THE THEORY OF PROBABILITY DISTRIBUTIONS OF POINTS ON A LATTICE[1]

By P. V. Krishna Iyer

University of Oxford

1. Introduction and summary. This paper discusses the theory of certain probability distributions arising from points arranged in the form of lattices in two, three and higher dimensions. The points are of k characters which for convenience are described as colors. A two-dimensional lattice will consist of $m \times n$ points in m columns and n rows. In a three-dimensional lattice there will be $l \times m \times n$ points in the form of a rectangular parallelopiped. Two situations arise for consideration. They are, to use the term of Mahalanobis, *free* and *non-free* sampling. In free sampling the color of each point is determined, on null hypothesis, independently of the color of the other points. The probabilities of the points belonging to the different colors, say black, white, etc. are $p_1, p_2 \cdots p_k$, such that $\sum_1^k p_r = 1$. In non-free sampling the number of points of each color is specified in advance, say $n_1, n_2 \cdots n_k$ so that $\sum_1^k n_r = mn$ or lmn according as the lattice is two- or three-dimensional. Only the arrangements of these points in the lattice are varied.

The distributions considered in this paper are the following:—

(i) the number of joins between adjacent points of the same color, say black-black joins,

(ii) the number of joins between adjacent points of two specified colors, say black-white joins, and

(iii) the total number of joins between points of different colors, along mutually perpendicular axes.

The methods used here are the same as those developed by the author [3] for the linear case. All the distributions tend to the normal form when l, m and n tend to infinity, provided the p's are not very small.

Before considering the various distributions, we shall have a brief review of the work done on this topic by other people. For free sampling, Moran [5] and [6] has discussed the distribution of black-white and black-black joins for an $m \times n$ lattice of points of two colors. For a three-dimensional lattice, he has given the first and the second moments for the distribution of black-white joins. Levene [4] has announced some results closely allied to those of Moran for a square of side N (with N^2 cells) each cell taking the characteristic A or B with probabilities p and $q = 1 - p$ respectively. Bose [2] has found the expectation of

$x =$ the number of black patches $-$ the number of embedded white patches,

[1] Part of a thesis approved for the degree of Doctor of Philosophy, Oxford University.

for a square divided into n^2 small cells, having p and $q = 1 - p$ as the probability of the cells being black or white. An embedded white patch is one that lies completely inside a black patch.

The above review shows that the work done so far is confined entirely to the free sampling distributions, the points taking only two characters. As mentioned in the beginning of this article, we shall deal here with the free and non-free sampling distributions for points possessing k characters or colors.

2. Two dimensional lattice. Let an $m \times n$ rectangular lattice consist of mn points of k colors with probabilities p_1, p_2, $\cdots$ p_k, such that $\Sigma p_r = 1$. (When there are only two colors, p_1 and p_2 are taken as p and q respectively.) All the problems dealt with for the linear lattice (Krishna Iyer, [3]) can be investigated here also. But the most important of them is the distribution for the total number of joins between points of different colors. This takes into consideration the relative position of points of all colors in the lattice. Distributions for the number of black-black or black-white joins are not based on the arrangement of all the points in the lattice and therefore cannot be considered to be adequate for testing the random distribution of the points in the lattice. Therefore the distribution of the total number of joins between points of different colors has been dealt with in some detail. As the actual distributions are very complicated they are discussed by means of cumulants. The first and the second moments for the other distributions have also been given.

2.1. *First and second moments for the distribution of black-black joins for two or more colors.* The first and the second moments for free sampling have been obtained by Moran [5] and [6]. In order to give an idea of the methods used in this paper for obtaining the moments and also to facilitate the derivation of the corresponding moments for non-free sampling, they have been obtained again for both black-black and black-white joins.

(a) *Free Sampling.* In the course of similar investigations on the distribution of black-black joins arising from points on a line, the author [3] has found that the rth factorial moment is $r!$ times the sum of expectations of the different ways of obtaining r joins. This finding is true for the rectangular lattice also. This may be established as follows.

Define variates $u_{ij'}$ ($i = 1, 2, \cdots, n$; $j' = 1, 2, \cdots, m - 1$) to be one if the (i, j) and $(i, j + 1)$ positions are black and zero otherwise; then $E(u_{ij'}) = p^2$, and the higher factorial moments are zero. Similarly, define $v_{i'j}$ ($i' = 1, 2, \cdots, n - 1$; $j = 1, 2, \cdots, m$) to be one when the (i, j) and $(i + 1, j)$ positions are black and zero otherwise; then $E(v_{i'j}) = p^2$, and the higher factorial moments are zero. Further, $u_{ij'}$ is independently distributed of all u's and v's except $u_{i,j'-1}$, $u_{i,j'+1}$, $v_{i-1,j'}$, $v_{i+1,j'}$, $v_{i-1,j'+1}$, $v_{i+1,j'+1}$, and $v_{i'j}$ is independently distributed of all u's and v's excepting two vertically adjacent v's and four horizontally adjacent u's. If

$$s = \Sigma\, u_{ij'} + \Sigma v_{i'j}\,,$$

then

$$E(s) = \sum_{i,j'} p^2 + \sum_{i',j} p^2$$
$$= (2mn - m - n)\, p^2$$

and $E(s^{(2)}) = 2E$ (the number of ways of selecting any two of the ones included in $\Sigma u_{ij'} + \Sigma v_{i'j}$)

$$= 2\, E\, (uu + uv + vv)$$

involves only the cross products since $E(u^{(2)}) = 0 = E(v^{(2)})$. For products of dependent pairs the expectation is p^3, while for independent pairs it is p^4. Hence one merely needs to count the number of dependent and independent products. Similarly for the third factorial moment one needs consider only products of three first powers of the variates (with expectation p^6), those with two dependent and one independent variates (with expectation p^5), and those with three dependent variates (with expectation p^4).

