen that asymptotically the test based on the non-circular

$$\bar{R}_h = \sum_{i=1}^{n-h} x_i x_{i+h}$$

has the same properties as that based on $R_h$. We find

$$E^0 \bar{R}_h = \frac{n-h}{n(n-1)} (A_1^2 - A_2),$$

$$V^0 \bar{R}_h = \frac{n-h}{n(n-1)} (A_2^2 - A_4) + \frac{2(n-2h)}{n(n-1)(n-2)} (A_1^2 A_2 - A_2^2 - 2A_1A_3 + 2A_4)$$

$$+ \frac{(n-h-1)(n-h-2) + 2(h-1)}{n(n-1)(n-2)(n-3)} (A_1^4 - 6A_1^2A_2 + 8A_1A_3 + 3A_2^2 - 6A_4)$$

$$- \frac{(n-h)^2}{n^2(n-1)^2} (A_1^2 - A_2)^2.$$

**3. Asymptotic normality of $R_h$ under randomization.** Let the set of chance variables $X_1, \cdots, X_n$ be defined on the $n!$ equally likely permutations of $n$ numbers $\mathfrak{A}_n = (a_1, \cdots a_n)$. Then we have

THEOREM 1: *The distribution of $R_h^0$ tends to the normal distribution with mean* 0 *and variance* 1 *as $n \to \infty$ provided*

$$\frac{\sum_{i=1}^{n} (a_i - \bar{a})^r}{\left[\sum_{i=1}^{n} (a_i - \bar{a})^2\right]^{r/2}} = o[n^{(2-r)/4}], \qquad (r = 3, 4, \cdots), \tag{3.1}$$

*where* $\bar{a} = n^{-1} \sum_{i=1}^{n} a_i$.

REMARK: The set $\mathfrak{A}_n$ need not be a subset of $\mathfrak{A}_{n+1}$.

The proof of this theorem will be omitted, since it is very similar to the proof of another theorem by the author [4].

THEOREM 2: *If the $a_1, a_2, \cdots$ are independent observations on a chance variable $X$ having positive variance and a finite absolute moment of order $4 + \delta$, $\delta > 0$, condition* (3.1) *is satisfied unless possibly an event of probability* 0 *has occurred.*

The proof of this theorem will be based on Markoff's method for proving the central limit theorem in the Liapounoff form.[2] Thus we shall show that there exists a sequence of sequences $\mathfrak{B}_n = (b_{n1}, \cdots, b_{nn})$ such that unless possibly an event of probability 0 has occured, (i) there exists an index $n'$ (depending

[2] See, for example, Uspensky [5], pp. 388-95.

on the given sequence) such that for $n > n'$, $\mathfrak{A}_n = \mathfrak{B}_n$, and (ii) the sequences $\mathfrak{B}_n$ satisfy condition (3.1) expressed in terms of the $b_{ni}$, $(i = 1, \cdots, n)$.

It is no restriction to assume that $EX = 0$, since the addition of one and the same constant to every $a_i$ does not change (3.1). Let

$$N = N(n) = n^{1/(4+\delta/2)},$$

and define for $i = 1, \cdots, n$

$$\begin{aligned} b_{ni} &= a_i, & c_{ni} &= 0, & &\text{if } a_i \leq N(n), \\ &= 0, & &= a_i, & &\text{if } a_i > N(n), \end{aligned}$$

so that $a_i = b_{ni} + c_{ni}$. Then $b_{ni}$ and $c_{ni}$ can be considered as observations on chance variables $Y_n$ and $Z_n$, respectively, where

$$\begin{aligned} Y_n &= X, & Z_n &= 0, & &\text{if } X \leq N(n), \\ &= 0, & &= X, & &\text{if } X > N(n). \end{aligned}$$

Further let $p_n = P\{Z_n = X\}$, $\alpha_r(U) = EU^r$, $\beta_r(U) = E \mid U \mid^r$ where $U = X$, $Y_n$, $Z_n$ and $r$ is positive integral, if these moments exist, $\beta_{4+\delta} = E \mid X \mid^{4+\delta}$, and finally, let $F(x)$ be the *cdf* of $X$.

In order to prove (i) consider the infinitely dimensional sample space $\Omega$ with the generic point $\omega = \omega(a_1, a_2, \cdots)$ and let $E_n = \{\omega \mid a_n > N(n)\}$, $(n = 1, 2, \cdots)$. Then $E_n$ has probability measure $p_n$. We shall show that $\sum_{n=1}^{\infty} p_n$ converges. Since

$$\beta_{4+\delta} = \int_{-\infty}^{\infty} \mid x \mid^{4+\delta} dF(x) \geq N^{4+\delta} \left[ \int_{-\infty}^{-N} dF(x) + \int_{+N}^{+\infty} dF(x) \right] \geq N^{4+\delta} p_n,$$

we find

$$p_n \leq \beta_{4+\delta} \frac{1}{N^{4+\delta}} = \beta_{4+\delta} \frac{1}{n^{(4+\delta)/(4+\delta/2)}}.$$

Now $(4 + \delta)/(4 + \delta/2) > 1$ and the infinite sum converges. It follows that the set $E$ of points which belong to infinitely many sets $E_n$ has probability measure 0. Thus for every point $\omega \in \Omega$ except those in a set of measure 0 there exists an index $n_\omega$ (depending on $\omega$) such that for $n > n_\omega$

$$a_n \leq N(n). \tag{3.2}$$

Further, since $n_\omega$ is finite and $N(n) \to \infty$, it follows that for these points there exists a second index $n'_\omega \geq n_\omega$ such that in addition to (3.2) $a_n \leq N(n'_\omega)$, $(n = 1, \cdots, n_\omega)$. Thus except on a set of measure 0 the sequences $\mathfrak{B}_n$ are identical with the sequences $\mathfrak{A}_n$ for $n > n'_\omega$. This proves (i).

