GENERALIZED HIT PROBABILITIES WITH A GAUSSIAN TARGET, II

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1. Summary. In a recent paper [2] the author developed a discrete distribution and several derived limiting distributions for the number of "hits" on a k-dimensional Gaussian target. The purpose of the present paper is to apply these results to the two-dimensional problem considered by Cunningham and Hynd [1]. A general expression and two limiting forms are obtained for the probability of at least one hit. The numerical evaluation using the data in [1] is considered for $n = 5$ rounds, and the probability of at least one hit is plotted in Fig. 1 for various combinations of aiming and dispersion error. For a given over-all time interval the evaluation for large n is discussed in Section 5 and illustrated using the data from [1].

2. Introduction. In 1946 Cunningham and Hynd considered a problem in aerial gunnery: to find the probability of hitting a moving target at least once. The various factors entering into the problem may be described as follows. The point at which the gun is aiming is found to wander back and forth across the target; its successive positions when n rounds are fired can be represented by a multivariate normal distribution with independence between the horizontal and vertical coordinates. We let the coordinates of this point of aim for the ith round be x_{1i}, x_{2i} (the point of aim is called a prediction in [2]). The dispersion error of the gun is also assumed to be normal; we let the trajectory coordinates be y_{1i}, y_{2i}.

In [1] the target was taken to be circular. Here we assume that it is Gaussian diffuse; that is, the probability of a hit is given up to a constant factor by a Gaussian p.d.f. of the coordinates of the trajectory. Because of the irregular outline of a plane and the sharp "drop off" of the p.d.f. proceeding out from the center, this is not an unreasonable assumption.

3. The k-dimensional problem. The k-dimensional problem treated in [2] may be summarized as follows. A series of n predictions $\{\bar{X}_i ; i = 1, \cdots, n\}$ is considered; a prediction $\bar{X}_i$ is a random vector in k-dimensional Euclidean space R^k and we let $\bar{X}_i = \{X_{\mu i} ; \mu = 1, \cdots, k\}$. The distribution of the n predictions is assumed to be Gaussian with independence between the n values of any coordinate and the n values of any other. Letting $\{X_{\mu i} ; i = 1, \cdots, n\}$ be Gaussian with mean $\{m_{\mu i} ; i = 1, \cdots, n\}$ and covariance matrix $\| \sigma_{ij}^{(\mu)} \|$, then $\{X_{\mu i} ; i = 1, \cdots, n\}$ and $\{X_{\nu i} ; i = 1, \cdots, n\}$ are assumed independent for $\mu \neq \nu$. A prediction $\bar{X}_i = \bar{x}_i = (x_{1i}, \cdots, x_{ki})$ becomes a successful prediction with probability given by $s_i(\bar{x}_i)$, the success function. In [2] $s_i(\bar{x}_i)$ has the

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following Gaussian form:

$$s_i(\bar{x}_i) = C_i \exp - \tfrac{1}{2} \sum_{\mu\nu} \tau^{\mu\nu}_{(i)}\, x_{\mu i} x_{\nu i}, \tag{3.1}$$

where $0 \leqq C_i \leqq 1$, $\| \tau^{\mu\nu}_i \|$ is positive definite, and μ, ν range over the set $\{1, \cdots, k\}$. The general distribution obtained in [2] is the distribution of R, the number of successful predictions.

In applying the results obtained in [2] to the Cunningham and Hynd problem, it is found that the success function is not immediately available in terms of the prediction $\bar{x}$; rather, it is given in terms of a vector $\bar{y}$ which has a Gaussian distribution about $\bar{x}$. The following lemma shows that, if the success function is Gaussian of form (3.1) in terms of $\bar{y}$, then it is also Gaussian in terms of $\bar{x}$.

LEMMA. *If $(Y_1, \cdots, Y_k)$ has a Gaussian distribution with mean $(x_1, \cdots, x_k)$ and covariance matrix $\| G_{\mu\nu} \| = \| G^{\mu\nu} \|^{-1}$, and if the success function in terms of $(y_1, \cdots, y_k)$ is $B \exp - \frac{1}{2}\sum_{\mu\nu} T^{\mu\nu} y_\mu y_\nu$, then the success function in terms of $(x_1, \cdots, x_k)$ is also Gaussian diffuse and has the form $C \exp - \frac{1}{2}\sum_{\mu\nu} \tau^{\mu\nu} x_\mu x_\nu$ where*

$$C = B \,|\, \delta_{\mu\nu} \sum_{\mu'} G^{\mu\mu'} T^{\mu'\nu} \,|^{-\frac{1}{2}},$$

$$\tau^{\mu\nu} = G^{\mu\nu} - \sum_{\mu'\nu'} \| G^{\mu'\nu'} + T^{\mu'\nu'} \|^{-1} G^{\mu'\mu} G^{\nu'\nu}.$$