Thus the second factorial moment can be obtained by counting the number of ways of obtaining two black-black joins from (i) three adjacent points and (ii) two pairs of adjacent points. They are explained below diagrammatically for a 5 x 4 lattice.

$$(1)\quad \begin{array}{ccccccccc}
\cdot & & \cdot & & \cdot & & \cdot & & \cdot \\
\cdot & & X & — & X & — & X & & \cdot \\
\cdot & & \cdot & & \cdot & & \cdot & & \cdot \\
\cdot & & \cdot & & \cdot & & \cdot & & \cdot
\end{array}$$

$$(2)\quad \begin{array}{ccccccccc}
\cdot & & \cdot & & \cdot & & \cdot & & \cdot \\
\cdot & & X & — & X & & \cdot & & \cdot \\
 & & & & | & & & & \\
\cdot & & \cdot & & X & & \cdot & & \cdot \\
\cdot & & \cdot & & \cdot & & \cdot & & \cdot
\end{array}$$

$$(3)\quad \begin{array}{ccccccccc}
\cdot & & \cdot & & \cdot & & \cdot & & \cdot \\
\cdot & & X & — & X & & \cdot & & \cdot \\
\cdot & & \cdot & & \cdot & & X & & \cdot \\
 & & & & & & | & & \\
\cdot & & \cdot & & \cdot & & X & & \cdot
\end{array}
\quad \text{or} \quad
\begin{array}{ccccccccc}
\cdot & & \cdot & & \cdot & & \cdot & & \cdot \\
\cdot & & X & — & X & & \cdot & & \cdot \\
\cdot & & \cdot & & \cdot & & X & — & X \\
 & & & & & & & & \\
\cdot & & \cdot & & \cdot & & \cdot & & \cdot
\end{array}$$

'X' denotes a black point.

'$\cdot$' denotes any point other than black. The expectations for items (1), (2) and (3) indicated above are

$$\begin{aligned}
&[(m-2)n + (n-2)m]p^3, \\
(2.1.1)\quad &4(m-1)\,(n-1)p^3, \\
&\tfrac{1}{2}\,[4m^2n^2 - 4mn(m+n) + m^2 + n^2 - 12mn + 13(m+n) - 8]\,p^4,
\end{aligned}$$

respectively. Thus

(2.1.2) $$\mu'_{[2]} = 2[6mn - 6(m + n) + 4]p^3 + [4m^2n^2 - 4mn(m + n) + m^2n^2 - 12mn + 13(m + n) - 8]p^4.$$

It can now be seen that

(2.1.3) $$\mu'_1 = (2mn - m - n)p^2,$$

(2.1.4) $$^2\mu_2 = (2mn - m - n)p^2 + 2(6mn - 6m - 6n + 4)p^3 - (14mn - 13m - 13n + 8)p^4.$$

Putting $m + n = a$, and $mn = b$, the above expressions reduce to

(2.1.5) $$\mu'_1 = (2b - a)p^2,$$

(2.1.6) $$\mu_2 = (2b - a)p^2 + 2(6b - 6a + 4)p^3 - (14b - 13a + 8)p^4.$$

These substitutions have been continued throughout this Section.

(b) *Non-free sampling.* The chances of obtaining r black points in free and non-free sampling are p^r and $n_1^{(r)}/b^{(r)}$ respectively. Therefore it is obvious that the rth factorial moment about zero for non-free sampling distribution of black-black joins can be reduced by substituting $n_1^{(r)}/b^{(r)}$ for p^r in $\mu'_{[r]}$ for free sampling. This substitution gives

(2.1.7) $$\mu'_{1(n_1,n_2)} = \frac{(2b - a)\, n_1^{(2)}}{b^{(2)}},$$

(2.1.8) $$\mu_{2(n_1,n_2)} = \frac{(2b - a)n_1^{(2)}}{b^{(2)}} + \frac{2(6b - 6a + 4)n_1^{(3)}}{b^{(3)}} - \frac{\{(14b - 13a + 8) - (2b - a)^2\}n_1^{(4)}}{b^{(4)}} - \left\{\frac{(2b - a)n_1^{(2)}}{b^{(2)}}\right\}^2,$$

where $\mu_{r(n_1,n_2)}$ represents the rth moment with n_1 black and n_2 white points on the lattice.

2.2. *Cumulants for the distribution of black-white joins for two colors.* For m points on a line, the author [3] has shown that the first four cumulants of the free and non-free sampling distribution of black-white joins can be obtained from the non-free distributions for $(1, m - 1)$, $(2, m - 2)$, $(3, m - 3)$ and $(4, m - 4)$ black and white points distributed at random. This method is applicable for two and three dimensional lattices also. This can be established from the following considerations.

(i) The rth moment about zero for the free sampling distribution is

$$\sum_0^b p^s q^{b-s} \sum x^r f_x,$$

[2] This result differs slightly from that given by Moran. The correct result is the one given here.

where $b = mn$ and $\Sigma x^r f_x$ is the rth moment for the non-free distribution with s black and $(b - s)$ white points.

(ii) $\Sigma x^r f_x$ is the same for the two distributions arising from (1) s black and $(b - s)$ white points and (2) $(b - s)$ black and s white points.

(iii) The rth moment is a polynomial in pq of degree r. This can be seen from the fact that the factorial moment is the sum of the expectations of the different ways of obtaining r black-white joins. The probability of r independent black-white joins is $(2pq)^r$ and this is the highest power of pq.

In view of the above conditions, (i) reduces to

$$\begin{aligned} A'_{1r}pq(p+q)^{(b-2)} + A'_{2r}p^2q^2(p+q)^{b-4} \cdots + A'_{rr}p^rq^r(p+q)^{b-2r} \\ = A'_{1r}pq + A'_{2r}p^2q^2 \cdots + A'_{rr}p^rq^r, \end{aligned} \tag{2.2.1}$$

where A'_{1r}, A'_{2r} etc. are determined from the following relations:—

$$\left\{ \begin{aligned} S_{r(1,b-1)} &= A'_{1r}, \\ S_{r(2,b-2)} &= A'_{2r} + \binom{b-2}{1} A'_{1r}, \\ S_{r(3,b-3)} &= A'_{3r} + \binom{b-4}{1} A'_{2r} + \binom{b-4}{2} A'_{1r}, \\ S_{r(4,b-4)} &= A'_{4r} + \binom{b-6}{1} A'_{3r} + \binom{b-6}{2} A'_{2r} + \binom{b-6}{3} A'_{1r}, \end{aligned} \right. \tag{2.2.2}$$

where $S_{r(t,b-t)}$ is the rth moment about zero for the non-free distribution with t black and $(b - t)$ white points. This is obvious by comparing the coefficients of p^tq^{b-t} in (i) with (2.2.1).

Therefore the first four cumulants can be calculated by finding the frequency distributions of black-white joins for $(1, b - 1)$, $(2, b - 2)$, $(3, b - 3)$ and $(4, b - 4)$ black and white points. These distributions were determined by a systematic examination of the number of black-white joins in all the possible arrangements for the given number of black and white points. The moments of these distributions enable us to determine the A's.

