ON SOME THEOREMS IN COMBINATORICS RELATING TO INCOMPLETE BLOCK DESIGNS

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1. Summary. In this paper we have studied certain combinatorial properties of incomplete block designs and efficient necessary conditions for the existence of affine resolvable balanced incomplete block (b.i.b., for abbreviation) designs. Two theorems give combinatorial properties of certain b.i.b. designs. The well known inequality of Fisher between the number of varieties and number of blocks is shown in this paper to hold under very general conditions. An intrinsic characteristic property of symmetrical b.i.b. designs is given in another theorem. In the last two theorems we have deduced efficient necessary conditions for the existence of affine resolvable b.i.b. designs. Besides these there are some minor results. Utilizing the simple yet very fruitful idea of associating an incidence matrix with a design, all the results are deduced with the help of arguments of algebra of matrices and linear equations. The last theorem requires the use of the celebrated four square theorem of Lagrange and a result due to Legendre in number theory.

2. Introduction. An arrangement of objects of v different varieties into b blocks (or sets) in such a way that (i) no two blocks are identical (i.e., contain the same varieties), (ii) a variety occurs at most once in a block, (iii) no block contains all the varieties, may be called an incomplete block design. If an incomplete block design satisfies the further conditions that (iv) all the blocks are of equal size (i.e., every block contains the same number of objects or varieties, say, k) and (v) any pair of varieties occurs together in the same number, say λ (where $\lambda \neq 0$), of blocks it is called a balanced incomplete block design. These designs were introduced by Yates in applied statistics. It easily follows that every variety is replicated (i.e., occurs in the whole design) the same number, say r, of times in a b.i.b. design. For, consider the r_i blocks in which the ith variety occurs. Each of the other varieties must occur together with the ith variety in λ of the r_i blocks considered. The total number of objects in these blocks is therefore $(v - 1)\lambda + r_i$. Also this total number equals $r_i k$, so that

$$r_i = \frac{\lambda(v-1)}{k-1} = r, \tag{2.1}$$

say. Counting the total number of objects in the whole design in two ways we get

$$vr = bk. \tag{2.2}$$

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A b.i.b. design in which $b = v$ is called a symmetrical b.i.b. design. (2.1) and (2.2) are trivial necessary conditions for the existence of a b.i.b. design with the parameters v, b, r, k, λ.

A $v \times b$ matrix A completely characterizing an incomplete block design can be constructed in the following manner. List the varieties in a column and the blocks in a row. Insert 1 or 0 at the intersection of ith row and jth column according as the ith variety occurs in the jth block or not; $i = 1, 2, \cdots v$; $j = 1, 2, \cdots b$. The sum of the elements of a row of an incidence matrix for a b.i.b. design is r while the similar column sum is k. The scalar product of any two row vectors is λ. Hence for a b.i.b. design,

$$AA' = \begin{bmatrix} r & \lambda & \cdots & \lambda \\ \lambda & r & \cdots & \lambda \\ \cdots & \cdots & \cdots & \cdots \\ \lambda & \lambda & \cdots & r \end{bmatrix}. \tag{2.3}$$

$$|AA'| = (r - \lambda)^{v-1}(r + \lambda(v-1)) \tag{2.4}$$

$$|A|^2 = |AA'| = (r-\lambda)^{v-1}r^2 \text{ for a symmetrical b.i.b. design.} \tag{2.5}$$

3. General nature of Fisher's inequality. Fisher's inequality [1] namely, $b \geqq v$, for a b.i.b. design has been proved by several authors by different methods of which [2] may be noted. The following discussion reveals the very general nature of the inequality.

The rank of an arbitrary $v \times b$ matrix A can exceed neither the number of its rows nor the number of its columns and

$$\text{rank } A = \text{rank } A' = \text{rank } (A'A) = \text{rank } (AA'). \tag{3.1}$$

If rank $(AA') = t$ then $t \leqq \min(b, v)$, so that for the inequality $b \geqq v$ to hold it is enough that $|AA'| \neq 0$. Consider, for instance, the matrix A which satisfies the conditions,