In proving (ii) let $B_{nr} = \sum_{i=1}^{n} b_{ni}^r$, $(n, r = 1, 2, \cdots)$. We first note that under the assumptions of the theorem $n^{-1}A_r \to \alpha_r(X)$ for $r = 1, 2, 3, 4$ except on a set of measure 0. Thus except on a set of measure 0

$$\bar{a} = n^{-1}A_1 = o(1), \qquad A_2 = \Omega(n),^3 \qquad A_3 = O(n), \qquad A_4 = O(n),$$

[3] A function $f(n)$ is said to be of order $\Omega(n^k)$, $k$ real, if $f(n) = O(n^k)$ and $\liminf_n \mid f(n)/n^k \mid > 0$.

and therefore by the argument used in proving (i) again except on a set of measure 0

$$\bar{b}_n = n^{-1}B_{n1} = o(1), \qquad B_{n2} = \Omega(n), \qquad B_{n3} = O(n), \qquad B_{n4} = O(n).$$

It follows that in order to prove (ii) it is sufficient to show that

$$B_{nr} = o[n^{(r+2)/4}], \qquad (r = 5, 6, \cdots), \tag{3.3}$$

except on a set of measure 0.

Now for $r \geq 5$

$$\alpha_r(Y_n) \leq \beta_r(Y_n) \leq N^{r-4}\beta_4(Y_n) \leq N^{r-4}\beta_4(X),$$

and therefore

$$\alpha_r(Y_n) = O(N^{r-4}) = O[n^{(r-4)/(4+\delta/2)}].$$

It follows that

$$EB_{nr} = n\alpha_r(Y_n) = O[n^{(r+\delta/2)/(4+\delta/2)}]$$

and

$$\text{var } B_{nr} = n \text{ var } Y_n^r = n[\alpha_{2r}(Y_n) - \alpha_r^2(Y_n)] = O[n^{(2r+\delta/2)/(4+\delta/2)}],$$

so that

$$\sigma(B_{nr}) = O[n^{(r+\delta/4)/(4+\delta/2)}].$$

Assume now that for some $r \geq 5$ (3.3) is not satisfied on a set $F_r$ having measure $\epsilon_r > \epsilon > 0$. We shall show that this assumption leads to a contradiction, and that therefore (3.3) is true.

Choose $e$ such that

$$1/2 < e < (16 + r\delta)/(32 + 4\delta). \tag{3.4}$$

Since $r \geq 5$, (3.4) can always be satisfied. Then the infinite sum $\sum_{n=1}^{\infty} (1/n^{2e})$ converges, and a positive constant $d$ can be found in such a way that

$$p = \frac{1}{d^2} \sum_{n=1}^{\infty} \frac{1}{n^{2e}} < \epsilon.$$

If we then write the Tchebysheff inequality

$$P\{| B_{nr} - EB_{nr} | > dn^e\sigma(B_{nr})\} \leq 1/d^2n^{2e},$$

it is seen that except on a set having at most measure $p$

$$B_{nr} = O\{\max[n^{(r+\delta/2)/(4+\delta/2)}, \qquad n^e n^{(r+\delta/4)/(4+\delta/2)}]\}.$$

Now for $r \geq 5$

$$(r + \delta/2)/(4 + \delta/2) < r/4$$

and by (3.4)

$$\begin{aligned} e + (r + \delta/4)/(4 + \delta/2) &= e + r/4 + (\delta/4 - r\delta/8)/(4 + \delta/2) \\ &< r/4 + (16 + 2\delta)/(32 + 4\delta) = (r + 2)/4, \end{aligned}$$

so that the measure of the set $F_r$ is not even equal to $\epsilon$. This contradicts our assumption, thus proving Theorem 2.

**4. Consistency.** To prove consistency of tests based on permutations of observations $a_1, \cdots, a_n$ the following procedure can be applied. Let the test statistic be $S_n = S(x_1, \cdots, x_n)$ and denote by $E_n^0 = E^0(a_1, \cdots, a_n)$ and $V_n^0 = V^0(a_1, \cdots, a_n)$ the expected value and variance of $S_n$ under the assumption that the set of random variables $X_1, \cdots, X_n$ is restricted to the subpopulation consisting of the $n!$ equally likely permutations of the observations. Assume that for the alternatives under consideration large values of $S_n$ are critical. Then we reject the null hypothesis whenever $(S_n - E_n^0)/\sqrt{V_n^0} > k$ where $k$ is some positive constant depending on the limiting distribution of $S_n$ under the assumption of equally likely permutations and the level of significance. Thus in order to prove consistency we have to show that

$$\lim_{n\to\infty} P\left\{\frac{S_n - E_n^0}{\sqrt{V_n^0}} > k \mid H_1\right\} = 1. \tag{4.1}$$

(4.1) will be satisfied if for some $\epsilon > 0$

$$\lim_{n\to\infty} P\left\{\frac{S_n - E_n^0}{\sqrt{nV_n^0}} > \epsilon \mid H_1\right\} = 1.$$

Thus we shall have proved consistency, if we can show that when $H_1$ is true, $E_n^0/\sqrt{nV_n^0}$ converges in probability to 0 and there exists some $\epsilon > 0$ such that $\lim_{n\to\infty} P\{S_n/\sqrt{nV_n^0} > \epsilon \mid H_1\} = 1$.

