# ABRAHAM WALD'S CONTRIBUTIONS TO ECONOMETRICS[1]

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The untimely death of Professor Wald has deprived econometrics of a man who has made many important contributions to mathematical economics and statistics. For an account of Wald's life and work we refer to the moving and brilliant essay by Morgenstern [20].[2] Many of Wald's numerous publications are of direct or indirect interest to econometricians. They may be conveniently classified into the following groups:

(1) The existence of meaningful solutions for systems of equations in mathematical economics.

(2) Indifference systems and cost of living index numbers.

(3) The minimax principle and its relationship to nonstatic economics.

(4) Elimination of the seasonal variation.

(5) The problem of identification.

(6) Stochastic difference equations.

There are also other contributions to mathematical statistics which may be of interest to econometrics. Among these we mention only: his criticism of the variate difference method {24}[2], (cf. Tintner, [24], pp. 14 ff.); his method of deriving linear relationships between variables which are subject to error {41}, (cf. Bartlett, [5]) and his nonparametric test for autocorrelations {59}.

**1. Meaningful solutions of systems of equations in mathematical economics.** Wald deals first with the equations of the theory of production {15}, {21}, {23} (cf. Stigler, [23], pp. 243 ff.). Each product is produced by a combination of various factors of production. There is no saving. The coefficients of production are constants. The commodities may be free goods, if there is a surplus. The theory yields an answer to the question: which of them will ultimately be free goods? There are demand functions for all the products which depend upon the prices.

Under which conditions has this system a unique and nonnegative solution in the quantities of products, the prices of the products, the prices of the factors of production and the surpluses?

The conditions under which this is the case are formulated by Wald as follows: (1) a positive amount of each factor of production is available; (2) the amounts of the factors of production which are used for the production of the products are not negative; (3) there is at least one factor of production which is necessary for the production of each product; (4) the relationship between the prices and the quantities demanded of the products is continuous; (5) the

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[2] References in brackets (e.g., [20]) are listed at the end of this paper; references in braces (e.g., {1}) are listed in "The publications of Abraham Wald," pp. 29–33 of this issue.

demand for each product is zero only for infinite price. Condition (6) is imposed upon the demand functions for the products. It is derived from a proposition in value theory: the marginal utility of each commodity is much more influenced by changes in the quantity of this commodity than of any other commodity. (7) The rank of the matrix of the constant coefficients of production is equal to the number of the factors of production used.

The second case considered by Wald is the equations of exchange in their Walrasian form {21}, {23}. They are static equations. We assume free competition. Each individual maximizes his utility. He is restricted by budget equations and the markets are all cleared (total demand equals total supply). Under which conditions will the equations of exchange have a unique solution where the prices are positive and the final quantities after the exchange are also positive?

The conditions are: (1) all initial quantities possessed are nonnegative; (2) each individual starts with a positive amount of at least one good; (3) for each good there is at least one individual who starts with a positive amount of this commodity. The conditions (4) and (5) restrict the form of the utility functions somewhat. They are identical with the Walrasian assumption that the marginal utility of each commodity is independent of all other commodities and decreases with increasing amounts of this good. The elasticity of the marginal utility of each good is infinite for the amount zero of this commodity and less than one throughout.

The third case treated deals with the Cournot case of duopoly {23}. Costs are neglected. Each duopolist considers the supply of his competitor as given and adjusts his own supply in such a fashion as to maximize his profit.

The results are as follows. (1) Assume that the demand function for the commodity in question cuts the quantity and price axis and has a continuous first derivative. Then there is at least one equilibrium point which is reached from any initial situation. (2) The demand function cuts the quantity and the price axis. It has a first and a second derivative. The first derivative is negative, the second not positive. Then the reaction function is unique, continuous and decreases monotonically. There is exactly one equilibrium point which is stable.

The great importance of these investigations for mathematical economics should not be underestimated. It would be most important to obtain analogous results for other cases of economic systems, for example, various market organizations.

