of the *Annals*. These values are more accurate than those heretofore available. A corrected Table I based on these values is as follows:

n	d_2	d_4	A_2	A_3	A_4	n
2	1.1284	.8256	1.8800	2.6951	3.0411	2
3	1.6926	.7480	1.0233	1.8258	3.0902	3
4	2.0588	.7012	.7286	1.5218	3.1330	4
5	2.3259	.6690	.5768	1.3629	3.1699	5
6	2.5344	.6449	.4832	1.2634	3.2020	6
7	2.7043	.6260	.4193	1.1945	3.2303	7
8	2.8472	.6107	.3725	1.1434	3.2556	8
9	2.9700	.5978	.3367	1.1038	3.2784	9
10	3.0775	.5868	.3083	1.0720	3.2992	10

ABSTRACTS OF PAPERS

(*Abstracts of papers presented at the Berkeley meeting of the Institute, August* 5, 1950)

1. Sampling from Populations with Overlapping Clusters. Z. W. Birnbaum, University of Washington, Seattle.

In cluster sampling it is usually assumed that the clusters are disjoint. In this paper situations are considered in which this assumption is not fulfilled. Let the population π consist of N individuals "j", having the variates $V[j], j = 1, 2, \cdots, N$, and let K clusters $C[i], i = 1, 2, \cdots, K$, be such that each "$j$" belongs to at least one cluster. Let $s[j] \geq 1$ be the number of different clusters to which "j" belongs (the multiplicity of "j"). The cluster $C[i]$ contains N_i individuals with the variates $V[i, t]$, $t = 1, 2, \cdots, N_i$; $i = 1, 2, \cdots, K$. In a sampling procedure, let sub-sample sizes $n[i]$ be given for each $C[i]$, and weights $\lambda[i, t]$ for each $V[i, t]$; a random sample of k clusters $C[i_u]$, $u = 1, 2, \cdots, k$ is obtained, then $n[i_u]$ individuals are sampled from $C[i_u]$, and for each of them its variate and its multiplicity are recorded. Necessary and sufficient conditions are derived for $S = \Sigma_{u=1}^{k} \Sigma_{v=1}^{n[i_u]} V[i_u, t_v] \lambda[i_u, t_v]$ being an unbiased estimate of $\bar{V} = \frac{1}{N} \Sigma_{j=1}^{N} V_j$. The variance of S is found, the weights are studied which minimize this variance, and some practically important special cases are derived.

2. A Simple Nonparametric Test of Independence. Nils Blomqvist, University of Stockholm.

Consider a sample of size n from a two-dimensional distribution $F(x, y)$. Let $\tilde{x}$ and $\tilde{y}$ denote the two sample medians and compute the number of individuals, say k, satisfying the inequality $x < \tilde{x}$, $y < \tilde{y}$ (the trivial difficulty arising when n is an odd number can easily be overcome). A test of independence based on k is nonparametric. As a matter of fact one has under the null hypothesis that

$$P(k) = \binom{m}{k}^2 \Big/ \binom{2m}{m},$$

where $m = [n/2]$. In the case of normal F with correlation coefficient ρ it is possible to show, by studying the asymptotic behavior of the power function of the test in the neighborhood of $\rho = 0$, that the asymptotic efficienty of the test is $(2/\pi)^2$, or about 41%. This result is based on the fact that k has an asymptotically normal distribution if some regularity conditions are fulfilled. In spite of its low efficiency it is suggested that the test be used in cases where some information can be neglected in favor of the simplicity of the method.

3. On Minimax Statistical Decision Procedures and Their Admissibility. Colin R. Blyth, University of California, Berkeley.

The problem considered is that of using a sequence of observations on a random variable X to make a decision. Two loss functions W_1 and W_2, each depending on the distribution F of X, the number n of observations taken, and the decision δ made, are assumed given. Minimax problems can be stated for weighted sums of W_1 and W_2, or for either one subject to an upper bound on the expectation of the other. Under suitable conditions it is shown that solutions of the first type of problem provide solutions for all problems of the latter types, and that admissibility for a problem of the first type implies admissibility for problems of the latter types. Two examples are given: estimation of $E\,X$ when X is (1) normal with known variance, (2) rectangular with known range. The two loss functions are in each case $W_1 = n$ and an arbitrary nondecreasing function $W_2(\mid \delta - \theta \mid)$. Admissible minimax estimates are obtained. Extensions to any function $W_1(n)$ are indicated; two examples are given for the normal case where the sample size must be randomised among more than a consecutive pair of integers.