Applying this method to our problem and noting that a corresponding procedure could have been used in the case when small values of $S_n$ are critical, we obtain

THEOREM 3: *The test based on $R_h$ is consistent with respect to alternatives for which*

$$\frac{E^0 R_h}{\sqrt{nV^0 R_h}} \underset{\text{pr}}{\to} 0 \tag{4.2}$$

*and there exists some $\epsilon > 0$ such that*

$$\lim_{n\to\infty} P\left\{\left|\frac{R_h}{\sqrt{nV^0 R_h}}\right| > \epsilon\right\} = 1, \tag{4.3}$$

*where $E^0R_h$ and $V^0R_h$ are given by* (2.1) *and* (2.2), *respectively.*

In what follows it will always be assumed that under the alternative hypothesis observations are independent from chance variables $X_n$ with continuous *cdf*'s $F_n(x)$, $(n = 1, 2, \cdots)$. We shall often have the opportunity to make use of the fact that the test is not changed if one and the same constant is subtracted from every observation. This will be helpful in reducing our problem to one for which (4.2) is true.

Let $a_i$ be the rank of the observation $x_i$ on the chance variable $X_i$, $(i =$

1, $\cdots$, $n$). Then it is no restriction to assume that these ranks take the special form

$$-(n-1)/2, \qquad -(n-3)/2, \cdots, (n-1)/2,$$

so that $A_1 = 0$, $A_2 = \frac{1}{12}(n^2 - 1)n = \Omega(n^3)$ and

$$V^0 R_h \sim \frac{1}{n} A_2^2 \sim \frac{1}{144} n^5 = \Omega(n^5) \tag{4.4}$$

and therefore (4.2) is always satisfied.

Before we can find conditions under which (4.3) is satisfied, we have to investigate the expected value and variance of $R_h$ when $H_1$ is true. For this purpose write $a_i = \sum_{j=1}^n y_{ij}$, $(i = 1, \cdots, n)$,

$$\begin{aligned} y_{ij} &= -1/2 \quad \text{if } x_i > x_j, \\ &= \phantom{-}1/2 \quad \text{if } x_i < x_j, \end{aligned} \qquad y_{ii} \equiv 0. \tag{4.5}$$

Then if $P\{X_i < X_j\} = p_{ij}$, $(i, j = 1, \cdots, n)$, we find

$$Ey_{ij} = \tfrac{1}{2}p_{ij} - \tfrac{1}{2}(1 - p_{ij}) = p_{ij} - \tfrac{1}{2} = \epsilon_{ij}, \qquad \text{(say)}.$$

Further,

$$R_h = \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n y_{ij} y_{i+h,k}, \qquad y_{n+l,k} = y_{lk}. \tag{4.6}$$

Therefore

$$E(R_h \mid H_1) = \sum_i \sum_j \sum_k \epsilon_{ij} \epsilon_{i+h,k} + O(n^2) \tag{4.7}$$

and

$$\begin{aligned} \operatorname{var} R_h &= E \sum_{ijk} \sum_{\alpha\beta\gamma} y_{ij} y_{i+h,k} y_{\alpha\beta} y_{\alpha+h,\gamma} - E \sum_{ijk} y_{ij} y_{i+h,k} E \sum_{\alpha\beta\gamma} y_{\alpha\beta} y_{\alpha+h,\gamma} \\ &= \sum_{ijk} \sum_{\alpha\beta\gamma} (E y_{ij} y_{i+h,k} y_{\alpha\beta} y_{\alpha+h,\gamma} - E y_{ij} y_{i+h,k} E y_{\alpha\beta} y_{\alpha+h,\gamma}). \end{aligned} \tag{4.8}$$

In (4.8) the expression in parentheses is 0 unless one of the Greek indices (including $\alpha + h$) equals one of the Roman indices. Therefore var $(R_h \mid H_1) = O(n^5)$.

It then follows from (4.4) that

$$R_h/\sqrt{nV^0 R_h} \sim \frac{12}{n^3} R_h \underset{pr}{\rightarrow} 12 \lim_{n\to\infty} \frac{1}{n^3} E(R_h | H_1),$$

and we can state the following corollary to Theorem 3:

Corollary: *When using ranks, the test based on* $R_h$ *is consistent, if under the alternative hypothesis*

$$\frac{1}{n^3} \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \epsilon_{ij} \epsilon_{i+h,k} = \Omega(1), \tag{4.9}$$

*where* $\epsilon_{ij} = P\{X_i < X_j\} - \frac{1}{2}$.