The equations in (2.2.2) give

$$\begin{aligned} A'_{11} &= 2(2b - a), \\ A'_{12} &= 2(8b - 7a + 4), \\ A'_{13} &= 2(32b - 37a + 36), \\ A'_{14} &= 2(128b - 175a + 220), \\ A'_{22} &= 4(a^2 - 4ab + 4b^2 + 13a - 14b - 8), \\ A'_{23} &= 4(21a^2 - 66ab + 48b^2 + 210a - 156b - 228), \\ A'_{33} &= 8(-a^3 + 6a^2b - 12ab^2 + 8b^3 - 39a^2 + 120ab - 84b^2 - 272a + 184b \\ &\quad + 312), \\ A'_{24} &= 4(295a^2 - 760ab + 448b^2 + 2305a - 1304b - 3428), \end{aligned}$$

$$A'_{34} = 8(-42a^3 + 216a^2b - 360ab^2 + 192b^3 - 1410a^2 + 3612ab - 2016b^2 - 7884a + 3648b + 12720),$$
$$A'_{44} = 16(a^4 - 8a^3b + 24a^2b^2 - 32ab^3 + 16b^4 + 78a^3 - 396a^2b + 648ab^2 - 336b^3 + 1643a^2 - 4196ab + 2252b^2 + 7926a - 3084b - 13464),$$

where $a = m + n$, and $b = mn$.

The above values of A''s give the first four moments for free sampling about zero. The cumulants reduce to the following expressions:

$$(2.2.3) \quad \kappa_1 = 2(2b - a)pq,$$

$$(2.2.4) \quad \kappa_2 = 2(8b - 7a + 4)pq - 4(14b - 13a + 8)p^2q^2,$$

$$(2.2.5) \quad \kappa_3 = 2(32b - 37a + 36)pq - 8(90b - 111a + 114)p^2q^2 + 64(29b - 37a + 39)p^3q^3,$$

$$(2.2.6) \quad \kappa_4 = 2(128b - 175a + 220)pq - 4(1784b - 2617a + 3476)p^2q^2 + 32(1548b - 2361a + 3228)p^3q^3 - 32(3126b - 4899a + 6828)p^4q^4.$$

As indicated for black-black joins, the first and the second moments for non-free sampling can be calculated by substituting

$$p^rq^s = n_1^{(r)}n_2^{(s)}/b^{(r+s)}$$

in the uncorrected moments about the origin for free sampling. This is true for all the distributions considered in this paper.

Before proceeding to discuss the limiting form of the distribution, it may be noted that the first four cumulants for the free-sampling distribution of black-white joins are linear expressions in a and b. This result is similar to what has been established for the linear lattice (Krishna Iyer, [3]). When the points lie on a line, all the cumulants of the distribution of the number of joins (black-black or black-white) are linear in m (the number of points on the line). This suggests that the higher order cumulants for the distribution of joins in a rectangular lattice also will be linear in a and b, i.e. the rth cumulant will be of the form

$$\sum_{s=1}^{r} (L_{rs}b + M_{rs}a + N_{rs})p^sq^s,$$

where L, M and N are independent of a and b. It has not been possible to obtain a formal proof for this statement.

The limiting form of the distribution of the number of black-white joins is now examined on the basis of the cumulants given above. Since κ_2, κ_3 and κ_4 are linear in a and b, γ_1 and γ_2 tend to the limit zero as m and n tend to infinity. That the higher order γ's also tend to the limit zero can be seen from the fact that all the cumulants will be linear functions in a and b. Hence the distribution of

$$y = \frac{x - 2(2b - a)pq}{\sqrt{2(8b - 7a + 4)pq - 4(14b - 13a + 8)p^2q^2}}$$

tends to the normal form as m and n tend to infinity, where x is the observed number of black-white joins in a given arrangement of the points.

When $p = q = \frac{1}{2}$, the first, second and third cumulants are equal to those obtained for a binomial distribution whose 'n' is $(2b - a)$.

As in the case of linear lattices, the distribution of the number of black-white joins in an $m \times n$ rectangular lattice for non-free sampling also will tend to the normal form as m and n tend to infinity.

TABLE 1

Distribution of the number of black-white joins for 2×3 lattice

No. of B-W joins	No. of black points 0	1	2	3	4	5	6	Total
0	1	—	—	—	—	—	1	2
1	—	—	—	—	—	—	—	—
2	—	4	2	—	2	4	—	12
3	—	2	4	6	4	2	—	18
4	—	—	5	8	5	—	—	18
5	—	—	4	4	4	—	—	12
6	—	—	—	—	—	—	—	—
7	—	—	—	2	—	—	—	2
								64

$$\kappa_1 = 7/2, \quad \kappa_2 = 7/4, \quad \kappa_3 = 0, \quad \kappa_4 = \frac{17}{8}.$$

In order to have an idea of the nature of the distribution of the number of black-white joins when $p = q$ or otherwise, the complete distributions for the lattices 2×3, 2×4, 3×3 and 3×4 are given in Tables 1, 2, 3, and 4.

The distributions tabulated in Tables 1, 2, 3 and 4 show that the probability of getting 1 and $(2b - a - 1)$ black-white joins is zero, while for 0 and $(2b - a)$ joins it is not so. But this abnormality will not affect the limiting form of the distribution when m and n tend to infinity because the probability for 0 and $(2b - a)$ black-white joins also tends to zero.

2.3. *First and second moments for the distribution of black-white joins for k colors. Free sampling.* Taking p_1 and p_2 as the probabilities that a point in the lattice is black or white, the expected number of black-white joins is

$$2(2b - a)\, p_1 p_2 . \tag{2.3.1}$$

TABLE 2

Distribution of the number of black-white joins for 2 × 4 lattice

No. of B-W joins	No. of black points									Total
	0	1	2	3	4	5	6	7	8	
0	1	—	—	—	—	—	—	—	1	2
1	—	—	—	—	—	—	—	—	—	—
2	—	4	2	—	2	—	2	4	—	14
3	—	4	4	4	—	4	4	4	—	24
4	—	—	8	12	8	12	8	—	—	48
5	—	—	12	16	24	16	12	—	—	80
6	—	—	2	12	20	12	2	—	—	48
7	—	—	—	8	8	8	—	—	—	24
8	—	—	—	4	6	4	—	—	—	14
9	—	—	—	—	—	—	—	—	—	—
10	—	—	—	—	2	—	—	—	—	2
										256

$\kappa_1 = 5, \quad \kappa_2 = 5/2, \quad \kappa_3 = 0, \quad \kappa_4 = \frac{13}{4}.$