4. Sufficient Statistics and Unbiased Estimates for "Selected" Distributions. Douglas G. Chapman, University of Washington, Seattle.

A family of distributions obtained from any given family by fixed selection may be called a "selected" family. Tukey's theorem that such selected families admit the same set of sufficient statistics as the parent family is proved for an extended class of distributions. Further if the selection does not involve truncation the existence of minimum variance unbiased estimates of parameters of the parent family ensures the existence of similar estimates for the selected family. Some results are derived for minimum variance unbiased estimates for truncated distributions.

5. The Unattainability of Certain Lower Bounds by Product Densities. R. C. Davis, U. S. Naval Ordnance Testing Station, China Lake.

Under weak regularity conditions it is shown that for the case in which the sample size is a nonrandom variable, certain lower bounds are unattainable. Consider a univariate chance variable X, possessing an absolutely continuous distribution function $F(x, \theta)$, in which θ is the unknown parameter. Under quite general regularity conditions Barankin has proved the existence and uniqueness of the locally best unbiased estimate of a function $g(\theta)$ for a specified parameter value θ_0. The criterion of bestness is the minimization of the s^{th} absolute central moment $(s > 1)$ of the estimate about $g(\theta_0)$, and Barankin has obtained an expression for the lower bound both in the general case and in particular for a case which yields a generalization of the Cramer-Rao inequality valid for any s^{th} absolute central moment. It is the latter lower bound with which we are concerned. With an additional weak assumption concerning the density function of X, it is shown that if $\varphi_s(x_1, x_2, \cdots, x_n)$ is the locally best unbiased estimate of $g(\theta_0)$ (obtained by Barankin) for each fixed sample size n and for each $s > 1$, then there exists no probability distribution $F(x, \theta)$ except for $s = 2$ yielding a sequence $\{\varphi_s(x_1, x_2, \cdots, x_n)\}\,(n = 1, 2, \cdots, \text{ad inf.})$ in which $x_1, x_2, \cdots, x_n$ are for each n independently and identically distributed chance variables and for which $\varphi_s(x_1, x_2, \cdots, x_n)$ attains for each n the special lower bound given by Barankin. Obviously in the case $s = 2$, the lower bound is achieved by an efficient statistic if one exists.

6. A Note on the Power of the Sign Test. T. A. Jeeves and Robert Richards, University of California, Berkeley.

Values obtained by using the normal approximation to the noncentral t-distribution given by Johnson and Welch were compared with exact values given by Neyman and Tokarska. The comparison indicated that efficiencies of the sign test computed from the approximation would be consistently higher than the true efficiencies. To avoid this bias the sign test was randomized so that levels of significance of $\alpha = .05$ and $\alpha = .01$ were obtained and the exact values of the noncentral t used. Efficiencies were computed using various measures of equivalence of the power functions: (1) balancing the area (Walsh), (2) minimizing the maximum difference, (3) equalizing the power at certain fixed points. The various measures of equivalence yielded no marked differences in efficiencies. Tables were given of the efficiencies for small n. The efficiency for $\alpha = .05$ was about .7 for n between 6 and 20 and somewhat higher for $\alpha = .01$. The efficiency slowly approaches the asymptotic value of $2/\pi = .6366$ as n increases.

7. About Some Classes of Sequential Procedures for Obtaining Confidence Intervals of Given Length. (Preliminary Report). Werner R. Leimbacher, University of California, Berkeley.