(i) the scalar product of any two, without loss of generality, say, of its first c row vectors is λ_1, where $0 < \lambda_1 < \min(r_1, r_2, \cdots, r_c$, and where r_i denotes the square of the length of the ith row vector. Similarly the scalar product of any two of its other $v - c$ row vectors is λ_2 where

$$0 < \lambda_2 < \min(r_{c+1}, r_{c+2}, \cdots, r_v),$$

(ii) the scalar product of any of the first set of c row-vectors with any of the second set of $v - c$ row vectors is λ where $\lambda^2 \leqq \lambda_1\lambda_2$. Then

$$|AA'| = \begin{vmatrix} r_1 & \lambda_1 & \cdots & \lambda_1 & \lambda & \lambda & \cdots & \lambda \\ \lambda_1 & r_2 & \cdots & \lambda_1 & \lambda & \lambda & \cdots & \lambda \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda_1 & \lambda_1 & \cdots & r_c & \lambda & \lambda & \cdots & \lambda \\ \lambda & \lambda & \cdots & \lambda & r_{c+1} & \lambda_2 & \cdots & \lambda_2 \\ \lambda & \lambda & \cdots & \lambda & \lambda_2 & r_{c+2} & \cdots & \lambda_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda & \lambda & \cdots & \lambda & \lambda_2 & \lambda_2 & \cdots & r_v \end{vmatrix} \tag{3.2}$$

To evaluate this determinant subtract the cth row from the previous rows and similarly the vth row from all rows above it and below the cth. Add suitable multiples of the first, second, $\cdots$ $(c - 1)$st column to the cth column so as to make its first $(c - 1)$ elements 0. Similarly reduce to 0 the $(c + 1)$st, $\cdots$, $(v - 1)$st elements of the last column. Thus

$$(3.3)\quad \begin{aligned} | AA' | &= \prod_{i=1}^{c-1} (r_i - \lambda_1) \prod_{i=c+1}^{v-1} (r_i - \lambda_2) \\ &\quad \cdot \{r_c r_v - \lambda^2 + de(\lambda_1\lambda_2 - \lambda^2) + d(r_v\lambda_1 - \lambda^2) + e(r_c\lambda_2 - \lambda^2)\}, \end{aligned}$$

where

$$(3.4)\qquad d = \sum_{i=1}^{c-1} \frac{r_c - \lambda_1}{r_i - \lambda_1}, \qquad e = \sum_{i=1}^{v-c-1} \frac{r_v - \lambda_2}{r_{c+i} - \lambda_2}.$$

Clearly $| AA' | > 0$ and consequently $b \geqq v$. We observe that this is a general result concerning the shape of a matrix when (i) and (ii) are satisfied. In particular, if we suppose A to be an incidence matrix for an incomplete block design (when of course the scalar products and lengths of row-vectors will have obvious interpretations in terms of varieties and blocks), we get the inequality $b \geqq v$ for a more general class of designs from the above result provided we replace (ii) by the condition $\lambda \leqq \min (\lambda_1, \lambda_2)$. Plainly, imposing diverse and more general conditions on A we can ensure that $| AA' | \neq 0$.

Before leaving this topic we note that if l_{ij} is the number of varieties common in the ith and jth blocks, $i = 1, 2, \cdots b$; $j = 1, 2, \cdots b$ in any incomplete block design, then the rank of the matrix (l_{ij}) is equal to the rank of the incidence matrix. Trivially, every $v \times v$ minor of the determinant $| l_{ij} |$, situated symmetrically about the main diagonal is a perfect square. For a b.i.b. design at least one of these minors is a nonzero perfect square since at least one set of v columns of the corresponding incidence matrix is independent.

4. A characteristic property of symmetrical b.i.b. designs. What is the nature of an incomplete block design in which every pair of varieties occurs together in λ blocks and every pair of blocks has λ' varieties common? The answer is given in the following.

THEOREM 1. *If in an incomplete block design every pair of varieties occurs together in λ blocks and any two blocks have λ' common varieties, then the design is a symmetrical b.i.b. design. (In case $\lambda = 1$ we further assume that there are at least two blocks, each containing at least 3 varieties.)*

We observe that not both λ and λ' can be 0. We consider two cases according as $\lambda = 1$ or $\lambda > 1$.

CASE I. $\lambda = 1$. We give an indication of the proof. For this case use the terms "points" and "lines" instead of varieties and blocks.

When $\lambda = 1$, λ' is also 1 as can be seen by considering two intersecting lines. Now, there cannot be any line with only two points on it. For, it is easy to see by drawing a diagram or by considering the incidence matrix that this implies that the system consists of a set of concurrent lines and a transversal, each line except the transversal having only two points on it. But by our assumption we have at least two lines with at least 3 points on each. Hence every line must contain at least 3 points. Thus the system becomes a finite plane projective geometry of Veblen and our theorem is a well known result in that geometry. We notice here that the third postulate of that geometry, namely, every line contains at least 3 points, can be replaced by the postulate that there are at least two lines each containing at least three points.