PROOF. We calculate the probability of a "hit" as a function of the point of aim $(x_1, \cdots, x_k)$.

$$\begin{aligned}
s(\bar{x}) &= E\{B \exp -\tfrac{1}{2} \sum_{\mu\nu} T^{\mu\nu} Y_\mu Y_\nu\} \\
&= B \frac{| G^{\mu\nu} |^{\frac{1}{2}}}{(2\pi)^{k/2}} \int \exp \{ -\tfrac{1}{2} \sum_{\mu\nu} G^{\mu\nu}(y_\mu - x_\mu)(y_\nu - x_\nu) - \tfrac{1}{2} \sum_{\mu\nu} T^{\mu\nu} y_\mu y_\nu \} \prod_\mu dy_\mu \\
&= B \frac{| G^{\mu\nu} |^{\frac{1}{2}}}{(2\pi)^{k/2}} \exp \{ -\tfrac{1}{2} \sum_{\mu\nu} G^{\mu\nu} x_\mu x_\nu \} \\
&\qquad \cdot \int \exp \{ -\tfrac{1}{2} \sum_{\mu\nu} (G^{\mu\nu} + T^{\mu\nu}) y_\mu y_\nu + \sum_\mu [\sum_{\mu'} G^{\mu\mu'} x_{\mu'}]\, y_\mu \} \prod_\mu dy_\mu \\
&= B \frac{| G^{\mu\nu} |^{\frac{1}{2}}}{| G^{\mu\nu} + T^{\mu\nu} |^{\frac{1}{2}}} \exp -\tfrac{1}{2} [\sum_{\mu\nu} G^{\mu\nu} x_\mu x_\nu - \sum_{\mu\nu\mu'\nu'} \| G^{\mu'\nu'} + T^{\mu'\nu'} \|^{-1} G^{\mu'\mu} G^{\nu'\nu} x_\mu x_\nu] \\
&= B \,|\, \delta_{\mu\nu} + \sum_{\mu'} G_{\mu\mu'} T^{\mu'\nu} \,|^{-\frac{1}{2}} \\
&\qquad \cdot \exp \{ -\tfrac{1}{2} \sum_{\mu\nu} [G^{\mu\nu} - \sum_{\mu'\nu'} \| G^{\mu'\nu'} + T^{\mu'\nu'} \|^{-1} G^{\mu'\mu} G^{\nu'\nu}] x_\mu x_\nu \}.
\end{aligned}$$

This completes the proof.

4. The Cunningham and Hynd problem. Cunningham and Hynd were interested in the probability P of at least one hit in a series of n rounds. Using

(3.2) in [2] we have the following expression for P:

(4.1) $$P = E_1 - E_2 + \cdots + (-1)^{n+1}E_n,$$

where

(4.2) $$E_r = \sum_{\beta_r} E_{\beta_r}$$

for which the summation is taken over all sets β_r of r integers chosen from the first n integers and E_{β_r} is the probability that all the rounds designated by the elements of β_r will be hits. Our problem is thus to calculate E_r or E_{β_r} and the above formulas will give the desired probability P. Any other probabilities for the distribution of R will be obtained from formula (3.3) in [2].

We now consider the two-dimensional problem [1] and introduce the following notation and formulation.

The target is given by the probability of a hit on the ith round in terms of the trajectory coordinates (with the center of the target as origin).

$$Pr\{\text{Hit} \mid (y_{1i}, y_{2i})\} = \exp\left\{-\tfrac{1}{2}\frac{y_{1i}^2 + y_{2i}^2}{\sigma_{(t)}^2}\right\},$$

where $\sigma_{(t)}$ is a measure of the effective radius of the target. In the notation of the lemma in Section 3, we have

$$B = 1,$$

$$\| T^{\mu\nu} \| = \begin{Vmatrix} \sigma_{(t)}^{-2} & 0 \\ 0 & \sigma_{(t)}^{-2} \end{Vmatrix}.$$

The dispersion error is given by the covariance matrix on the ith round (independence of coordinates being assumed as in [1]).

$$\| G_{\mu\nu}^{(i)} \| = \begin{Vmatrix} \sigma_{1i}^{2(g)} & 0 \\ 0 & \sigma_{2i}^{2(g)} \end{Vmatrix},$$

where $\sigma_{\mu i}^{2(g)}$ is the variance of the μth coordinate of the dispersion error.