The mathematical methods used in the proofs are essentially topological and related to the theory of sets. The command of advanced methods is unmatched except for articles by Menger [18], [19] and von Neumann [28], [29]. It is quite comparable to the beautiful theory of von Neumann and Morgenstern [30]. Wald has also contributed an important review of this theory {74} and used it in connection with his own ideas on statistical inference ({94}, pp. 24 ff.).

It should also be mentioned that methods similar to those used by Wald in this field have recently been employed by a number of authors in the discussion of activity analysis of production and allocation (Koopmans, [13]). This important field, which is related to Leontief's earlier verification of the *tableau économique* [15] is of supreme importance for economic theory and policy.

**2. Indifference systems and the cost of living index.** Three related investigations of Wald deal with this problem. Following the work of Konues [12], Haberler [9], Leontief [14], and Staehle [22], he improved the limits of cost of living index numbers {28}.

Assume that there are two situations, called one and two. The market prices of all commodities consumed and the quantities consumed are known. The indifference system is the same in the two situations. What are the limits of the cost of living index numbers which correspond to situation one or situation two?

Wald derives all limits which hold for all possible cases. He also shows that these limits cannot be improved, if nothing else is known.

He succeeded in narrowing down the limits if the following condition holds. Consider two situations which lie on the same indifference curve. It is assumed that not all commodities have a larger or all commodities a smaller marginal utility in one situation than in the other.

This theory gives us limits for the cost of living index numbers. If we make the assumption that the utility function can be approximated by a polynomial of the second degree in a small region we can actually obtain an approximation to the true cost of living index, as Wald has shown {28}, {36}. His work is related to earlier efforts by Bowley and Frisch (Frisch, [6]).

The above assumption implies the linearity of the Engel curves. The Engel curves show the dependence of the consumption of each commodity upon income, prices being constant. Let $E$ be the expenditure (or income) and $q_1^i$ the quantities of good $i$ consumed in situation one. Then the Engel curves in situation one are:

$$q_1^i = a_1^i E + b_1^i ,$$

where $a_1^i$ and $b_1^i$ are constants which can be determined by a statistical study of family budgets (see, e.g., Allen and Bowley, [1]).

Let $q_2^i$ be the consumption of commodity $i$ in situation two. Then the Engel curves are in situation two:

$$q_2^i = a_2^i E + b_2^i .$$

The approximation to the true cost of living index number is then:

$$\sqrt{\frac{\sum a_1^i p_2^i}{\sum a_2^i p_1^i}} + \frac{\sum b_1^i p_2^i - [\sum b_2^i p_1^i \sqrt{\sum a_1^i p_2^i \sum a_2^i p_1^i} / \sum a_2^i p_1^i]}{E_1 (1 + \sqrt{\sum a_1^i p_2^i \cdot \sum a_2^i p_1^i})},$$

where $p_1^i$ and $p_2^i$ are the prices of commodity $i$ in periods one and two, $E_1$ is the total expenditure in situation one. The summations are extended over all commodities.

Wald shows that the index number of the double expenditure method (Frisch, [6]) is a special case of his own cost of living index number, under certain conditions. If the Engel curves are straight lines through the origin then both formulae are equivalent and lead to the celebrated ideal index number of Fisher. This is the geometric mean of Laspeyres' and Paasche's index numbers.

The importance of these results for theory and policy is great, as pointed out recently by Hotelling [10]. It is deplorable that they have not become more widely known among economists and economic statisticians interested in the cost of living index number problem (see however Ulmer, [27], pp. 81 ff.).

Wald's work on the empirical determination of indifference surfaces {39} is closely related to his writings on index numbers. He assumes again the existence in a certain region of a utility function or indicator which is a second order polynomial in the quantities consumed. The Engel curves are hence linear. From a knowledge of the Engel curves we are able to find an approximation to the (integrable) utility function or indifference system. Tests for the stability conditions are given. The integrability conditions are applied. The calculation of the demand functions of the individual goods is indicated.