CASE II. $\lambda > 1$. Let $r_1, r_2, \cdots, r_v$ respectively denote the numbers of replications of the varieties and $k_1, k_2, \cdots, k_b$ respectively be the sizes of the blocks. If A is the incidence matrix of the design we have

$$A \begin{bmatrix} k_1 \\ k_2 \\ \cdot \\ \cdot \\ \cdot \\ k_b \end{bmatrix} = \begin{bmatrix} \lambda(v - 1) + r_1 \\ \lambda(v - 1) + r_2 \\ \cdot \\ \cdot \\ \cdot \\ \lambda(v - 1) + r_v \end{bmatrix}, \tag{4.1}$$

$$A' \begin{bmatrix} r_1 \\ r_2 \\ \cdot \\ \cdot \\ \cdot \\ r_v \end{bmatrix} = \begin{bmatrix} \lambda'(b - 1) + k_1 \\ \lambda'(b - 1) + k_2 \\ \cdot \\ \cdot \\ \cdot \\ \lambda'(b - 1) + k_b \end{bmatrix}, \tag{4.2}$$

and

$$AA' = \begin{bmatrix} r_1 & \lambda & \cdots & \lambda \\ \lambda & r_2 & \cdots & \lambda \\ \cdot & \cdot & \cdot & \cdot \\ \lambda & \lambda & \cdots & r_v \end{bmatrix}, \quad A'A = \begin{bmatrix} k_1 & \lambda' & \cdots & \lambda' \\ \lambda' & k_2 & \cdots & \lambda' \\ \cdot & \cdot & \cdot & \cdot \\ \lambda' & \lambda' & \cdots & k_b \end{bmatrix}. \tag{4.3}$$

Premultiply (4.1) by A' and use the value of $A'A$ from (4.3). We then get

$$\begin{bmatrix} k_1 & \lambda' & \cdots & \lambda' \\ \lambda' & k_2 & \cdots & \lambda' \\ \cdot & \cdot & \cdot & \cdot \\ \lambda' & \lambda' & \cdots & k_b \end{bmatrix} \cdot \begin{bmatrix} k_1 \\ k_2 \\ \cdot \\ \cdot \\ \cdot \\ k_b \end{bmatrix} = A' \begin{bmatrix} \lambda(v - 1) + r_1 \\ \lambda(v - 1) + r_2 \\ \cdot \\ \cdot \\ \cdot \\ \lambda(v - 1) + r_v \end{bmatrix} = \begin{bmatrix} k_1\lambda(v - 1) \\ k_2\lambda(v - 1) \\ \cdot \\ \cdot \\ \cdot \\ k_b\lambda(v - 1) \end{bmatrix} + A' \begin{bmatrix} r_1 \\ r_2 \\ \cdot \\ \cdot \\ \cdot \\ r_v \end{bmatrix}, \tag{4.4}$$

and using (4.2) we obtain

$$(4.5)\qquad \begin{bmatrix} k_1 & \lambda' & \cdots & \lambda' \\ \lambda' & k_2 & \cdots & \lambda' \\ \cdot & \cdot & \cdot & \cdot \\ \lambda' & \lambda' & \cdots & k_b \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ \cdot \\ \cdot \\ \cdot \\ k_b \end{bmatrix} = \begin{bmatrix} k_1\lambda(v-1) \\ k_2\lambda(v-1) \\ \cdot \\ \cdot \\ \cdot \\ k_b\lambda(v-1) \end{bmatrix} + \begin{bmatrix} \lambda'(b-1)+k_1 \\ \lambda'(b-1)+k_2 \\ \cdot \\ \cdot \\ \cdot \\ \lambda'(b-1)+k_b \end{bmatrix}$$

The ith equation from (4.5) is

$$(4.6)\qquad k_i^2 + \lambda' \sum_j k_j - \lambda' k_i = k_i\lambda(v-1) + \lambda'(b-1) + k_i; \quad i = 1, 2, \cdots, b,$$

that is,

$$(4.7)\qquad k_i^2 - k_i(\lambda' + \lambda v - \lambda + 1) + \lambda' m - \lambda'(b-1) = 0; \quad i = 1, 2, \cdots, b$$

where m is the total number of objects in the design. If α, β are the roots of (4.7) then $k_i = \alpha$ or β. We now show that either all k_i's are equal to α or all are equal to β. If possible, let $k_i = \alpha$; $k_j = \beta$. In that case,

$$(4.8)\qquad v \geqq k_i + k_j - \lambda' = \alpha + \beta - \lambda' = \lambda(v-1) + 1$$

which is absurd unless $\lambda = 1$. Hence all the block sizes are equal, to k, say. From (4.1) or from (2.1) we get

$$(4.9)\qquad r_i = \frac{\lambda(v-1)}{k-1} = r(\text{say}), \qquad i = 1, 2, \cdots v.$$