TABLE 3

Distribution of the number of black-white joins for 3 × 3 lattice

No. of B-W joins	No. of black points										Total
	0	1	2	3	4	5	6	7	8	9	
0	1	—	—	—	—	—	—	—	—	1	2
1	—	—	—	—	—	—	—	—	—	—	—
2	—	4	—	—	—	—	—	—	4	—	8
3	—	4	8	4	—	—	4	8	4	—	32
4	—	1	6	4	12	12	4	6	1	—	46
5	—	—	12	24	12	12	24	12	—	—	96
6	—	—	10	26	36	36	26	10	—	—	144
7	—	—	—	12	36	36	12	—	—	—	96
8	—	—	—	10	13	13	10	—	—	—	46
9	—	—	—	4	12	12	4	—	—	—	32
10	—	—	—	—	4	4	—	—	—	-	8
11	—	—	—	—	—	—	—	—	—	—	—
12	—	—	—	—	1	1	—	—	—	—	2
											512

$\kappa_1 = 6, \quad \kappa_2 = 3, \quad \kappa_3 = 0, \quad \kappa_4 = 4.5.$

TABLE 4

Distribution of the number of black-white joins for 4 × 3 lattice

No. of B-W joins	No. of black points 0	1	2	3	4	5	6	7	8	9	10	11	12	Total
0	1	—	—	—	—	—	—	—	—	—	—	—	1	2
1	—	—	—	—	—	—	—	—	—	—	—	—	—	—
2	—	4	—	—	—	—	—	—	—	—	—	4	—	8
3	—	6	8	2	—	—	2	—	—	2	8	6	—	34
4	—	2	8	8	10	4	—	4	10	8	8	2	—	64
5	—	—	22	28	10	18	16	18	10	28	22	—	—	172
6	—	—	22	46	56	42	30	42	56	46	22	—	—	362
7	—	—	6	52	88	88	120	88	88	52	6	—	—	588
8	—	—	—	50	119	162	156	162	119	50	—	—	—	818
9	—	—	—	28	104	184	186	184	104	28	—	—	—	818
10	—	—	—	6	58	134	192	134	58	6	—	—	—	588
11	—	—	—	—	32	88	122	88	32	—	—	—	—	362
12	—	—	—	—	16	46	48	46	16	—	—	—	—	172
13	—	—	—	—	2	14	32	14	2	—	—	—	—	64
14	—	—	—	—	—	8	18	8	—	—	—	—	—	34
15	—	—	—	—	—	4	—	4	—	—	—	—	—	8
16	—	—	—	—	—	—	—	—	—	—	—	—	—	—
17	—	—	—	—	—	—	2	—	—	—	—	—	—	2
														4096

$\kappa_1 = 8.5, \quad \kappa_2 = 4.25, \quad \kappa_3 = 0, \quad \kappa_4 = 6.875.$

TABLE 5

Frequency distribution of the total number of joins between points of different colors for 1 black, 1 white and (mn − 2) red points

No. of joins	Frequency
4	28
5	$4(5a - 26)$
6	$2(2a^2 - 25a + 4b + 56)$
7	$2(-4a^2 + 2ab + 17a - 6b - 12)$
8	$4a^2 - 4ab + b^2 - 4a + 3b - 12)$

As in the case of black-black joins, the second factorial moment about zero is twice the sum of the expectations of the different ways of forming two black-white joins and can be determined by the method described in section 2.1.

$$(2.3.2)\quad \begin{aligned} \mu'_{[2]} &= 2(6b - 6a + 4)p_1p_2(p_1 + p_2) \\ &\qquad + 4(a^2 - 4ab + 4b^2 + 13a - 14b - 8)p_1^2p_2^2 . \end{aligned}$$

From this, μ_2 works out to be

$$(2.3.3)\quad \begin{aligned} \mu_2 &= 2(2b - a)p_1p_2 + 2(6b - 6a + 4)p_1p_2(p_1 + p_2) \\ &\qquad - 4(14b - 13a + 8)p_1^2p_2^2 . \end{aligned}$$

2.4. *First and second moments for the distribution of the total number of joins between points of different colors for three colors.* The expectation for free sampling is

$$(2.4.1)\quad \mu'_1 = 2(2b - a)\Sigma p_rp_s .$$

The coefficients of pq and p^2q^2 in the second moment are the same as those for two colors. The coefficient of $p_1p_2p_3$ can be obtained from the frequency distribution of the total number of joins between points of different colors when there are 1 black, 1 white and $(mn - 2)$ red points in the lattice. See Table 5.

Defining $S_{2(1,1,b-2)} = \Sigma x^2f_x$ for the above distribution,

$$S_{2(1,1,b-2)} = 2(4a^2 - 30ab + 32b^2 + 55a - 54b - 32).$$

As in the case of two colors, the second moment about zero for three colors reduces to the form

$$\begin{aligned} &\mathrm{A}_{21}(p_1 + p_2 + p_3)^{b-2}\ \Sigma p_rp_s + A_{112}(p_1 + p_2 + p_3)^{b-3}\ p_1p_2p_3 + \\ &\qquad A_{22}(p + p_1 + p_3)^{b-4}\ \Sigma p_r^2p_s^2 = A_{21}\Sigma p_rp_s + A_{112}p_1p_2p_3 + A_{22}\ \Sigma p_r^2p_s^2 , \end{aligned}$$

since $p_1 + p_2 + p_3 = 1$.

The coefficient of $p_1^{b-2}p_2p_3$ on the left hand side of the above equation is equal to $S_{2(1,1,b-2)}$, i.e. $S_{2(1,1,b-2)}$ = sum of coefficients of $p_1^{b-2}p_2p_3$ in $A_{21}(p_1 + p_2 + p_3)^{b-2}\ \Sigma p_rp_s$ and $A_{112}(p_1 + p_2 + p_3)^{b-3}p_1p_2p_3$. Therefore the coefficient of $p_1p_2p_3$ in μ_2 is $S_{2(1,1,b-2)}$ − coefficient of $p_1^{mn-2}p_2p_3$ in $2(8b - 7a + 4)(p_1 + p_2 + p_3)^{b-2}\Sigma p_rp_s$ − coefficient of $p_1p_2p_3$ in

$$\begin{aligned} 4(2b - a)^2\ (\Sigma p_rp_s)^2 &= S_{2(1,1,b-2)} - 2(8b - 7a + 4)\ (2b - 3) - 8(2b - a)^2 \\ &= 4(17a - 19b - 10). \end{aligned}$$

It can now be seen that

$$(2.4.2)\quad \begin{aligned} \mu_2 &= 2(8b - 7a + 4)\Sigma p_rp_s - 4(14b - 13a + 8)\Sigma p_r^2p_s^2 \\ &\qquad - 4(19b - 17a + 10)p_1p_2p_3 . \end{aligned}$$