It is astonishing that until recently nobody has applied this theory to the determination of empirical indifference systems. Professor J. A. Nordin of Iowa State College has undertaken this task not long ago with good success. In view of the enormous amount of data available from budget studies it is to be hoped that his example will soon be followed by more econometricians.

**3. The minimax principle and its relationship to nonstatic economics.** The novel statistical ideas introduced by Wald into the theory of statistical inference are possibly of great importance in economic statistics. Generalizing the Neyman-Pearson theory of testing hypotheses, he bases himself upon a completely pragmatic approach {94}. The values of the losses which are incurred by committing errors of various kinds must be known. Then it is possible at least in principle to choose the action which is in a sense most advantageous; this action minimizes the risk among all possible actions. It should be mentioned that Wald advocated the minimax principle in a tentative way and because of certain formal advantages. I am informed that he was still interested in finding a less conservative and more satisfactory principle for statistical inference.

To my mind, it is somewhat doubtful if principles of this kind are really applicable in the social sciences [26]. They are without any doubt applicable in industrial applications (quality control, etc.). Wald has made many important and interesting contributions to this field; for example, his rightly celebrated sequential analysis {76}. But it is somewhat doubtful if a similar pragmatic approach is applicable in the field of social policy. How can we evaluate and measure the comparative advantage and disadvantage of various actions in this field, for example, gains and losses produced by the adoption of competing social policies? Arrow's recent work in this field [3] proves the nonexistence of a social welfare function, except if very special conditions hold. This result points definitely toward the impossibility of a pragmatic approach.

It should also be remarked that the minimax principle even if it is applicable leads to an extremely conservative policy. Wald used the simile of a zero-sum two person game (von Neumann and Morgenstern, [30]). He considers a game of two persons, Nature (in our case Society) against the statistician. It is postulated in the theory of games that each player gains what the other loses. We have to assume that Nature is trying to ruin the statistician, who has to take the action which will be the best possible even if Nature is trying to do its worst. But actually Nature is supremely indifferent to the statistician. Hence the actions of the statistician if based upon Wald's minimax principle will be very conservative (L. J. Savage, [21]; K. J. Arrow, [4]).

Hurwicz [11] and Marschak [16] have pointed out that Wald's principles may also be applicable to the theory of maximizing profits or utility in cases of incomplete information. This idea has never been fully elaborated. There is, however, not any doubt that indeed the principle is applicable, if we are willing to adopt the very conservative attitude which it implies.

In connection with his work on decision functions Wald considered also a problem which is of paramount importance in economic statistics. This is the problem of multiple choice ({76}, pp. 138 ff.). Here we have to choose not between two hypotheses as in the Neyman-Pearson theory of testing hypotheses but between any number of hypotheses. Even if we are unwilling to use the minimax principle in this field it is of the utmost importance that the statistician's interest has been directed toward these crucial problems. Two examples which are of great importance in econometrics are as follows.

Suppose we want to fit a polynomial trend to a series of given data. We do not know if a straight line, a parabola, a cubic, a quartic, etc. gives the most adequate fit. The methods given by Fisher give no adequate answer to the problem of the degree of the polynomial which ought to be chosen. This is essentially a multiple choice problem.

Another example is as follows. Let us assume we have a number of observations on a certain set of economic variables which are subject to errors. We want to know how many linear relationships exist between the systematic parts of our variables in the population from which our sample is taken. Again, the methods proposed by Geary [7], Anderson [2], and the author [25] are inadequate because we are faced with a multiple choice problem.

Wald's work in this field opens up the possibility that econometricians and statisticians may be able to deal with problems of multiple choice in a more rational manner than before.

**4. Elimination of the seasonal variation.** This is perhaps the earliest work of Wald concerned with economic statistics {24}. It has been very well presented by Mendershausen [17]. Hence we can summarize it briefly.

The following assumptions underlie the method. The given time series is the sum of three components: one component represents the nonseasonal movement, that is, the combined effect of trend and cycle; the second part is the season; the third a random element.