The success function is given, using the lemma in Section 3, by the following:

$$C_i = \frac{\sigma^{2(t)}}{\sigma_{1i}^{(t+g)}\,\sigma_{2i}^{(t+g)}},$$

$$\| \tau_{(i)}^{\mu\nu} \| = \begin{Vmatrix} \sigma_{1i}^{-2(t+g)} & 0 \\ 0 & \sigma_{2i}^{-2(t+g)} \end{Vmatrix},$$

where $\sigma_{\mu i}^{2(t+g)} = \sigma^{2(t)} + \sigma_{\mu i}^{2(g)}$, the addition of variances.

The aiming error is given by the variance of that error for μth coordinate on the ith round, $\sigma_{ii}^{(\mu)} = \sigma_{\mu i}^{2(a)}$, and the correlation between the values of the μth coordinate for rounds i and j, $\rho_{ij}^{(\mu)}$. We are assuming that the mean is equal to zero; that is, there is no bias in the aiming.

Using Theorem 3 in [2] we obtain the following expression for E_{β_r}:

$$E_{\beta_r} = \prod_{p \epsilon \beta_r} C_p \prod_{\mu=1,2} | \delta_{pq} + \sigma_{pq}^{(\mu)} \tau_{(p)}^{\mu\mu} |^{-\frac{1}{2}}$$

$$(4.3) \qquad = \prod_{p \epsilon \beta_r} \frac{\sigma^{2(t)}}{\sigma_{1p}^{(t+g)} \sigma_{2p}^{(t+g)}} \prod_{\mu} | \delta_{pq} + \sigma_{\mu p}^{(a)} \sigma_{\mu q}^{(a)} \rho_{pq}^{(\mu)} \sigma_{\mu p}^{-2(t+g)} |^{-\frac{1}{2}}$$

$$= \prod_{p \epsilon \beta_r} \frac{\sigma^{2(t)}}{\sigma_{1p}^{(a)} \sigma_{2p}^{(a)}} \prod_{\mu=1,2} \left| \rho_{pq}^{(\mu)} + \delta_{pq} \frac{\sigma_{\mu p}^{2(t+g)}}{\sigma_{\mu p}^{2(a)}} \right|^{-\frac{1}{2}}.$$

Considering now the application of the Type I limiting distribution, we have the following result:

$$(4.4) \quad E_r = \frac{1}{r!\,T^r} \int_0^T \cdots \int_0^T \prod_{p=1}^{r} \frac{n\sigma^{2(t)}}{\sigma_{1(t_p)}^{(a)} \sigma_{2(t_p)}^{(a)}} \prod_{\mu} \left| p_{(t_p t_q)}^{(\mu)} + \delta_{pq} \frac{\sigma_{\mu(t_p)}^{2(t+g)}}{\sigma_{\mu(t_p)}^{2(a)}} \right|^{-\frac{1}{2}} \prod_{1}^{r} dt_p .$$

The duration of the burst is T, and $\sigma_{1(t_p)}^{2(a)}$, for example, is the variance of the aiming error at time t_p. From the conditions of Theorem 4 in [2], (4.4) will be a valid approximation for (4.2) with (4.3) if $\sigma^{2(t)}$ is of order $1/n$ with respect to $\sigma^{2(g)}$ and n is large.

Similarly the Type II limiting distribution gives the following:

$$(4.5) \qquad E_r = \frac{1}{r!\,T^r} \int_0^T \cdots \int_0^T \prod_{p=1}^{r} \frac{n\sigma^{2(t)}}{\sigma_{1(t_p)}^{(a)} \sigma_{2(t_p)}^{(a)}} \prod_{\mu=1,2} \left| \rho_{(t_p t_q)}^{(\mu)} \right|^{-\frac{1}{2}} \prod_{p=1}^{r} dt_p .$$

The limiting conditions are that $\sigma_{1i}^{2(t+g)}$, $\sigma_{2i}^{2(t+g)}$ be of order $1/n$,

$$\frac{\sigma^{2(t)}}{\sigma_{1i}^{(t+g)} \sigma_{2i}^{(t+g)}}, \qquad \sigma_{1i}^{2(a)}, \text{ and } \sigma_{2i}^{2(a)} \text{ of order 1, and } n \text{ large.}$$

5. Evaluation for $n = 5$. The evaluation of the probability P requires all values of E_r or a sufficient number to estimate P from a truncated portion of the series (4.1). This direct evaluation can readily be carried out if n is small as in the example ($n = 5$) in this section; we defer to Section 6 a consideration of the evaluation for large n.