Since $\epsilon_{ij} = -\epsilon_{ji}$, we can write

$$\sum_i \sum_j \sum_k \epsilon_{ij}\epsilon_{i+h,k} = \sum_k \sum_i \sum_{j>i} \epsilon_{ij}(\epsilon_{i+h,k} - \epsilon_{j+h,k}) = L, \quad \text{(say)},$$

and the test is consistent if

$$\lim_{n\to\infty} \frac{1}{n^3} L \neq 0. \tag{4.10}$$

4.1. *Downward (upward) trend.* Assume that for $i < j$ and all $k$

$$\epsilon_{ij} < 0 \tag{4.11}$$

and

$$\epsilon_{ik} \leq \epsilon_{jk} . \tag{4.12}$$

These requirements are equivalent to $P\{X_i < X_j\} < 1/2$ and $P\{X_i < X_k\} \leq P\{X_j < X_k\}$ and are satisfied if the alternative to randomness is a downward trend in the sense that $F_i(x) \leq F_j(x)$, $(-\infty < x < \infty, i < j)$, with at least one interval of strict inequality.

(4.11) and (4.12) are not sufficient for (4.10) to be true. Thus assume in addition that there exist a positive integer $n'$ and a number $\epsilon < 0$ such that $\text{l.u.b.}_{\,j-i\geq n'}\, \epsilon_{ij} = \epsilon$. Then

$$\lim_{n\to\infty} \frac{1}{n^3} L \geq \lim_{n\to\infty} \frac{1}{n^3} \sum_{k=1}^{n} \sum_{\substack{i\leq k-h-n' \\ j\geq k-h+n'}} \epsilon_{ij}(\epsilon_{i+h,k} - \epsilon_{j+h,k})$$

$$\geq 2\epsilon^2 \lim_{n\to\infty} \frac{1}{n^3} \sum_{k=1}^{n} (k - h - n')(n - k + h - n' + 1) = 2\epsilon^2(\tfrac{1}{2} - \tfrac{1}{3}) > O,$$

and the test is consistent.

The case of an upward trend can be treated in exactly the same way. The test is consistent with respect to alternatives for which for $i < j$ and all $k$, $\epsilon_{ij} > 0$, $\epsilon_{ik} \geq \epsilon_{jk}$, and $\text{g.l.b.}_{\,j-i\geq n'}\, \epsilon_{ij} = \epsilon$, where this time $\epsilon > 0$.

Another test of randomness, the so-called $T$-test, has been proposed by Mann [6] with exactly this alternative of a downward (upward) trend in mind. This $T$-test is also consistent provided certain general conditions are satisfied. Thus the question arises which of the two tests should be chosen if a downward (upward) trend is feared. This question will be considered in Section 7.

4.2. *Cyclical movement.* Let the class of alternatives be specified by

$$\epsilon_{lg+\alpha, mg+\beta} = \epsilon_{\alpha\beta}, \qquad (\alpha, \beta = 1, \cdots, g > 1; l, m = 0, 1, \cdots), \tag{4.13}$$

in other words, assume that the statistic $R_h$ is used to test for randomness while under the alternative hypothesis there exists a regular cyclical movement with a period of length $g$. It is sufficient to consider the case $h \leq g$.

If (4.13) is true,

$$\sum_{ijk=1}^{n} \epsilon_{ij}\epsilon_{i+h,k} = n^2 \sum_{i=1}^{n} \epsilon_{i.}\epsilon_{i+h,.} + O(n^2) = n^3\eta + O(n^2), \tag{4.14}$$

where

(4.15) $$\epsilon_{i.} = \frac{1}{g} \sum_{\alpha=1}^{g} \epsilon_{i\alpha}$$

and

(4.16) $$\eta = \frac{1}{g} \sum_{\alpha=1}^{g} \epsilon_{\alpha.} \epsilon_{\alpha+h,.}.$$

Thus in view of (4.9) the test is consistent if $\eta \neq 0$.

If $h = g$, $\eta$ reduces to a sum of squares and is therefore $> 0$ if some $\epsilon_{\alpha.} \neq 0$. However it is possible that some or even all $\epsilon_{\alpha\beta} \neq 0$, $(\alpha \neq \beta)$, and still $\epsilon_{\alpha.} \equiv 0$. If this happens, the test is inconsistent, otherwise it is consistent. If under $H_1$ the populations from which consecutive observations are drawn differ only in location, the above mentioned exceptional case cannot happen, and the test is always consistent with respect to this class of alternatives.

If $h < g$, it is not difficult to construct an example where $\sum_{\alpha=1}^{g} \epsilon_{\alpha.}\epsilon_{\alpha+h,.} \neq 0$ while $\sum_{\alpha=1}^{g} \epsilon_{r_\alpha.}\epsilon_{r_{\alpha+h}.} = 0$, where the $r_\alpha$ are a permutation of the numbers $1, \cdots, g$. Thus in this case it is not sufficient that some $\epsilon_{\alpha.} \neq 0$ for the test to be consistent. Consistency may also depend on the order of the elements of a period.

We may conclude that if $g$ is known, we should always choose $h = g$. If $g$ is not known, we may as well take $h = 1$.

*4.3. Change in location.* Turning now to the case when the test is performed on the basis of the original observations, it will often be appropriate to assume that under the alternative hypothesis the distribution remains the same except for a location parameter. We shall consider only the case of a cyclical movement.

Thus let

$$F_n(x) = F(x - m_n) \qquad (n = 1, 2, \cdots),$$

where $F(x)$ is the *cdf* of a chance variable $U$ having mean 0, and $m_n$ is a location parameter. It will also be assumed that $U$ has the positive variance $\sigma^2$ and a finite fourth moment.