Finally $|AA'| = (r-\lambda)^{v-1}(r + \lambda(v-1)) \neq 0$;

$$|A'A| = (k-\lambda')^{b-1}(k + \lambda'(b-1)) \neq 0.$$

Therefore, rank $(AA') = v$ and rank $(A'A) = b$ and thus $b = v$. From the relation $vr = bk$, it then follows that $r = k$ and from the relations $\lambda(v-1) = r(k-1)$ and $\lambda'(b-1) = k(r-1)$ we get $\lambda = \lambda'$. This proves that the design is a symmetrical b.i.b. design.

5. Two combinatorial properties of certain b.i.b. designs. Let us now consider the following solution of the b.i.b. design $v = 16$, $b = 24$, $r = 9$, $k = 6$, $\lambda = 3$ given by K. N. Bhattacharya [5].

(1, 2, 7, 8, 14, 15)	(3, 5, 7, 8, 11, 13)	(2, 3, 8, 9, 13, 16)
(3, 5, 8, 9, 12, 14)	(1, 6, 7, 9, 12, 13)	(2, 5, 7, 10, 13, 15)
(3, 4, 7, 10, 12, 16)	(3, 4, 6, 13, 14, 15)	(4, 5, 7, 9, 12, 15)
(2, 4, 9, 10, 11, 13)	(3, 6, 7, 10, 11, 14)	(1, 2, 3, 4, 5, 6)
(1, 4, 7, 8, 11, 16)	(2, 4, 8, 10, 12, 14)	(5, 6, 8, 10, 15, 16)
(1, 6, 8, 10, 12, 13)	(1, 2, 3, 11, 12, 15)	(2, 6, 7, 9, 14, 16)
(1, 4, 5, 13, 14, 16)	(2, 5, 6, 11, 12, 16)	(1, 3, 9, 10, 15, 16)
(4, 6, 8, 9, 11, 15)	(1, 5, 9, 10, 11, 14)	(11, 12, 13, 14, 15, 16)

where 1, 2, $\cdots$, 16 denote the 16 varieties. If we compare the numbers of varieties common to the 20th and 24th blocks (counting is done in a vertical way) and other blocks, and similarly for the 6th and 10th blocks, we obtain the following tables.

Block	Number of varieties common with	
	Block 20	Block 24
1–6	2	2
7	3	3
8–10	2	2
11	3	3
12–13	2	2
14–15	3	3
16–19	2	2
21–23	2	2

Block	Number of varieties common with	
	Block 6	Block 10
1–5	2	2
7–9	2	2
11–12	2	2
13	3	1
14–18	2	2
19	1	3
20	2	2
21	3	1
22	1	3
23–24	2	2

The two right-hand columns are identical in the first table. A superficial inspection of the second table reveals that the sum of its two right-hand columns gives a constant. This led the author to conjecture Theorems 2 and 3. The first proofs were deduced from the main theorem given in the doctoral dissertation [6] of Connor. Later on independent proofs were obtained and these are given below.

THEOREM 2. *The necessary and sufficient condition for the existence in a b.i.b. design of two blocks such that the numbers of varieties common to any other block and these two are equal, is that these two blocks contain* $k + \lambda - r$ *common varieties.*

Let two blocks (which we take to be the first and second blocks) contain c common varieties. Let the ith block contain x_i varieties which occur in the first but not in the second block. Similarly it has y_i varieties which occur in the second but not in the first block. Considering the combinations of varieties taken two at a time we have

$$(5.1) \quad \sum_{i=3}^{b} x_i(x_i - 1) = \sum_{i=3}^{b} y_i(y_i - 1) = (k - c)(k - c - 1)(\lambda - 1)$$

$$(5.2) \quad \sum_{i=3}^{b} x_i y_i = (k - c)^2\lambda$$

$$(5.3) \quad \sum_{i=3}^{b} x_i = \sum_{i=3}^{b} y_i = (k - c)(r - 1).$$

Using these we have

$$(5.4) \quad \sum_{i=3}^{b} (x_i - y_i)^2 = 2(k - c)(c + r - \lambda - k).$$

If $c = k + \lambda - r$ we must have $x_i = y_i$ for all i and conversely. Hence the theorem.