2.5. *First and second moments for the distribution of the total number of joins between points of different colors for k colors.* As in the previous cases, the expectation for free sampling is

$$(2.5.1)\quad 2(2b - a)\Sigma p_rp_s .$$

The coefficients of $\Sigma p_r p_s$, $\Sigma p_r p_s p_t$ and $\Sigma p_r^2 p_s^2$ in the second moment are the same as those for three colors. The coefficient of $\Sigma p_r p_s p_t p_u$ is determined by finding the distribution of joins between points of different colors when there are 1 black, 1 white, 1 red and $mn - 3$ green points in the lattice. See Table 6.

$$S_{2(1,1,1,mn-3)}$$
$$= 2(12a^2b - 69ab^2 + 72b^3 - 36a^2 + 330ab - 342b^2 - 408a + 348b + 240).$$

The coefficient of $\Sigma p_r p_s p_t p_u$ in μ_2 can be obtained on the same lines as explained for three colors and is equal to $S_{2(1,1,1,mn-3)}$ $-$ coefficient of $p_r^{(mn-3)} p_s p_t p_u$ in the homogeneous expression of degree mn in μ_2' for three colors $+ 8(2b - a)^2$

$$= 8(14b - 13a + 8).$$

TABLE 6

Frequency distribution of the total number of joins between points of different colors when there are 1 black, 1 white, 1 red and (mn − 3) green points

No. of joins	Frequency
6	240
7	$12(19a - 112)$
8	$12(6a^2 - 78a + 7b + 208)$
9	$4(2a^3 - 57a^2 + 15ab + 310a - 66b - 444)$
10	$6(-4a^3 + 2a^2b + 36a^2 - 21ab + 2b^2 - 86a + 36b + 72)$
11	$6(4a^3 - 4a^2b + ab^2 - 6a^2 + 8ab - 2b^2 - 10a - 40)$
12	$(-8a^3 + 12a^2b - 6ab^2 + b^3 - 24a^2 + 18ab - 3b^2 + 44a - 34b + 192)$

It follows now that

$$\mu_2 = 2(8b - 7a + 4)\Sigma p_r p_s - 4(19b - 17a + 10)\Sigma p_r p_s p_t - 4(14b - 13a + 8)\Sigma p_r^2 p_s^2 + 8(14b - 13a + 8)\Sigma p_r p_s p_t p_u \,. \tag{2.5.2}$$

In general the cumulants[3] for free sampling involve b and a in the first degree only, and therefore, when m and n are large, the distribution tends to the normal form. If x is the observed total number of joins between points of different colors, the distribution of

$$\frac{x - 2(2b - a)\Sigma p_r p_s}{\sqrt{b}}$$

[3] The author has recently obtained the third and fourth cumulants for this distribution. They are linear functions of the dimensions of the lattice. The results will be published in an early issue of the *Ind. J. Agric. Stat.*

tends to the normal form with

$$16\Sigma p_r p_s - 76\Sigma p_r p_s p_t - 56\Sigma p_r^2 p_s^2 + 112\Sigma p_r p_s p_t p_u ,$$

as its variance for large values of m and n.

For non-free sampling also, the distribution of

$$\frac{x - 2(2mn - m - n)\Sigma e_r e_s}{\sqrt{mn}},$$

where $e_r = n_r/mn$, approaches the normal form having

$$4\Sigma e_r e_s e_t + 8\Sigma e_r^2 e_s^2 - 16\Sigma e_r e_s e_t e_u$$

as its variance. The error of this variance will be about 5% or less when m and n are greater than 35.

3. Three- and higher-dimensional lattices. This section deals with the first and the second moments for the distribution of black-black, black-white and the total number of joins between points of different colors for three- and higher-dimensional lattices. Besides these, the third and the fourth cumulants for the distribution of black-white joins in a three-dimensional lattice with points of two colors are also given.

3.1. *First and second moments for the distribution of black-black joins. Free sampling.* Let $E_3(1)$ be the expectation of the number of black-black joins for a lattice of sides l, m and n. Further let A_2 and A_3 be the number of ways of obtaining a black-black join in $m \times n$ and $l \times m \times n$ lattices. Then

$$\mathrm{E}_3(1) = A_3 p_1^2 ,$$

$$A_3 = A_2 l + mn(l - 1),$$

and

$$A_2 = (2mn - m - n).$$

Therefore

(3.1.1) $$E_3(1) = (3lmn - lm - mn - nl)\, p_1^2 .$$

For the sake of convenience all the results for the three-dimensional lattice are expressed after making the following substitutions:

$$c = l + m + n,$$

$$d = lm + mn + nl,$$

$$e = lmn.$$

$\mathrm{E}_3(1)$ in terms of c, d and e is

$$(3e - d)p_1^2 .$$

The expectation of the number of black-black joins for a lattice of r dimensions $(l_1 \times l_2 \times \cdots l_r)$ is given by

$$E_r(1) = (rl_1l_2 \cdots l_r - \Sigma l_1l_2 \cdots l_{r-1})\, p_1^2, \tag{3.1.2}$$

where $\Sigma l_1l_2 \cdots l_{(r-1)}$ is the sum of the product of the sides taken $(r - 1)$ at a time.

It has been pointed out before that the second factorial moment is twice the sum of the expectations of the different ways of forming two black-black joins. Using this fact, if $2B_2$, $2B_3$, etc. are the coefficients of p^3 in the second factorial moment for two-, three- and higher-dimensional lattices, it will be found by direct enumeration made in succession from lattices of lower dimensions that

$$B_r = B_{(r-1)}l_r + 4A_{(r-1)}(l_r - 1) + l_1l_2 \cdots l_{(r-1)}(l_r - 2).$$

This can be established from the following considerations. 1) Two black-black joins can be obtained from three black points situated close to one another and the chance of having three black points in a specified manner is p^3. 2) The number of ways of getting two black-black joins from three points in the lattice is

$$B_{(r-1)}l_r + 4A_{(r-1)}(l_r - 1) + l_1l_2 \cdots l_{(r-1)}(l_r - 2).$$

C_r, the coefficient of p^4 in the corrected second moment, is given by the equation

$$C_r = -(2B_r + A_r).$$

This follows from the fact that the sum of the coefficients of p^3 and p^4 in the uncorrected factorial moment, about zero, is twice the number of ways of selecting two joins from the total number of joins in the lattice which is $(A_r - 1)$. Thus

$$A_rp_1^2 + 2B_rp_1^3 + C_rp_1^4 \tag{3.1.3}$$

is the corrected second moment for the distribution of black-black joins in a lattice of r dimensions. For an $l \times m \times n$ lattice

$$\mu_2 = (3e - d)\, p_1^2 + 2(15e - 10d + 4c)\, p_1^3 - (33e - 21d + 8c)\, p_1^4. \tag{3.1.4}$$