In the cyclical case with period $g$

(4.17) $$m_{lg+\alpha} = m_\alpha \qquad (\alpha = 1, \cdots, g > 1; l = 0, 1, \cdots).$$

We shall find conditions under which our test is consistent with respect to alternatives of this kind. Obviously we can assume that $\sum_{\alpha=1}^{g} m_\alpha = g\bar{m} = 0$, since otherwise we could have subtracted $\bar{m}$ from every observation. Writing then $a_n = u_n + m_n$, $(n = 1, 2, \cdots)$, where $u_n$ can be considered as an observation on the previously defined chance variable $U$, we find

$$A_1 = \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} u_i + O(1),$$

$$A_2 = \sum_{i=1}^{n} u_i^2 + 2\sum_{i=1}^{n} u_i m_i + \sum_{i=1}^{n} m_i^2$$

$$= \sum_{i=1}^{n} u_i^2 + 2\sum_{\alpha=1}^{g} m_\alpha \sum_{l=0}^{n_\alpha} u_{lg+\alpha} + \left[\frac{n}{g}\right] \sum_{\alpha=1}^{g} m_\alpha^2 + O(1),$$

where $n_\alpha$ is the largest integer such that $n_\alpha g + \alpha \leq n$ and $[n/g]$ the largest integer $\leq n/g$. $A_3$ and $A_4$ are given by similar expressions. Since we assumed that $EU = 0$, $EU^2 = \sigma^2 > 0$, and $EU^4 < \infty$, we have with probability 1

$$\sum_{i=1}^n u_i = o(n), \qquad \sum_{i=1}^n u_i^2 = \Omega(n), \qquad \sum_{i=1}^n u_i^3 = O(n), \qquad \sum_{i=1}^n u_i^4 = O(n),$$

so that with the same probability

$$A_1 = o(n), \qquad A_2 = \Omega(n), \qquad A_3 = O(n), \qquad A_4 = O(n).$$

It follows that with probability 1

$$E^0 R_h = o(n), \qquad V^0 R_h \sim \frac{1}{n} A_2^2 = \Omega(n),$$

and condition (4.2) of Theorem 3 is satisfied.

Since further

$$(4.18) \quad \begin{aligned} \operatorname{var} R_h &= \sum_{i=1}^n \operatorname{var}(x_i x_{i+h}) + 2 \sum_{i=1}^n \operatorname{cov}(x_i x_{i+h}, x_{i+h} x_{i+2h}) \\ &= \sum_{i=1}^n \{(\sigma^2 + m_i^2)(\sigma^2 + m_{i+h}^2) - m_i^2 m_{i+h}^2\} \\ &\qquad + 2 \sum_{i=1}^n \{m_i m_{i+2h}(\sigma^2 + m_{i+h}^2) - m_i m_{i+h}^2 m_{i+2h}\} \\ &= \sum_{i=1}^n \{\sigma^4 + \sigma^2(m_i^2 + m_{i+h}^2 + 2 m_i m_{i+2h})\} = O(n) \end{aligned}$$

and therefore except on a set of probability measure 0

$$\frac{R_h}{\sqrt{n V^0 R_h}} \sim \frac{R_h}{A_2} = \frac{\frac{1}{n} R_h}{\frac{1}{n} A_2} \xrightarrow[pr]{} \frac{\lim_{n\to\infty} \frac{1}{n} E(R_h \mid H_1)}{\sigma^2 + \frac{1}{g} \sum_{\alpha=1}^g m_\alpha^2},$$

condition (4.3) is satisfied provided $\lim_{n\to\infty} n^{-1} E(R_h \mid H_1) \neq 0$. Now $E(R_h \mid H_1) = [n/g] \sum_{\alpha=1}^g m_\alpha m_{\alpha+h} + O(1)$, so that the test is consistent with respect to the class of alternatives (4.17) for which

$$\sum_{\alpha=1}^g (m_\alpha - \bar{m})(m_{\alpha+h} - \bar{m}) \neq 0,$$

where $\bar{m} = g^{-1} \sum_{\alpha=1}^g m_\alpha$. Thus by the same argument as in the case of ranks, the test is consistent whenever $h = g$, while it may or may not be consistent if $h < g$.

**5. Limiting distribution of $R_h$ under $H_1$ in case of ranks.** For the remaining two sections, it is of importance to know conditions under which $R_h$ based on ranks is asymptotically normal under the alternative hypothesis. Using the methods of moments, it can be shown that in this case the distribution of

$(R_h - ER_h)/\sigma(R_h)$ tends to the normal distribution with mean 0 and variance 1 provided var $R_h = \Omega(n^5)$.

Generalizing the method used in Section 4 in evaluating the variance of $R_h$, it is not difficult to see that $E(R_h - ER_h)^{2s+1} = O(n^{5s+2})$, $(s = 0, 1, \cdots)$. It follows that if var $R_h = \Omega(n^5)$, the odd moments are asymptotically zero. By means of a more careful analysis, it is also possible to show that $E(R_h - ER_h)^{2s} \sim (2s - 1)(2s - 3) \cdots 3(\text{var } R_h)^s$. This proves our statement.

**6. Ranks versus original observations.** We have seen in Section 4 that if the alternative hypothesis is characterized by a regular cyclical movement the test based on $R_h$ is consistent both for original observations and for ranks, provided $h = g$, where $g$ is the length of a cycle. The question arises which test is more efficient, the one based on original observations or the one based on ranks.