We use the data in [1] for a one-second burst of $n = 5$ rounds. As indicated in [1] we take the correlation $\rho_{ij}^{(\mu)}$ to be the same for the horizontal and vertical coordinates and to be dependent only on the time interval between the rounds. Then from the experimental correlations tabulated in [1] we have $\rho(0) = 1.00$, $\rho(.25) = .80$, $\rho(.50) = .62$, $\rho(.75) = .48$, $\rho(1.00) = .33$. From formulas (4.1), (4.2), and (4.3) we have that

$$(5.1) \quad \begin{aligned} P = \beta \frac{n}{\alpha} - \beta^2 \sum_{i_1 < i_2} \frac{1}{\alpha^2 - \rho^2(i_2 - i_1)} \\ + \cdots - (-\beta)^r \sum_{i_1 < \cdots < i_r} | \rho(i_p - i_q)(1 - \delta_{pq}) + \alpha\delta_{pq} |^{-1}, \end{aligned}$$

where

$$\beta = \frac{\sigma^{2(t)}}{\sigma^{2(a)}},$$

$$\alpha = 1 + \frac{\sigma^{2(t+g)}}{\sigma^{2(a)}},$$

and the variances $\sigma^{2(t)}$, $\sigma^{2(g)}$, $\sigma^{2(a)}$ are assumed to remain constant for the duration of the burst. The function P was calculated for a series of values of α and β and plotted in Fig. 1 with $\sigma^{(t)} = 1$.

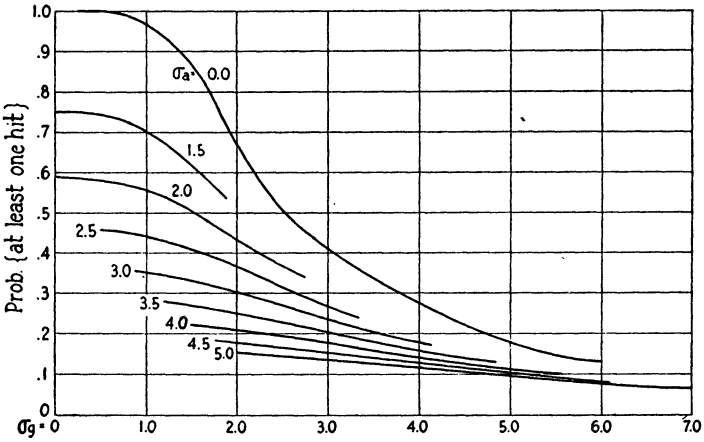

FIG. 1

6. Evaluation for large *n*. The direct computation of P for a series of large values of n would be excessive. We now introduce a procedure for approximation. Assuming that the correlation depends only on the time interval between the rounds, that the horizontal and vertical components have the same distribution, and that variances are constant over the time interval, then P has the form (5.1).

Consider for a moment the case in which correlation is absent ($\rho_{ij} = \delta_{ij}$), and let

$$F_r = E_r\,(\rho_{ij} = \delta_{ij})$$
$$= \binom{n}{r}\left[\frac{\beta}{\alpha}\right]^r.$$

The expression for P also simplifies:

$$P = \beta\frac{n}{\alpha} - \beta^2\binom{n}{2}\frac{1}{\alpha^2} + \cdots - (-\beta)^r\binom{n}{r}\frac{1}{\alpha} + \cdots.$$

This binomial expression has simple terms and it is natural to investigate the correction factor between terms in the correlated and uncorrelated series. We therefore define as follows:

$$E_r = F_r(1 + c_r), \qquad c_r \geqq 0, \qquad c_1 \equiv 0.$$

We now derive an expression for c_r from which it easily follows that c_r is non-negative.

$$c_r = \alpha^r \sum_{i_1 < \dots < i_r} \frac{|\alpha\delta_{pq} + \rho_{pq}(1 - \delta_{pq})|^{-1}}{\binom{n}{r}} - 1$$