THEOREM 3. *The necessary and sufficient condition for the existence in a b.i.b. design of two blocks, such that if any other block has s varieties in common with the first block it has* $2\lambda k/r - s$ *or* $2kr/b - s$ *varieties in common with the second, is that the two blocks have* $r - \lambda - k + 2\lambda k/r$ *or* $2kr/b - k$ *varieties in common, respectively.*

As before, let two blocks have c common varieties. Let the ith block have x_i varieties which occur in the first but not in the second block, y_i varieties which occur both in the first and second blocks, and z_i varieties which occur in the second but not in the first block. Then

$$(5.5) \quad \sum_{i=3}^{b} x_i(x_i - 1) = \sum_{i=3}^{b} z_i(z_i - 1) = (k - c)(k - c - 1)(\lambda - 1)$$

$$(5.6) \quad \sum_{i=3}^{b} x_i z_i = (k - c)^2\lambda$$

$$(5.7) \quad \sum_{i=3}^{b} x_i y_i = \sum_{i=3}^{b} y_i z_i = (k - c)(\lambda - 1)c$$

$$(5.8) \quad \sum_{i=3}^{b} y_i(y_i - 1) = c(c - 1)(\lambda - 2)$$

$$(5.9) \quad \sum_{i=3}^{b} x_i = \sum_{i=3}^{b} z_i = (k - c)(r - 1)$$

$$(5.10) \quad \sum_{i=3}^{b} y_i = c(r - 2).$$

We now form the sum $\sum_{i=3}^{b} (x_i + 2y_i + z_i - w)^2$ where w is the mean of the variable $x_i + 2y_i + z_i$, that is, $w = (2(r - 1)k - 2c)/(b - 2)$. Using the results (5.5) to (5.10) and (2.1) and (2.2) one finds that the cumbersome sum factorizes as

$$(5.11) \quad \frac{2b}{b-2}\left(r - \lambda - k + \frac{2\lambda k}{r} - c\right)\left(c - \frac{2kr}{b} + k\right).$$

If $c = r - \lambda - k + 2\lambda k/r$ we have $x_i + 2y_i + z_i = w = 2\lambda k/r$ for all i. Conversely if $x_i + 2y_i + z_i = 2\lambda k/r$ we must have $c = r - \lambda - k + 2\lambda k/r$ since $w = 2\lambda k/r$ in this case. As $x_i + 2y_i + z_i$ represents the sum of the numbers of varieties common to the ith block and the first and second blocks, the first part of the theorem follows. Similarly for the other part.

As a by-product, we get from (5.4) and (5.11) the inequality

$$\max\left(\frac{2rk}{b} - k, k + \lambda - r\right) \leqq l \leqq r - \lambda - k + \frac{2\lambda k}{r} \tag{5.12}$$

where l is the number of varieties common to any two blocks. This inequality was obtained by Connor [6] by a different method.

6. Necessary conditions for the existence of affine resolvable b.i.b. designs. If in a b.i.b. design the blocks are separable into groups such that the blocks in any group (or "replication") contain between them all the varieties, each variety occurring once and only once in the replication, the design is defined to be a resolvable balanced incomplete block (r.b.i.b.) design. Of course, all the replications necessarily contain the same number of blocks, say n, and then

$$v = nk; \quad b = nr. \tag{6.1}$$

Some properties of r.b.i.b. designs were studied by Bose [3]. These designs are of special importance in analysis of variance. Here we are interested only in their structural properties and will study them through incidence matrices.

In forming the incidence matrix A of a resolvable design we arrange the blocks in such a way that the consecutive columns of A correspond to the blocks in a replication. Thus the first n columns correspond to the n blocks of the first replication, the next n columns to those of the second replication and so on. An important piece of information about A is that on any row, the portion cut off by the n columns corresponding to a replication, contains 1 once and only once. This fact will be fully exploited in Theorems 6 and 7. Since the sum of the n column vectors corresponding to any replication is a column vector of 1's, we get Bose's inequality $b - r + 1 \geqq \text{rank } A = v$ by (2.4), for r.b.i.b. designs. The r.b.i.b. designs for which

$$b = v + r - 1 \tag{6.2}$$

or equivalently,

$$r = k + \lambda \tag{6.3}$$

are called affine r.b.i.b. designs. These possess some properties somewhat analogous to those of symmetrical b.i.b. designs. We give here an alternative proof for such a property due to Bose. Consider the n blocks constituting a replication. Since the number of varieties common between any two of these blocks is $k + \lambda - r$, that is, 0, Theorem 2 can be applied. If therefore any block not belonging to this replication has c varieties common to the first block, it must have c

varieties common to the second and so c varieties common to the third and so on successively. Since the n blocks are disjoint and contain all the varieties

$$nc = k, \quad \text{or,} \quad c = \frac{k}{n} = \frac{k^2}{v}. \tag{6.4}$$

Using (2.1), (6.1) to (6.4) it is not difficult to show that the parameters of an affine r.b.i.b. design can always be expressed in terms of two integral variables as

$$\begin{gathered} v = n(n^2t - nt + n), \qquad b = n(n^2t + n + 1), \qquad r = n^2t + n + 1, \\ k = n^2t - nt + n, \qquad \lambda = nt + 1. \end{gathered} \tag{6.5}$$