In trying to answer this question, we shall make use of a procedure due to Pitman[4], which allows us to compare two consistent tests of the hypothesis that some population parameter $\theta$ has the value $\theta^0$ against the alternatives $\theta > \theta^0$ using critical regions of size $\alpha$, $S_{in} \geq S_{in}(\alpha)$, $(i = 1, 2)$, where $S_{in}$ is a statistic having finite variance and $S_{in}(\alpha)$ is an appropriate constant. The relative efficiency of the second test with respect to the first test is defined as the ratio $n_1/n_2$ where $n_2$ is the sample size of the second test required to achieve the same power for a given alternative as is achieved by the first test using a sample of size $n_1$ with respect to the same alternative.

Let $E(S_{in} \mid \theta) = \psi_{in}(\theta)$, $\text{var}(S_{in} \mid \theta) = \sigma_{in}^2(\theta)$, and $\psi'_{in}(\theta^0)/\sigma_{in}(\theta^0) = H_i(n)$. Assuming that the alternative is of the form $\theta_n = \theta^0 + k/\sqrt{n}$ where $k$ is a positive constant, Pitman has shown that the asymptotic relative efficiency of the second test with respect to the first test is given by $\lim_{n\to\infty} [H_2^2(n)/H_1^2(n)]$, provided there exists a number $\epsilon > 0$ such that for $\theta^0 \leq \theta \leq \theta^0 + \epsilon$

$$\psi'_{in}(\theta) \text{ exists;} \tag{6.1}$$

as $\theta_n \to \theta^0$ with $n \to \infty$

$$\frac{\psi'_{in}(\theta_n)}{\psi'_{in}(\theta^0)} \to 1 \tag{6.2}$$

and

$$\frac{\sigma_{in}(\theta_n)}{\sigma_{in}(\theta^0)} \to 1; \tag{6.3}$$

(6.4) $\lim_{n\to\infty} \frac{1}{\sqrt{n}} H_i(n) = c_i$, where $c_i$ is some positive constant;

(6.5) the distribution of $[S_{in} - \psi_{in}(\theta)]/\sigma_{in}(\theta)$ tends to the normal distribution with mean 0 and variance 1 uniformly in $\theta$.

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[4] I should like to thank Professor Pitman for his kind permission to quote from his lectures on non-parametric statistical inference which he delivered at Columbia University during the spring semester 1948.

Condition (6.5) can be replaced by the weaker condition

(6.5′) the distribution of $[S_{in} - \psi_{in}(\theta_n)]/\sigma_{in}(\theta_n)$ tends to the normal distribution with mean 0 and variance 1 as $n \to \infty$.

In our case, in order to insure consistency, it will be assumed that $h = g$. Consider the parameter

$$\theta = \frac{1}{h} \sum_{\alpha=1}^{h} (m_\alpha - \bar{m})^2, \tag{6.6}$$

where as before $m_\alpha$ is the expected value of the $(lh + \alpha)$th observation, $(l = 0, 1, \cdots)$. We want to find the asymptotic relative efficiency of the test performed on ranks with respect to the test performed on original observations as $\theta \to 0$ with $n \to \infty$.

Again it is no restriction to assume that

$$\bar{m} = \frac{1}{h} \sum_{\alpha=1}^{h} m_\alpha = 0. \tag{6.7}$$

Assume further that the chance variable $U$ defined in 4.3 has a finite absolute moment of order $4 + \delta$, $\delta > 0$. Then $R_h^0 \sim \sqrt{n} R_h / A_2$ with probability 1 and, if the null hypothesis is true, it follows from Theorem 2 that with the same probability the statistic

$$Q_h = \frac{\sqrt{n} \sum_{i=1}^{n} x_i x_{i+h}}{\sum_{i=1}^{n} x_i^2}$$

has in the population of permutations of the observed sample values an asymptotically normal distribution with mean 0 and variance 1. This, however, is also the limiting distribution of $Q_h$ under random sampling when the null hypothesis is true, as follows from the results of Hoeffding and Robbins [3]. Thus it will be sufficient to find the asymptotic relative efficiency of the $R_h$-test for ranks with respect to the $Q_h$-test. In doing this, it will also be assumed that $U$ has a continuous density function $f(x) = F'(x)$, and, in order to simplify notation, that there are $nh$ observations instead of $n$.

In finding $H_Q(nh)$, let $x_{\alpha,j} = x_{\alpha j} = x_{(j-1)h+\alpha}$ and $u_{\alpha,j} = u_{\alpha j} = u_{(j-1)h+\alpha}$, $(\alpha = 1, \cdots, h; j = 1, \cdots, n)$. Then

$$\begin{aligned} \frac{1}{nh} A_2 &= \frac{1}{nh} \sum_{\alpha=1}^{h} \sum_{j=1}^{n} x_{\alpha j}^2 = \frac{1}{nh} \sum_{\alpha=1}^{h} \sum_{j=1}^{n} (u_{\alpha j} + m_\alpha)^2 \\ &= \frac{1}{nh} \sum_{\alpha=1}^{h} \left\{ \sum_{j=1}^{n} u_{\alpha j}^2 + 2m_\alpha \sum_{j=1}^{n} u_{\alpha j} + nm_\alpha^2 \right\} \underset{pr1}{\to} \sigma^2 + \theta. \end{aligned}$$