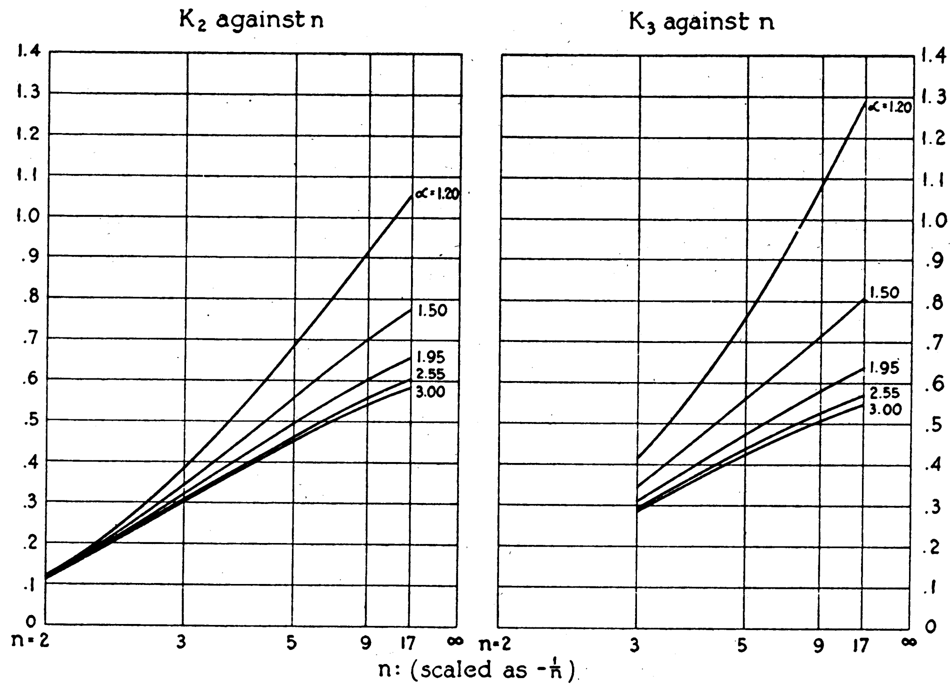

FIG. 2

$$\frac{1}{\binom{n}{r}} \sum_{\beta_r} \left[\frac{1}{\left| \delta_{pq} + \frac{\rho_{pq}}{\alpha}(1 - \delta_{pq}) \right|} - 1 \right].$$

From Theorem 4 in [2] we know that c_r approaches a limit as $n \to \infty$. This can also be seen directly since $\alpha > 1$ ($\alpha = 1$ would imply there was no target!). The limiting value is derived from Theorem 4, as

$$\lim_{n\to\infty} C_r = \frac{1}{T^r} \int_0^T \cdots \int_0^T \left| \delta_{pq} + \frac{\rho(t_p - t_q)}{\alpha}(1 - \delta_{pq}) \right|^{-1} \alpha t_1 \cdots \alpha t_r - 1.$$

Since c_r is stable for large n we now investigate the dependence on r. We note that $\lim_n c_r$ is the excess over 1 of the average value of the reciprocal of a determinant having 1's down the diagonal, and that these 1's are larger by $(\alpha - 1)/\alpha$ than elements which would be sufficient to make the matrix positive definite.

Thus assuming $\alpha > 1$ and expanding the determinant we find that the determinant is $1 - \sum_{p<q}\{\rho(t_p - t_q)/\alpha\}^2 \cdots$, where p, q range over $(i_1, \cdots, i_r)$. Formally taking the reciprocal, we find that the inverse of the determinant is $1 - \sum_{p<q}\{\rho(t_p - t_q)/\alpha\}^2 \cdots$. The excess over 1 of the average of such expressions will have a first term which is the sum of $\binom{r}{2}$ squares of the form

$$[\rho(t_p - t_q)/\alpha]^2.$$

This expression suggests replacing our correction factor c_r by k_r defined by $k_r = \alpha^2 c_r/\binom{r}{2}$. Thus we have $E_r = F_r[1 + \binom{r}{2} k_r/\alpha^2]$. The correction constant k_r has the following properties:

(1) k_r approaches a limit as n increases.

(2) k_r is to the first approximation independent of r and α for α large.

Values of k_2 and k_3 were calculated for a series of n, using the time interval $T = 1$ as in Section 5 and the correlation function $\rho(t)$ given in [1]. These are plotted in Fig. 2. It is to be noted that the approach to a limiting value as n increases seems very regular and k_2 and k_3 are quite similar except for the smaller values of α. This stability of the functions $\{k_r\}$ would facilitate the calculations if P were to be obtained for a series of large values of n.

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