3.2. *Cumulants for the distribution of black-white joins for two colors.* The first four cumulants for free and non-free sampling distributions in an $l \times m \times n$ lattice can be determined from the frequency distributions of black-white joins for $(1, lmn - 1)$, $(2, lmn - 2)$, $(3, lmn - 3)$ and $(4, lmn - 4)$ black and white points by the method described for linear rectangular lattices. If

$$\mu_r' = A_{1r}''pq + A_{2r}''p^2q^2 + \cdots + A_{rr}''p^rq^r,$$

the first three distributions give the coefficients of pq, p^2q^2 and p^3q^3 in the first three moments about zero. The three cumulants calculated from these moments are given below in terms of c, d, and e for free sampling.

(3.2.1) $\kappa_1 = 2(3e - \text{d})pq,$

(3.2.2) $\kappa_2 = 2(18e - 11\text{d} + 4c)pq - 4(33e - 21d + 8c)p^2q^2,$

(3.2.3)
$$\kappa_3 = 2(108e - 91d + 60e - 24)pq + 8(327e - 288d + 198c - 84)p^2q^2 + 32(219e - 197d + 138c - 60)p^3q^3.$$

The calculation of the fourth cumulant by the direct method of finding the frequency distribution of the number of black-white joins for 4 black and $(lmn-4)$ white points was found to be very laborious and therefore this has been calculated by a special method. The coefficients of pq, p^2q^2 and p^3q^3 have been determined, as in other cases, by finding $\Sigma x^4 f_x$ for the first three distributions. These coefficients reduce to a linear form in c, d and e. Now the fourth cumulant, being a linear function of these quantities, the coefficient of p^4q^4 involves c, d and e in the first degree only and therefore this can be assumed to be of the form

$$\alpha e + \beta d + \gamma c + \delta,$$

where α, β, γ and δ are constants. No simple proof can be given here regarding the linear assumption of the cumulants. It may be observed that this is true of the first four cumulants for linear and rectangular lattices. The author [3] has already provided a general proof of this assumption for the linear lattice and he hopes to extend this for the higher dimensional lattices in the near future.[4]

The constants α, β, γ, and δ can be determined by finding κ_4 for $p = q = \frac{1}{2}$ from the frequency distributions of black-white joins for $2 \times 2 \times 2$, and $2 \times 3 \times 3$ lattices for two colors as given in Tables 7 and 8.

When $p = q = \frac{1}{2}$, κ_4 reduces to the form $a'e + b'd + c'c + d'$, where a', b', c' and d' are constants. In view of this relation, if m and n are fixed, and l takes values 1, 2, 3, etc., the values of κ_4 for the different lattices should be in arithmetic progression. This can be seen by comparing the values of κ_4 for the lattices $1 \times 2 \times 2$, $2 \times 2 \times 2$ and $3 \times 2 \times 2$ which are 1, 7.5 and 14, respectively. Using this property, it is possible to find κ_4 for a lattice of any size from the complete distribution of the lattices $1 \times 2 \times 2$, $1 \times 2 \times 3$, $1 \times 3 \times 3$, and $2 \times 2 \times 2$ given before. Thus κ_4 for $2 \times 2 \times 2$, $2 \times 2 \times 3$, $3 \times 3 \times 2$ and $3 \times 3 \times 3$ lattices are 7.5, 14, 25.875 and 47.25 respectively. Now α, β, γ and δ can be obtained by equating the general expression for the fourth cumulant to the values given above for the corresponding values of l, m and n and putting $p = q = \frac{1}{2}$. The equations giving the values of α, β, γ and δ are

(3.2.4)
$$\begin{cases} 8\theta_1 + 12\theta_2 + 6\theta_3 + \theta_4 = 7.5, \\ 12\theta_1 + 16\theta_2 + 7\theta_3 + \theta_4 = 14.0, \\ 18\theta_1 + 21\theta_2 + 8\theta_3 + \theta_4 = 25.875, \\ 27\theta_1 + 27\theta_2 + 9\theta_3 + \theta_4 = 47.25, \end{cases}$$

[4] This proof has been obtained recently and will be published soon.

where $\theta_1 = \dfrac{32 \times 19176 + \alpha}{256}$, $\theta_2 = \dfrac{-32 \times 21638 + \beta}{256}$,

$\theta_3 = \dfrac{32 \times 20952 + \gamma}{256}$, and $\theta_4 = \dfrac{-32 \times 16128 + \delta}{256}$.

They give

$$\alpha = -32 \times 19143, \qquad \beta = 32 \times 21615,$$

$$\gamma = -32 \times 20940, \text{ and } \delta = 32 \times 16128.$$

TABLE 7

Frequency distribution of black-white joins, 2 × 2 × 2 lattice for two colors

No. of black-white joins	No. of black points									Total
	0	1	2	3	4	5	6	7	8	
0	1	—	—	—	—	—	—	—	1	2
1	—	—	—	—	—	—	—	—	—	—
2	—	—	—	—	—	—	—	—	—	—
3	—	8	—	—	—	—	—	8	—	16
4	—	—	12	—	6	—	12	—	—	30
5	—	—	—	24	—	24	—	—	—	48
6	—	—	16	—	32	—	16	—	—	64
7	—	—	—	24	—	24	—	—	—	48
8	—	—	—	—	30	—	—	—	—	30
9	—	—	—	8	—	8	—	—	—	16
10	—	—	—	—	—	—	—	—	—	—
11	—	—	—	—	—	—	—	—	—	—
12	—	—	—	—	—	—	—	—	—	2
										256

$$\kappa_1 = 6, \qquad \kappa_2 = 3, \qquad \kappa_3 = 0, \qquad \kappa_4 = 7.5$$

Thus the general formula for the fourth cumulant is

$$(3.2.5) \quad \begin{cases} \kappa_4 = 2(648e - 671d + 604c - 432)pq \\ \quad -4(9996e - 10857d + 10196c - 7632)p^2q^2 \\ \quad +32(9144e - 10167d + 9732c - 7416)p^3q^3 \\ \quad -32(19143e - 21615d + 20940c - 16128)p^4q^4 . \end{cases}$$

For a lattice of sides $l_1, l_2, \cdots l_r$ in r dimensions, the first two moments for the distribution of black-white joins for free sampling are as follows:

$$(3.2.6) \qquad \mu_1' = 2A_r pq,$$

$$(3.2.7) \qquad \mu_2 = 2(A_r + B_r)pq + 4C_r p^2 q^2.$$

Like the distributions for linear and rectangular lattices, when l, m and n tend to infinity, γ_1 and γ_2 will tend to zero and therefore the distribution of black-white joins for an $l \times m \times n$ lattice also tends to the normal form. The remarks made in connection with the distribution of black-white joins for a rectangular lattice are true here also. Here the frequencies for 1, 2, $[(3e - d) - 2]$ and $[(3e - d) - 1]$ black-white joins are zero, while for 0 and $(3e - d)$ they are two. But this irregularity will not affect the limiting form of the distribution since the relative frequencies tend to zero.