Further,

$$R_h = \sum_{\alpha=1}^{h} \left\{ \sum_{j=1}^{n} u_{\alpha j} u_{\alpha, j+1} + 2m_\alpha \sum_{j=1}^{n} u_{\alpha j} + nm_\alpha^2 \right\}$$

so that

$$EQ_h = E \frac{\frac{1}{\sqrt{nh}} R_h}{\frac{1}{nh} A_2} \sim \frac{\sqrt{nh}\,\theta}{\sigma^2 + \theta} = \psi_{Qn}(\theta).$$

Therefore

$$\psi'_{Qn}(\theta) = \sqrt{nh}\, \frac{\sigma^2}{(\sigma^2 + \theta)^2}.$$

Also by (4.18)

$$\operatorname{var} Q_h \sim \frac{nh\sigma^4 + 4n\sigma^2 \sum_{\alpha=1}^{h} m_\alpha^2}{nh(\sigma^2 + \theta)^2} = \frac{\sigma^4 + 4\sigma^2\theta}{(\sigma^2 + \theta)^2}$$

which converges to 1 as $\theta \to 0$. It follows that

$$H_Q(nh) = \psi'_{Qn}(0) = \frac{\sqrt{nh}}{\sigma^2}. \tag{6.8}$$

Conditions (6.1)–(6.5) are easily seen to be satisfied.

Considering now the $R_h$-test for ranks, we know that $(nh)^{-5/2}R_h$ has finite variance. From (4.7) and (4.14)–(4.16) it is found that

$$E[(nh)^{-5/2} R_h \mid \theta] \sim \sqrt{nh}\,\eta = \sqrt{nh}\, \frac{1}{h^3} \sum_{\alpha=1}^{h} \left( \sum_{\beta=1}^{h} \epsilon_{\alpha\beta} \right)^2 = \psi_{Rn}(\theta) \tag{6.9}$$

and after some computations

$$\psi'_{Rn}(0) = \sqrt{nh} \left[ \int_{-\infty}^{\infty} f^2(x)\, dx \right]^2. \tag{6.10}$$

From (4.4) and (6.10)

$$H_R(nh) = 12\sqrt{nh} \left[ \int_{-\infty}^{\infty} f^2(x)\, dx \right]^2.$$

Conditions (6.1)–(6.4) and (6.5′) can be shown to be satisfied.

Thus the asymptotic relative efficiency of the test based on ranks with respect to the test based on original observations is

$$H_{RQ} = \frac{144nh \left[ \int_{-\infty}^{\infty} f^2(x)\, dx \right]^4}{nh/\sigma^4} = 144 \left[ \sigma \int_{-\infty}^{\infty} f^2(x)\, dx \right]^4. \tag{6.11}$$

As is not difficult to see, this expression is independent of location and scale.

Let the chance variable $U$ have density function

$$f(x) = \begin{cases} 0, & x < -1, \quad x > 1, \\ \dfrac{1 + x}{1 + a}, & -1 \le x \le a, \\ \dfrac{1 - x}{1 - a}, & a \le x \le 1, \end{cases} \qquad -1 \le a \le 1,$$

i.e., let the graph of $f(x)$ be given by the two straight lines connecting the points $(-1, 0)$ and $(1, 0)$ with the point $(a, 1)$. Then $EU = a/3$, $\operatorname{var} U = \frac{1}{18}(3 + a^2)$, $\int_{-\infty}^{\infty} f^2(x)\,dx = 2/3$, and (6.11) becomes $[8(3 + a^2)/27]^2$. Thus $H_{RQ}$ increases with $|a|$. For $a = 0$, it is equal to 64/81; for $|a| = 1$, it is equal to $(32/27)^2$. It is equal to 1, for $a = \sqrt{3/8}$.

This example shows that the asymptotic relative efficiency of the rank test with respect to the test based on original observations may be $<1$, $=1$, or $>1$, depending on the density function $f(x)$. Unless $f(x)$ is explicitly given, no statement can be made as to which of the two tests is to be preferred.

We are now in a position to give at least a partial answer to a question raised in [1]. In concluding their paper, Wald and Wolfowitz note that the problem dealt with in this section can be posed not only when transforming to ranks, but also for any transformation carried out by means of a continuous and strictly monotonic function $h(x)$.

Let $t = h(x)$ be such a transformation, satisfying in addition the condition that Pitman's procedure remains applicable for the transformed distribution. Corresponding to $\sigma^2$ and $Q$ we shall use $\sigma_t^2$ and $Q_t$. Let $h(m_\alpha) = \mu_\alpha$, $h^{-1}\sum_{\alpha=1}^{h}(\mu_\alpha - \bar{\mu})^2 = \vartheta$. Then if $EQ_t \sim \psi_{Q_t n}(\theta)$, by (6.8), (6.9), and (6.10)

$$\begin{aligned} \left.\frac{d\psi_{Q_t n}(\theta)}{d\theta}\right|_{\theta=0} &= \left.\frac{d\psi_{Q_t n}}{d\vartheta}\frac{d\vartheta}{d\eta}\frac{d\eta}{d\theta}\right|_{\theta=0} \\ &= \frac{\sqrt{nh}}{\sigma_t^2}\,\frac{1}{\left\{\int_{-\infty}^{\infty} f^2[g(t)]g'^2(t)\,dt\right\}^2}\left[\int_{-\infty}^{\infty} f^2(x)\,dx\right]^2 = H_{Q_t}(nh), \end{aligned} \tag{6.12}$$

where $g(t)$ is the inverse of $h(x)$. Therefore by (6.8) and (6.12)

$$H_{Q_tQ} = \frac{\left\{\sigma\int_{-\infty}^{\infty} f^2(x)\,dx\right\}^4}{\left\{\sigma_t\int_{-\infty}^{\infty} f^2[g(t)]g'^2(t)\,dt\right\}^4},$$

and the asymptotic relative efficiency does not merely depend on $h(x)$, the operator defining the transformation, but also very essentially on the underlying distribution $f(x)$.