The following hypotheses are made concerning the individual parts of the series: (1) the difference between the first component (combined trend and cycle) and its moving twelve-month average is negligible; (2) the mean value of the random component and its moving twelve-month average is approximately zero; (3) the seasonal component can be represented as the product of a periodic function with a period of twelve months and another function which changes only slowly with time; (4) the twelve-month moving average of the seasonal component is approximately zero; (5) the slowly changing function is approximately constant in a short time interval; (6) the slowly changing component may be replaced by an approximation which assumes that it is constant over one year. The approximation is computed by the method of least squares.

The seasonal component is then computed in the following way. First the moving twelve-month averages of all items of the given monthly series are computed. The raw seasonal factor is the average of the difference between the original series and the corresponding twelve-month moving average for the same item. This average is formed for each month. The raw values are corrected so that their sum is equal to zero. This permits then the computation of the series which is approximately free from seasonal fluctuations.

Wald's method of eliminating the seasonal fluctuations has many rivals. But it seems preferable in most cases because it is based upon assumptions which are very reasonable from the point of view of economics. It will certainly serve to eliminate approximately the season if it is changing not too rapidly. But if the season should change very rapidly some more complicated adjustments should be used which are also given by Wald.

**5. The problem of identification.** The question of identification has come into the foreground of econometric discussion since the important pioneer investigations of Haavelmo [8]. The problem is as follows. Suppose we have a system of equations which describes the behavior of the individuals in an economy and may also include definitory relations. The behavior equations include random terms which are the result of certain variables which have not been included in the system. The general form of the joint distribution of these random variables is known. Then we may ask: under what assumptions can we determine uniquely the equations which express the behavior of the individuals in the economy?

In the fundamental work edited by Koopmans we find two contributions of Wald to this subject: a note on identification of economic relations {92}, and a contribution concerning the important problem of the estimation of unknown parameters in an incomplete system of equations {93}.

This field of inquiry is so new that it is not yet very easy to see its permanent importance. There are only very few empirical investigations which use the methods and take care of the problem of identification. But the contributions of Wald are of such generality that they are bound to play an important part in the future development of these ideas.

The importance of the question of identification has been shown very forcefully by Haavelmo [8]. It is of paramount significance for all investigations which deal with the econometric verification of economic theories. But it also has very important implications for economic policy. If measures of policy are based upon statistical information the question of identification cannot be neglected. Hence Wald's contributions are also of great interest to anybody concerned with economic policy.

**6. Stochastic difference equations.** Among the many important contributions of Wald in the field of economic statistics we mention only his treatment of linear stochastic difference equations, since this is of great value for econometrics {57}. He discussed several cases.

Suppose first we have a single linear stochastic difference equation. This is a linear difference equation which includes a random term. We assume that the random terms are independently (but not necessarily normally) distributed. The stochastic process is stationary. This excludes a trend. Then it can be shown that the application of the classical method of least squares leads to consistent estimates of the constants in the difference equation. In the limit, that is for large samples, the estimates are jointly normally distributed. The covariance matrix of the estimates is ascertained for the large-sample case. This permits the testing of hypotheses.

A more complicated case is a system of simultaneous linear stochastic difference equations. The random terms are not autocorrelated and the total system is stationary. If the matrix of the coefficients which involve no time lag is the unit matrix, then the maximum likelihood estimates are shown to follow from an application of the method of least squares to each single equation in the system. The large-sample limiting distribution of the estimates is derived. They are again jointly normally distributed and their covariance matrix is estimated.

It is also shown that in the general case there are no unique estimates. We need a priori restrictions on the coefficients of the system or on the covariance matrix of the random elements.

These contributions are of great importance in practical econometric work. It is to be hoped that they can be amplified and extended and that they will stimulate work leading to the derivation of small-sample distributions in this field. These would be more useful than the large-sample distributions derived by Wald but are much more difficult to obtain.

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