From the property in (6.4) it immediately follows that if we choose a block in an affine r.b.i.b. design and break up every other block (not belonging to the replication to which the chosen block belongs) into two subblocks, one containing the k^2/v varieties which occur in the chosen block and the other containing the residual varieties, then the first set of $b - n$ subblocks constitutes the r.b.i.b. design,

$$\begin{gathered} v' = n(nt - t + 1), \qquad b' = n(n^2t + n), \qquad r' = n^2t + n, \\ k' = nt - t + 1, \qquad \lambda' = nt. \end{gathered} \tag{6.6}$$

Though numerous combinatorial structures like b.i.b. designs, finite projective geometries, etc. have been constructed and a large number of sufficient conditions accumulated [4], yet even now very little is known about necessary conditions for their existence. Only recently Bruck, Schützenberger, Chowla, Ryser, Shrikhande, Hall, Mann and others have obtained some important necessary conditions for the existence of finite plane projective geometries, symmetrical b.i.b. designs and cyclic projective planes.

For symmetrical b.i.b. designs the following two theorems are known.

THEOREM 4. *If for given r, λ and even v a symmetrical b.i.b. design exists, then $r - \lambda$ is a perfect square.*

This theorem seems to have been obtained first by Schützenberger [7]. It was independently discovered by Chowla and Ryser [8], and Shrikhande [10]. The proof is almost trivial and follows immediately from (2.5).

THEOREM 5. (Chowla and Ryser). *If for given r, λ and odd v a symmetrical b.i.b. design exists, then*

$$\left(\frac{(-1)^{\frac{1}{2}(v-1)}\lambda}{p}\right) = 1,$$

where p is any odd prime which divides the square-free part of $r - \lambda$ and (m/n) is the Legendre symbol in number theory. In case p divides λ we will have

$$\left(\frac{(-1)^{\frac{1}{2}(v-1)}\lambda_1}{p}\right) = 1, \qquad \text{or} \qquad \left(\frac{(-1)^{\frac{1}{2}(v+1)}\lambda_1 a}{p}\right) = 1$$

according as the highest power of p which divides λ is even or odd. Here λ_1 and a respectively denote the greatest divisors of λ and $r - \lambda$, which are prime to p.

The proof given by Chowla and Ryser in their important paper [8] is remarkably ingenious and simple. By straightforward generalization of Bruck and Ryser's result [9], Shrikhande indepehdently deduced necessary conditions in terms of Hilbert's norm residue symbols. The last sentence in the statement of Theorem 5 does not occur in [8]. It is due to the present author and can be deduced from Chowla-Ryser's proof.

We now establish some necessary conditions for the existence of affine r.b.i.b. designs. Theorems 6 and 7 are precisely analogous to Theorems 4 and 5. These results were obtained some time ago. Shrikhande, working independently on the same problem, has obtained similar results, which he announced without proof in [12]. It may be mentioned here that the proofs given below are direct and self-contained.

THEOREM 6. *If for given b, r, k, λ and odd v an affine r.b.i.b. design exists, then k or k^2/v is a perfect square according as r is odd or even.*

Adjoin $r - 1$ row vectors to the incidence matrix A of the affine r.b.i.b. design so as to get a square matrix A^*. Let the ith of the new row vectors have 1's at the $(i - 1)n + 1$st, $(i - 1)n + 2$th, $\cdots$, $(i - 1)n + n$th positions in the vector while the other positions are occupied by 0's, $i = 1, 2, \cdots, r - 1$. Remembering the special nature of the incidence matrices of r.b.i.b. designs noted above we have

$$(6.7) \qquad A^*A^{*\prime} = \begin{bmatrix} r & \lambda & \cdots & \lambda & 1 & 1 & \cdots & 1 \\ \lambda & r & \cdots & \lambda & 1 & 1 & \cdots & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \lambda & \lambda & \cdots & r & 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 & n & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 & 0 & n & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & n \end{bmatrix}.$$

and

$$(6.8) \qquad \begin{aligned} |A^*|^2 &= |A^*A^{*\prime}| \\ &= (r - \lambda)^{v-1}\left(r - \frac{(r-1)v}{n} + \lambda(v - 1)\right)n^{r-1} = k^{v-r+1}v^{r-1} \end{aligned}$$

as

$$r - (r - 1)\frac{v}{n} + \lambda(v - 1) = r - (r - 1)k + r(k - 1) = k,$$

$$r - \lambda = k, \qquad n = \frac{v}{k}.$$

From (6.8) it follows that k is a perfect square when v and r are odd. When v is odd and r is even, v must equal a perfect square. From (6.4) we know that k^2/v is an integer and so when v is a perfect square, k^2/v is also so. This completes the proof.