**7. Comparison of the $R_h$- and $T$-tests.** The $T$-test by Mann [6] designed to test for randomness against a downward trend is based on the statistic

$$T = \sum_{i=1}^{n}\sum_{j>i}(y_{ij} + \tfrac{1}{2}) = \sum_i\sum_{j>i} y_{ij} + \tfrac{1}{4}n(n-1),$$

where $y_{ij}$ is defined by (4.5). Making the same assumptions as in 4.1, Mann shows that under the null hypothesis $T$ has a limiting normal distribution with mean $\frac{1}{4}n(n-1)$ and variance $\frac{1}{72}(2n^3+3n^2-5n)$, while under the alternative hypothesis

$$ET = \tfrac{1}{4}n(n-1)(2\zeta_n+1), \tag{7.1}$$

where $\zeta_n$ is defined by $\frac{1}{2}n(n-1)\zeta_n = \sum_i \sum_{j>i} \epsilon_{ij} < 0$.

Let

$$S_n = \frac{6}{n^{3/2}}\left[T - \tfrac{1}{4}n(n-1)\right].$$

When $H_0$ is true, $S_n$ is asymptotically normal with mean 0 and variance 1. If we then put $\phi(\lambda) = \frac{1}{\sqrt{2\pi}}\int_\lambda^\infty e^{-\frac{1}{2}x^2}\,dx$, a critical region for testing $H_0$ is given by $S_n \leq -\lambda$, where $\lambda$ is determined in such a way that $\phi(\lambda) = \alpha$, the level of significance.

When $H_1$ is true, we find from (7.1)

$$E(S_n \mid \zeta_n) \sim 3\sqrt{n}\,\zeta_n.$$

By paralleling the proof of asymptotic normality of $R_h$ under $H_1$ given in Section 5, it can be shown that $(S_n - ES_n)/\sigma(S_n)$ is asymptotically normal with mean 0 and variance 1 provided $\sigma(S_n) = \Omega(1)$. This is essentially the result obtained already by Hoeffding [7]. Thus the asymptotic power of the test based on $S_n$ is given by

$$P\{S_n \leq -\lambda\} \sim \phi\left(\frac{\lambda + 3\sqrt{n}\,\zeta_n}{\sigma(S_n)}\right) \tag{7.2}$$

converging to 1, provided $\lim_{n\to\infty} \sqrt{n}\,\zeta_n = -\infty$. This is the condition for consistency given by Mann.

We may ask for the asymptotic power of the $S_n$-test as $\zeta_n \to 0$ with $n \to \infty$. More exactly, instead of considering a certain alternative $\epsilon_{ij} = k_{ij}$, where the $k_{ij}$ are given constants, consider the alternative (changing with $n$)

$$\epsilon_{ij} = \frac{k_{ij}}{\sqrt{n}}. \tag{7.3}$$

If then as $n \to \infty$

$$\frac{2}{n(n-1)} \sum_i \sum_{j>i} k_{ij} \to k$$

and

$$\sigma(S_n) \to 1,$$

it follows from (7.2) that the asymptotic power of the $S_n$-test, and therefore of the $T$-test, for alternatives (7.3) is equal to

$$\phi(\lambda + 3k).$$

Now consider the same situation when the statistic $R_h$ is used instead of $T$. We know that when $H_0$ is true

$$R'_n = \frac{12}{n^{5/2}} R_h,$$

where $R_h$ is given by (4.6), is asymptotically normal with mean 0 and variance 1. Thus in this case the critical region is given by $R'_n \geq \lambda$. If we set $\xi_n = \frac{1}{n^3}\sum_{ijk} \epsilon_{ij}\epsilon_{i+h,k}$, we find

$$E(R'_n \mid \xi_n) \sim 12\sqrt{n}\xi_n,$$

and asymptotically the power of the $R'_n$-test is

$$P\{R'_n \geq \lambda\} \sim \phi\left(\frac{\lambda - 12\sqrt{n}\,\xi_n}{\sigma(R'_n)}\right), \tag{7.4}$$

provided $\sigma(R'_n) = \Omega(1)$. Thus the test is consistent if $\lim_{n\to\infty} \sqrt{n}\xi_n = \infty$. However, for the alternative (7.3), (7.4) tends to $\phi(\lambda) = \alpha$, provided that as $n \to \infty$

$$\sigma(R'_n) \to 1.$$

Thus the $R_h$-test is ineffective with respect to the alternative (7.3) in contrast to the $T$-test. This means that for this alternative the asymptotic relative efficiency of the $R_h$-test with respect to the $T$-test is 0.

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