The special class C_1 of such procedures indicated by A. Wald (*Sequential Analysis*, John Wiley & Sons, 1947, pp. 145–156) can be extended by generalizing and improving the inequality on which the procedures are based. It is shown that even in this larger class C_2, a procedure could possibly be optimum only under very special circumstances. The well known optimum procedure for a normal distribution $N(\theta, 1)$ can be obtained as the limit of a sequence of procedures from C_2. For the suggested sequence, however, the limit no longer belongs to C_2. In order to eliminate various deficiences of C_2, a modified class C_3 is proposed which contains the well known optimum procedures for the normal and rectangular distributions. The method indicated seems suggestive for the general case of estimating location parameters by confidence intervals.

8. On the Stochastic Independence of Symmetric and Homogeneous Linear and Quadratic Statistics. Eugene Lukacs, U. S. Naval Ordnance Testing Station, China Lake.

It is known that the sampling distributions of the mean and of the variance are stochastically independent if and only if the parent distribution is normal. This was proven by R. C. Geary (*Jour. Roy. Stat. Soc., Suppl.*, Vol. 3 (1936)) and using a different method by E. Lukacs (*Annals of Math. Stat.*, Vol. 13 (1942)). The question arises whether there are any distributions having the property that the sampling distributions of the mean and of a symmetric and homogeneous quadratic statistic are independent. It can be shown that there are only the following possibilities: (1) the parent distribution is normal, (2) the parent distribution is degenerate with a single saltus of one, (3) the parent distribution is a step function with two steps, located symmetrically with respect to zero, (4) the parent distribution is a gamma distribution.

9. The Distribution of the Maximum Deviation between Two Sample Cumulative Step Functions. Frank J. Massey, Jr., University of Oregon.

Let $x_1 < x_2 < \cdots < x_n$ and $y_1 < y_2 < \cdots < y_m$ be the ordered results of two random samples from populations having continuous cumulative distribution functions $F(x)$ and

$G(x)$ respectively. Let $S_n(x) = k/n$ when k is the number of observed values of X which are less than or equal to x, and similarly let $S'_m(y) = j/m$ where j is the number of observed values of Y which are less than or equal to y. The statistic $d = \max_x | S_n(x) - S'_m(x) |$ can be used to test the hypothesis $F(x) \equiv G(x)$, where the hypothesis would be rejected if the observed d is significantly large. In this paper a method of obtaining the exact distribution of d for small samples is described, and a short table for equal size samples is included. The general technique is that used by the author for the single sample case. There is a lower bound to the power of the test against any specified alternative. This lower bound approaches one as n and m approach infinity proving that the test is consistent.

10. An Iterative Construction of the Optimum Sequential Decision Procedure with Linear Cost Function. LINCOLN E. MOSES, Stanford University.

Where the cost of taking n observations is proportional to n, define a sequential decision procedure D_T by means of its associated "stopping region" T; T is the set of a posteriori probability distributions $\xi(\theta)$ for which D_T instructs the statistician to take no observation and to make the decision which minimizes the Bayes risk. Now let D_T be any sequential decision procedure which has uniformly bounded average risk for every a priori distribution, $\xi(\theta)$. Define T as the derived region of T: T' is the set of $\xi(\theta)$ such that the Bayes risk of stopping at $\xi(\theta)$ is not greater than the risk of taking one observation and then using D_T. Define $T^{(n+1)} = T^{(n)\prime}$. Then it is shown that the sequence of regions $\{T^{(n)}\}$ $n = 1, 2, \cdots$ is monotonically decreasing to a limit region T^{∞}, and that $D_{T^{\infty}}$ is the optimum sequential decision procedure. Some numerical examples are given where the exact solution is obtained and the convergence of the iteration is examined. (This paper was prepared under the sponsorship of the Office of Naval Research.)

11. On the Law of the Iterated Logarithm for Dependent Random Variables. STANLEY W. NASH, University of California, Berkeley.

The order of the remainder term is evaluated in the distribution function of the asymptotically normal sum S_n of dependent random variables of a certain class considered by Loève. Bounds are found for the probability that $\max_{k \leq n} | S_n | \geq B_n^{\frac{1}{2}}x$, where B_n is the sum of the variances of components of S_n. Given an infinite sequence of events A_n, a necessary and sufficient condition is found for the probability that infinitely many A_n occur to equal one. This criterion extends criteria due to Borel. With these results established, the law of the iterated logarithm is shown to hold for a wide subclass of Loève's class of dependent random variables. Within this class the partial sum $S_n - S_i$ may approach normality with a speed which depends in a certain functional way on the previous sum S_i, and which may be arbitrarily slow for some values of S_i. The conclusions generalize earlier results due to W. Doeblin and N. A. Sapogov.