TABLE 8

Frequency distribution of black-white joins for 2 × 3 × 3 lattice for two colors

No. of black-white joins	No. of black points													Total
	0	1	2	3	4	5	6	7	8	9	10	11	12	
0	1	—	—	—	—	—	—	—	—	—	—	—	1	2
1	—	—	—	—	—	—	—	—	—	—	—	—	—	—
2	—	—	—	—	—	—	—	—	—	—	—	—	—	—
3	—	8	—	—	—	—	—	—	—	—	—	8	—	16
4	—	4	8	—	2	—	—	—	2	—	8	4	—	28
5	—	—	8	8	—	—	—	—	—	8	8	—	—	32
6	—	—	24	20	8	8	12	8	8	20	24	—	—	132
7	—	—	24	48	40	40	16	40	40	48	24	—	—	320
8	—	—	2	52	81	56	68	56	81	52	2	—	—	450
9	—	—	—	40	104	112	144	112	104	40	—	—	—	656
10	—	—	—	44	100	188	160	188	100	44	—	—	—	824
11	—	—	—	8	88	144	176	144	88	8	—	—	—	656
12	—	—	—	—	36	108	162	108	36	—	—	—	—	450
13	—	—	—	—	24	88	96	88	24	—	—	—	—	320
14	—	—	—	—	12	28	52	28	12	—	—	—	—	132
15	—	—	—	—	—	8	16	8	—	—	—	—	—	32
16	—	—	—	—	—	4	20	4	—	—	—	—	—	28
17	—	—	—	—	—	8	—	8	—	—	—	—	—	16
18	—	—	—	—	—	—	—	—	—	—	—	—	—	—
19	—	—	—	—	—	—	—	—	—	—	—	—	—	—
20	—	—	—	—	—	—	2	—	—	—	—	—	—	2
														4096

$\kappa_1 = 10, \quad \kappa_2 = 5, \quad \kappa_3 = 0, \quad \kappa_4 = 14$

3.3. *First and second moments for the distribution of black-white joins for k colors in an r-dimensional lattice.* The results for free sampling follow easily from a consideration of the expectations of the various ways of obtaining one and two black-white joins. The expectation of the number of black-white joins is

$$2\, A_r p_1 p_2 . \tag{3.3.1}$$

The expectation for two black-white joins is

$$B_r p_1 p_2 (p_1 + p_2) + 4 \left\{ \frac{A_r(A_r - 1)}{2} - B_r \right\} p_1^2 p_2^2.$$

From this it will follow that the second moment

$$\mu_2 = 2A_r p_1 p_2 + 2B_r p_1 p_2 (p_1 + p_2) + 4C_r p_1^2 p_2^2. \tag{3.3.2}$$

3.4. *First and second moments for the distribution of the total number of joins between points of different colors for an $l \times m \times n$ lattice for three colors.* The expectation for free sampling is

$$2(3e - d)\Sigma p_r p_s . \tag{3.4.1}$$

TABLE 9

Distribution of joins between points of different colors for 1 black, 1 white and (lmn − 2) red points

No. of joins	Frequency for lattices			
	2×2×2	2×2×3	2×3×3	3×3×3
5	24	16	8	—
6	32	56	80	104
7	—	56	104	144
8	—	4	96	276
9	—	—	18	112
10	—	—	—	66
Total..................	56	132	306	702
$\Sigma x^2 f_x$ about zero	1752	5416	15778	44136

The second moment will involve terms in $\Sigma p_r p_s$, $p_1 p_2 p_3$ and $\Sigma p_r^2 p_s^2$. The coefficients of $\Sigma p_r p_s$ and $\Sigma p_r^2 p_s^2$ are the same as those for two colors. The coefficient of $p_1 p_2 p_3$ can be determined by finding the frequency distribution of joins between points of different colors when the lattice consists of 1 black, 1 white and $(lmn - 2)$ red points. But this straightforward method is cumbersome and hence the coefficient of $p_1 p_2 p_3$ has been determined by finding the distribution for the special lattices $2 \times 2 \times 2$, $2 \times 2 \times 3$, $2 \times 3 \times 3$, and $3 \times 3 \times 3$. These results are shown in Table 9.

The coefficients of $p_1 p_2 p_3$ in the corrected second moment for the above lattices are obtained by subtracting $2(18e - 11d + 4e)\ (2e - 3) + 8(3e - d)^2$ from the moments noted above. This can be seen to be so by comparing the above expression with the quantity subtracted from the uncorrected second moment for a two dimensional lattice in section 2.4. The coefficients so obtained for $2 \times 2 \times 2$,

$2 \times 2 \times 3$, $2 \times 3 \times 3$, and $3 \times 3 \times 3$ lattices are -336, -640, -1184 and -2142 respectively. Now the coefficient of $p_1p_2p_3$ in the corrected second moment is of the form

$$\alpha' e + \beta' d + \gamma' c + \delta'.$$

The equations obtained by equating this expression to -336, -640, -1184 and -2142 for the respective lattices give $\alpha' = -174$, $\beta' = 108$, $\gamma' = -40$

TABLE 10

Distribution of joins between points of different colors when there are 1 black, 1 white, 1 red and (lmn-3) green points

No. of joins	Frequency for lattices			
	$2 \times 2 \times 2$	$2 \times 2 \times 3$	$2 \times 3 \times 3$	$3 \times 3 \times 3$
7	144	48	—	—
8	144	312	288	72
9	48	480	912	1344
10	—	432	1344	2664
11	—	48	1560	4392
12	—	—	720	4584
13	—	—	72	3168
14	—	—	—	1206
15	—	—	—	120
Total............	336	1320	4896	17550
$\Sigma x^2 f_x$ about zero ...	20160	110208	531312	2370168

and $\delta' = 0$. Thus the second moment for a lattice with points in three colors is

$$\begin{aligned} &2(18e - 11d + 4c)\Sigma p_r p_s \\ &-2(87e - 54d + 20c)p_1p_2p_3 \\ &-4(33e - 21d + 8c)\Sigma p_r^2 p_s^2 . \end{aligned} \tag{3.4.2}$$