As an application of the theorem consider the design $v = 45$, $b = 66$, $r = 22$, $k = 15$, $\lambda = 7$ obtained from (6.5) by putting $n = 3$, $t = 2$. An affine r.b.i.b. design with these parameters cannot exist as v and k^2/v are not perfect squares though v is odd and r is even. Similarly an affine r.b.i.b. design with the parameters $v = 63$, $b = 93$, $r = 31$, $k = 21$, $\lambda = 10$ is impossible. The impossibility of the latter design was demonstrated in [11] by a different method.

THEOREM 7. *If for given v, b, k, λ and $r \equiv 2$ or 3 (mod 4) an affine r.b.i.b. design exists, then every prime of the form $4a + 3$ which divides v/k, occurs to an even exponent in the standard form of v/k (i.e. when v/k is expressed as a product of distinct prime powers).*

Introduce a column vector X of rational variables $x_1, x_2, \cdots, x_{v+r-1}$. Put

$$X'A^* = (u_1, u_2, \cdots, u_{v+r-1}). \tag{6.9}$$

Then from (6.7) we have

$$\sum_{i=1}^{v+r-1} u_i^2 = X'A^*A^{*\prime}X = (r-\lambda)\sum_{i=1}^{v} x_i^2 + n\sum_{i=v+1}^{v+r-1} x_i^2 + \lambda\left(\sum_{i=1}^{v} x_i\right)^2 + 2\left(\sum_{i=1}^{v} x_i\right)\left(\sum_{i=v+1}^{v+r-1} x_i\right), \tag{6.10}$$

that is,

$$\sum_{i=1}^{v+r-1} u_i^2 = k\sum_{i=1}^{v} x_i^2 + n\sum_{i=v+1}^{v+r-1} x_i^2 + \lambda\left(\sum_{i=1}^{v} x_i\right)^2 + 2\left(\sum_{i=1}^{v} x_i\right)\left(\sum_{i=v+1}^{v+r-1} x_i\right). \tag{6.11}$$

CASE I. v odd, $r \equiv 3$ (mod 4). In this case $k = m^2$ where m is an integer by Theorem 6 and so $k\sum_{i=1}^{v} x_i^2 = \sum_{i=1}^{v} (mx_i)^2$. Now by Lagrange's four square theorem every positive integer can be expressed as a sum of four integer squares. So let $n = a^2 + b^2 + c^2 + d^2$ where a, b, c, d, are integers. Then

$$\begin{aligned} n\sum_{i=v+1}^{v+r-1} x_i^2 = n(x_{v+r-2}^2 + x_{v+r-1}^2) + \sum_{i=0}^{\frac{r-7}{4}} &\cdot\{(ax_{v+4i+1} + bx_{v+4i+2} + cx_{v+4i+3} + dx_{v+4i+4})^2 \\ &+ (bx_{v+4i+1} - ax_{v+4i+2} + dx_{v+4i+3} - cx_{v+4i+4})^2 \\ &+ (-cx_{v+4i+1} + dx_{v+4i+2} + ax_{v+4i+3} - bx_{v+4i+4})^2 \\ &+ (dx_{v+4i+1} + cx_{v+4i+2} - bx_{v+4i+3} - ax_{v+4i+4})^2\}. \end{aligned} \tag{6.12}$$

From (6.9), $X' = (u_1, u_2, \cdots, u_{v+r-1})A^{*-1}$. Consequently the x's in (6.11) can be replaced by the u's. Utilizing (6.12) we can write (6.11) as

$$\text{(6.13)} \quad \sum_{i=1}^{v+r-1} u_i^2 = \sum_{i=1}^{v+r-3} y_i^2 + n(y_{v+r-2}^2 + y_{v+r-1}^2) + \lambda y_{v+r}^2 + 2y_{v+r}y_{v+r+1}$$

where the y's are linear forms in the u's with rational coefficients.