12. Conditional Expectation and the Efficiency of Estimates. PAUL G. HOEL, University of California, Los Angeles.

A probability density function, $f(x; \theta)$, is considered for which the range of x does not depend on θ and for which there exists a sufficient statistic for θ. It is shown that under certain regularity conditions, there exists a unique unbiased sufficient estimate of θ among those sufficient estimates which can be expressed as functions of a particular sufficient statistic. This result, together with results of other authors, is used to show that for the class of statistics satisfying the regularity conditions, the method of Blackwell for improving an unbiased estimate of θ does not yield an essentially better estimate than a well known estimate.

13. Optimum Estimates for Location and Scale Parameters. RAYMOND P. PETERSON, University of California and National Bureau of Standards, Los Angeles.

Let $h_i(W \mid E, \theta) = W(\theta_i^*, \theta)p(E \mid \theta)$, where $p(E \mid \theta)$ is the joint probability density function of the n (not necessarily independent) sample values $x_1, \cdots, x_n$ which may be represented as a point $E = (x_1, \cdots, x_n)$ in the n-dimensional Euclidean sample space M. The unknown parameters, $\theta_1, \cdots, \theta_s$, may be represented as a point $\theta = (\theta_1, \cdots, \theta_s)$ in the s-dimensional Euclidean parameter space Ω. $W(\theta_i^*, \theta)$ is a real-valued, nonnegative, measurable weight function, defined for all E in M and θ in Ω, which represents the relative seriousness of taking the estimate $\theta_i^*(E)$ as the value of θ_i for any particular sample point E. Let $G(\theta)$ be the unknown cumulative distribution function of θ. Then $\theta_i^*(E)$ is defined to be a best estimate of θ_i, provided that, if $\bar{\theta}_i(E)$ is any other estimate of θ_i in the class under consideration, $\bar{I} - I^* \geq 0$, where

$$I = \int_\Omega \int_M h_i(W \mid E, \theta) dE\, dG(\theta).$$

Let

$$r_i(\theta) = \int_M h_i(W \mid E, \theta)\, dE, \qquad \varphi_i(E) = \int_\Omega h_i(W \mid E, \theta)\, d\theta.$$

A general theorem is proved to the effect that if $h_i(W \mid E, \theta)$ is measurable over the product space $M \times \Omega$ and if $r_i(\theta)$ and $\varphi_i(E)$ are uniformly convergent integrals, then a best estimate $\theta_i^*(E)$ of θ_i exists provided that $r_i(\theta)$ is constant and that $\theta_i^*(E)$ minimizes $\varphi_i(E)$ for all points E in M. General methods are obtained for constructing best estimates for location and scale parameters, separately or jointly, and for functions of location and scale parameters from several populations. As special cases, results are derived which are analogous to converses of Theorems 1 and 2 in Kallianpur's, "Minimax Estimates of Location and Scale Parameters", Abstract, (*Annals of Math. Stat.*, Vol. 21 (1950), pp. 310–311).

NEWS AND NOTICES

Readers are invited to submit to the Secretary of the Institute news items of interest.

Personal Items

Professor William Feller of Cornell University has been appointed Eugene Higgins Professor of Mathematics at Princeton University.

Dr. Leonard Kent, formerly on the staff at the University of Chicago in the School of Business, is now with the firm of Alderson and Sessions, 1905 Walnut Street, Philadelphia 3, Pennsylvania.

Dr. G. B. Oakland has resigned an associate professorship of statistics at the University of Manitoba to accept the position as Head of Biometrics Unit, Division of Administration, Department of Agriculture, Ottawa.

Dr. Norman Rudy has accepted an appointment as Assistant Professor at Sacramento State College, Sacramento, California.

Professor G. R. Seth has returned to India to accept the position of Professor of Statistics and Deputy Statistical Advisor to the Indian Council of Agricultural Research, New Delhi.