3.5. *First and second moments for the distribution of the total number of joins between points of different colors in an $l \times m \times n$ lattice for four or more colors.* The expectations for free sampling are given by the same expression as for three colors. The coefficients of $\Sigma p_r p_s$, $\Sigma p_r p_s p_t$ and $\Sigma p_r^2 p_s^2$ in the corrected second moment are also the same as in section 3.4. The coefficient of $\Sigma p_r p_s p_t p_u$ can be determined by the method described in section 3.4 for $\Sigma p_1 p_2 p_3$ from the frequency distributions of joins (Table 9) between points of different colors for $2 \times 2 \times 2$, $2 \times 2 \times 3$, $2 \times 3 \times 3$ and $3 \times 3 \times 3$ lattices when they consist of 1 black, 1 white, 1 red and $(e - 3)$ green points.

The coefficient of $\Sigma p_r p_s p_t p_u$ in the corrected second moment is obtained by subtracting (obtained in the same way as for the two dimensional lattice in section 2.5)

$$\begin{aligned}&6(18e - 11d + 4c)(e - 2)^2\\&+(3e - 8)[2(-87e + 54d - 20c) + 8(3e - d)^2]\\&-8(3e - d)^2\end{aligned}$$

from the uncorrected values. The values so obtained for the four lattices are 480($2\times 2 \times 2$), 928($2 \times 2 \times 3$), 1736($2 \times 3 \times 3$) and 3168($3 \times 3 \times 3$). The coefficient of $p_r p_s p_t p_u$, as in other cases, being of the form

$$\alpha''e + \beta''d + \gamma''c + \delta'',$$

α'', β'', γ'' and δ'' can be determined by equating the above expression to 480, 928, 1736 and 3168 for the respective lattices. The coefficient so obtained is

$$8(33e - 21d + 8c).$$

Hence the second moment for free sampling when the lattice contains points of four or more colors is

$$\begin{aligned}&2(18e - 11d + 4c)\Sigma p_r p_s\\&-2(87e - 54d + 20c)\Sigma p_r p_s p_t\\&-4(33e - 21d + 8c)\Sigma p_r^2 p_s^2\\&+8(33e - 21d + 8c)\Sigma p_r p_s p_t p_u .\end{aligned} \tag{3.5.1}$$

In general, it will be found that the cumulants involve terms in c, d, e and an absolute term only. Therefore when l, m and n tend to infinity and $p_1, p_2, p_3 \cdots$ are finite, the distribution of $R - 2(3e - d)\Sigma p_r p_s$, where R is the total number of joins of points of different colors, tends to the normal form. When l, m and n are large,

$$\frac{R - 2(3e - d)\Sigma p_r p_s}{\sqrt{e}}$$

can be considered to be normally distributed with

$$36\Sigma p_r p_s - 174\Sigma p_r p_s p_t - 132\Sigma p_r^2 p_s^2 + 264\Sigma p_r p_s p_t p_u \tag{3.5.2}$$

as its variance.

The distribution for non-free sampling here also tends to the normal form for the same reasons given for the rectangular lattice. As in free sampling, for large values of l, m and n

$$\frac{R - 2(3e - d)\Sigma e_r e_s}{\sqrt{e}}$$

is distributed normally with the variance

$$6\Sigma e_r e_s e_t + 12\Sigma e_r^2 e_s^2 - 24\Sigma e_r e_s e_t e_u , \tag{3.5.4}$$

where R is the observed number of joins for a given distribution of the points and $e_r = \frac{n_r}{lmn}.$ The error in this variance will be about 5% or less when l, m and n are greater than 36.

We may conclude this section by giving the first and the second moments for free sampling with k colors for an r-dimensional lattice.

$$\mu_1' = 2A_r\Sigma p_r p_s, \tag{3.5.5}$$

$$\begin{aligned} \mu_2 = {} & 2(A_r + B_r)\Sigma p_r p_s \\ & + 2(3B_r + 4C_r)\Sigma p_r p_s p_t \\ & + 4C_r\Sigma p_r^2 p_s^2 - 8C_r\Sigma p_r p_s p_t p_u , \end{aligned} \tag{3.5.6}$$

where A_r , B_r and C_r are as defined in section 3.1.

This can be seen from the following facts:

(1) The coefficients of $\Sigma p_r p_s$ and $\Sigma p_r^2 p_s^2$ are the same as for two colors.

(2) The coefficient of $\Sigma p_r p_s p_t$ is the number of ways of getting two joins of different colors from combination of points not included in $\Sigma p_r p_s p_t p_u$. This can be had from three points of three different colors close together and four points of three different colors separated into groups of two each such that each group will give one join. The number of arrangements of the first kind is $3!B_r$. For the second kind it is $8(A_r^2 + C_r)$. Subtracting from the total number, the contribution of $\Sigma p_r p_s p_t$ in the correction factor $4A_r^2(\Sigma p_r p_s)^2$, the coefficient of $\Sigma p_r p_s p_t$ in the second moment works out to be

$$2(3B_r + 4C_r).$$

(3) The coefficient of $\Sigma p_r p_s p_t p_u$, as in all other cases dealt before, is twice that of $\Sigma p_r^2 p_s^2$ with an opposite sign.

Acknowledgements. The author's thanks are due to Dr. D. J. Finney for suggesting this problem and for all the facilities and help given to him in carrying out the investigations discussed in this paper. The author is also grateful to Mr. P. A. P. Moran for explaining the results of his investigations on this problem and for the interest taken by him in the course of the research.

REFERENCES

[1] V. Bortkiewicz, *Die Iteration*, Berlin, 1917.

[2] R. C. Bose, "The patch number problem," *Sci. Cult.*, Vol. 12 (1946), p. 199.

[3] P. V. Krishna Iyer, "The theory of probability distribution of points on a line," *Indian J.Agric. Stat.*, Vol. 1 (1948).

[4] H. Levene, "A test of randomness in two dimensions," *Ann. Math. Stat.*, Vol. 17 (1946), p. 500.

[5] P. A. Moran, "Random associations on a lattice," *Proc. Cam. Phil. Soc.*, Vol. 43 (1947), p. 321.

[6] P. A. Moran, "The interpretation of statistical maps," *J. Roy. Stat. Soc.*, B, Vol. 10 (1948), p. 243.