Now set $y_1 - u_1 = 0$ if the coefficient of u_1 in y_1 is not one, otherwise set $y_1 + u_1 = 0$. Next set $y_2 - u_2 = 0$ if on elimination of u_1 from it by means of the first equation the coefficient of u_2 in the resulting expression does not vanish; otherwise put $y_2 + u_2 = 0$. Similarly set $y_3 - u_3 = 0$ if the coefficient of u_3 in it after the elimination of u_1 and u_2 by using the first two equations, is not 0, otherwise put $y_3 + u_3 = 0$. Proceed in this way up to the $v + r - 3$rd stage. Finally, eliminate $u_1, u_2, \cdots, u_{v+r-3}$ from $y_{v+r} = 0$ using the other equations. The resulting equations are equivalent to a system of the following form:

$$\text{(6.14)} \quad \begin{aligned} a_{11}u_1 + a_{12}u_2 + , \cdots , + a_{1v+r-1}u_{v+r-1} &= 0 \\ a_{22}u_2 + , \cdots , + a_{2v+r-1}u_{v+r-1} &= 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ a_{tt}u_t + a_{tt+1}u_{t+1} &= 0, \end{aligned}$$

where $t = v + r - 2$ and $a_{ii} \neq 0$, $i = 1, 2, \cdots, t - 1$. A little reflection on (6.14) shows that there exist integral solutions $(u_1, u_2, \cdots, u_{v+r-1})$ for which at least one of u_{v+r-2} and u_{v+r-1} is a nonzero integer. Thus, $y_i^2 = u_i^2$, $i = 1, 2, \cdots, v + r - 3$, and $y_{v+r} = 0$. Consequently, with this choice of the u's and because of the homogeneity of the relation (6.13) we arrive at the nontrivial relation,

$$\text{(6.15)} \quad u_{v+r-2}^2 + u_{v+r-1}^2 = n(p^2 + q^2)$$

where all the quantities are integral.

Case II. v even, $r \equiv 3 \pmod 4$. Considering various combinations of n, t (mod 4) it is seen from (6.5) that $v \equiv 0 \pmod 4$ for this case. Express $k \sum_{i=1}^{v} x_i^2$ and $n \sum_{i=v+1}^{v+r-1} x_i^2$ as sums of squares of linear forms in the x's as in Case I and proceed as before. Ultimately we get a nontrivial relation exactly similar to (6.15).

Case III. $r \equiv 2 \pmod 4$. From (6.5) it follows that $v \equiv 1 \pmod 4$. By Theorem 6, $kn = s^2$ where s is an integer. Multiply (6.11) throughout by n and express $n \sum_{i=1}^{v+r-3} u_i^2$ as a sum of squares of $v + r - 3$ linear forms y_i in the u's (i.e., in the x's by (6.9)). We then get

$$\text{(6.16)} \quad \begin{aligned} \sum_{i=1}^{v+r-3} y_i^2 + n(u_{v+r-2}^2 + u_{v+r-1}^2) &= \sum_{i=1}^{v} (sx_i)^2 + \sum_{i=v+1}^{v+r-1} (nx_i)^2 \\ &+ \lambda n \left(\sum_{i=1}^{v} x_i\right)^2 + 2n \left(\sum_{i=1}^{v} x_i\right)\left(\sum_{i=v+1}^{v+r-1} x_i\right). \end{aligned}$$

Choosing x's such that $y_i = \pm s x_i$; $i = 1, 2, \cdots, v$, $y_i = \pm n x_i$; $i = v + 1, v + 2, \cdots, v + r - 3$ and $\sum_{i=1}^{v} x_i = 0$ exactly as in Case I we obtain a nontrivial relation

$$n(u_{v+r-2}^2 + u_{v+r-1}^2) = p_1^2 + q_1^2 \tag{6.17}$$

where all the quantities are integers.

So for all the cases it follows that n can be expressed as the sum of two integer squares. An appeal to a result of Legendre in number theory completes the proof.

As an application let us consider a design with the parameters $v = 216, b = 258, r = 43, k = 36, \lambda = 7$. An affine resolvable b.i.b. design with these parameters cannot exist by Theorem 7 since here $v/k = 2 \cdot 3$. An affine r.b.i.b. design corresponding to $n = 21, t = 1$ in (6.5) cannot exist—this is decided by Theorem 7 though the parameters satisfy the condition of Theorem 6. Manifestly these two theorems are independent though in many cases both lead to the same conclusion. Finally we observe that the powerful and general theorem of Minkowski and Hasse [9] on the arithmetical reduction of quadratic forms, when applied to (6.7) in conjunction with Theorem 6, gives us Theorem 7 and nothing more.

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