$\beta - n\epsilon > \alpha + n\epsilon$. It then follows by (3), (4), and (5) that if $\mu_1, \cdots, \mu_n$; $\mu_1', \cdots, \mu_n'$ satisfies (1), then $P(u < \delta) > P(v < \delta)$. Hence, by property 4, $f(\mu_1, \cdots, \mu_n) \neq f(\mu_1', \cdots, \mu_n')$.

On the other hand, let $\mu_1, \cdots, \mu_n$; $\mu_1', \cdots, \mu_n'$ be such that

$$V = \sum_{i=1}^{n} (\bar{\mu} - \mu_i)^2 = \sum_{i=1}^{n} (\mu_1' - \bar{\mu}')^2. \tag{6}$$

Suppose $a = f(\mu_1, \cdots, \mu_n)$, $b = f(\mu_1', \cdots, \mu_n')$, and that $a \neq b$. Let C_1 and C_2 be continuous curves joining $(\mu_1, \cdots, \mu_n)$ to $(\mu_1', \cdots, \mu_n')$ such that for every $(\mu_1^{(1)}, \cdots, \mu_n^{(1)}) \varepsilon C_1$, not an end-point, $\sum_{i=1}^{n} (\mu_i^{(1)} - \bar{\mu}^{(1)})^2 < V$, and for every $(\mu_1^{(2)}, \cdots, \mu_n^{(2)}) \varepsilon C_2$, not an end-point, $\sum_{i=1}^{n} (\mu_i^{(2)} - \bar{\mu}^{(2)})^2 > V$. Since $f(\mu_1, \cdots, \mu_n)$ is continuous, there are points $(\mu_1^{(1)}, \cdots, \mu_n^{(1)}) \varepsilon C_1$ and $(\mu_1^{(2)}, \cdots, \mu_n^{(2)}) \varepsilon C_2$ for which

$$f(\mu_1^{(1)}, \cdots, \mu_n^{(1)}) = f(\mu_1^{(2)}, \cdots, \mu_n^{(2)}) = \tfrac{1}{2}(a + b). \tag{7}$$

But (7) contradicts the fact, already established, that $\sum_{i=1}^{n} (\mu_i - \bar{\mu})^2 \neq \sum_{i=1}^{n} (\mu_i' - \bar{\mu}')^2$ implies $f(\mu_1, \cdots, \mu_n) \neq f(\mu_1, \cdots, \mu_n)$. We have now proved that $f(\mu_1, \cdots, \mu_n) = f(\mu_1', \cdots, \mu_n')$ if and only if $\sum_{i=1}^{n} (\mu_i - \bar{\mu})^2 = \sum_{i=1}^{n} (\mu_i' - \bar{\mu}')^2$. But this is simply another way of saying that there is an $F(x)$ such that $F(V) = f(\mu_1, \cdots, \mu_n)$.

Conversely, it is easy to prove that:

If $F(x)$ is continuous, monotonically increasing, and $F(0) = 0$, then $f(\mu_1, \cdots, \mu_n) = F(V)$ has properties (i)–(iv).

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ON MILL'S RATIO FOR THE TYPE III POPULATION

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1. Introduction and summary. Mills [1], Gordon [2], Birnbaum [3], and the author [4] have studied the ratio of the area of the standardized normal curve from x to ∞ and the ordinate at x. The object of this note is to establish the monotonic character of, and to obtain lower and upper bounds for, the ratio of the ordinate of the standardized Type III curve at x and the area of the curve from x to ∞. This ratio, as shown by Cohen [5] and the author [6], has to be calculated for several values of x when solving approximately the equations involved in the problem of estimating the parameters of Type III populations from truncated samples. It was found by the author that, for large values of

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x, when the ordinates and areas are small, either this ratio cannot be obtained from existing tables prepared by Salvosa [7] or that very few significant digits are available for its calculation. It was thus found desirable to obtain lower and upper bounds which could satisfactorily locate this ratio. The monotonic behavior of this ratio and the inequalities obtained may also prove useful in checking the accuracy of the tables in [7], and in studying the nature of the tail of the chi square distribution.

2. Derivations. The standardized Type III population is given by

$$Cf(x)\,dx, \qquad -2/\alpha_3 \leqq x \leqq \infty, \qquad 0 \leqq \alpha_3 \leqq 2, \tag{1}$$

where

$$f(x) = \left\{1 + \frac{\alpha_3}{2}x\right\}^{(4/\alpha_3^2)-1} e^{-(2/\alpha_3)x}, \tag{2}$$

and

$$C = (4/\alpha_3^2)^{(4/\alpha_3^2)-\frac{1}{2}} e^{-4/\alpha_3^2}[\Gamma(4/\alpha_3^2)]^{-1}.$$

We define

$$\mu(x) = f(x)[G(x)]^{-1}, \tag{3}$$

where

$$G(x) = \int_x^\infty f(t)\,dt. \tag{4}$$

We have

$$\mu(-2/\alpha_3) = 0, \qquad \mu(\infty) = 2/\alpha_3,$$

and

$$\frac{d}{dx}\mu(x) = \mu(x)[G(x)]^{-1}\phi_1(x), \tag{5}$$

where

$$\phi_1(x) = f(x) - \left(x + \frac{\alpha_3}{2}\right)\left(1 + \frac{\alpha_3}{2}x\right)^{-1} G(x). \tag{6}$$

Since

$$\phi_1(-2/\alpha_3) = \infty, \qquad \phi_1(\infty) = 0,$$

and

$$\frac{d}{dx}\phi_1(x) = -\left(1 + \frac{\alpha_3}{2}x\right)^{-2}\left(1 - \frac{\alpha_3^2}{4}\right)G(x) \leqq 0,$$

it follows that $\mu(x)$ is monotonically increasing and that

$$\mu(x) \geqq \mu_1(x) = \left(x + \frac{\alpha_3}{2}\right)\left(1 + \frac{\alpha_3}{2}x\right)^{-1}. \tag{7}$$

Again, considering

$$\phi_2(x) = -\left(x + \frac{\alpha_3}{2}\right)\left(1 + \frac{\alpha_3}{2}x\right)^{-1} f(x)$$
$$(8) \qquad + \left[\left(x + \frac{\alpha_3}{2}\right)^2 + \frac{\alpha_3}{2}\left(\frac{2}{\alpha_3} - \frac{\alpha_3}{2}\right)\right]\left(1 + \frac{\alpha_3}{2}x\right)^{-2} G(x),$$

we have

$$\phi_2(-2/\alpha_3) = \infty, \qquad \phi_2(\infty) = 0,$$

TABLE I

Values of $\mu(x)$, $\mu_1(x)$, $\mu_2(x)$, and $\mu_3(x)$

$\alpha_3 = 1.0$

x	$\mu_1(x)$	$\mu(x)$	$\mu_3(x)$	$\mu_2(x)$
−.50	0.000	0.692	0.869	∞
.00	0.500	0.901	1.000	2.000
.50	0.800	1.059	1.117	1.400
1.00	1.000	1.180	1.215	1.333
1.50	1.143	1.275	1.298	1.357
2.00	1.250	1.351	1.366	1.400
2.50	1.330	1.413	1.423	1.444
3.00	1.400	1.464	1.472	1.486
3.50	1.455	1.507	1.513	1.523
4.00	1.500	1.544	1.549	1.556

and

$$\frac{d}{dx}\phi_2(x) = -\alpha_3\left(\frac{2}{\alpha_3} - \frac{\alpha_3}{2}\right)\left(1 + \frac{\alpha_3}{2}x\right)^{-2}[G(x)]^{-1}\left[\mu(x) - x\left(1 + \frac{\alpha_3}{2}x\right)^{-1}\right] \leqq 0,$$

so that

$$(9) \quad \mu(x) \leqq \mu_2 x = \frac{\left(x + \frac{\alpha_3}{2}\right)^2\left(1 + \frac{\alpha_3}{2}x\right)^{-2} + \frac{\alpha_3}{2}\left(\frac{2}{\alpha_3} - \frac{\alpha_3}{2}\right)\left(1 + \frac{\alpha_3}{2}x\right)^{-2}}{\left(x + \frac{\alpha_3}{2}\right)\left(1 + \frac{\alpha_3}{2}x\right)^{-1}},$$

$$x \geqq -\frac{\alpha_3}{2}.$$

Combining (7) and (9) we have the inequalities

$$\mu_1(x) \leqq \mu(x) \leqq \mu_2(x).$$

A better estimate for the upper inequality can be obtained from Jensen's inequality.

$$\phi\left[\int_a^b tg(t)\,dt \Big/ \int_a^b g(t)\,dt\right] \leqq \int_a^b \phi(t)g(t)\,dt \Big/ \int_a^b g(t)\,dt, \tag{10}$$

where $\phi(t)$ is convex and $g(t) \geqq 0$ in (a, b). Setting $a = x$, $b = \infty$, $\phi(t) = (1 + (\alpha_3 t)/2)(t + \alpha_3/2)^{-1}$, and $g(t) = (t + \alpha_3/2)(1 + (\alpha_3 t)/2)^{(4/\alpha_3^2)-2}e^{-2t/\alpha_3}$, in (10), we have

$$-\left(1 + \frac{\alpha_3}{2}x\right)\mu^2(x) + x\mu(x) + 1 \geqq 0, \tag{11}$$

from which it follows that

$$\mu(x) \leqq \mu_3(x) = 2\left[-x + \sqrt{x^2 + 4\left(1 + \frac{\alpha_3}{2}x\right)}\right]^{-1}. \tag{12}$$

As a check on our results, by putting $\alpha_3 = 0$ in (7), (9) and (12), we obtain inequalities given in [2], [3] and [4]. Incidently, the function

$$\phi_3(x) = \left(1 + \frac{\alpha_3}{2}x\right)\mu(x) - x,$$

used by Cohen [5] can be shown to be monotonically decreasing. For,

$$\phi_3(-2/\alpha_3) = 2/\alpha_3, \qquad \phi_3(\infty) = 0,$$

and

$$\frac{d}{dx}\phi_3(x) = \left(1 + \frac{\alpha_3}{2}x\right)\mu^2(x) - x\mu(x) - 1 \leqq 0 \quad \text{from (11)}.$$

The closeness with which these inequalities can locate $\mu(x)$ is illustrated by Table I, where $\mu(x)$ is calculated from the tables in [7].

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