SOME THEOREMS FOR PARTIALLY BALANCED DESIGNS

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Summary. This paper generalizes certain results which are known for balanced incomplete block designs and group divisible designs to partially balanced incomplete block (PBIB) designs with m associate classes. Some of the results are for general m but others are for $m = 2, 3,$ or 4.

1. Introduction. Let N be the incidence matrix of a PBIB design with m associate classes. Then the determinant $| NN' |$ may be written as

$$| NN' | = rk(r - z_1)^{\alpha_1} \cdots (r - z_t)^{\alpha_t}, \qquad \textstyle\sum_{u=1}^{t} \alpha_u = v - 1, \qquad t \leqq m,$$

where the z's are distinct, $r - z_u$, $(u = 1, \cdots, t)$, are factors of $| NN' |$, and α_u, $(u = 1, \cdots, t)$, are their respective multiplicities. For any m the factors, and for $m = 2, 3$, and 4 the multiplicities, are expressed in terms of the parameters of the design.

For m general it is observed for $v > b$ that $| NN' |$ is zero, which implies that one of the factors is zero, a slight modification of a condition of Nair [10]; and for $v = b$ that $| NN' |$ is an integral square, a generalization of Shrikhande's [11] and Chowla and Ryser's [7] result for balanced incomplete block designs, and of Bose and Connor's result for group divisible designs [3].

The special case $m = 2$ is studied at length, with calculation of $| NN' |$ for group divisible designs, which was first done in [3], triangular designs, and Latin square designs with i constraints. Corollaries to the general theorems mentioned in the preceding paragraph are stated in detail, and several necessary conditions for v even and odd are developed from consideration of the integral nature of the α's. These latter theorems are very useful in showing that certain sets of parameters which satisfy the necessary conditions given by Bose and Nair [4], and quoted in (2.2) below, do not correspond to constructible designs.

For general m, lower bounds are developed for b, a generalization of Fisher's work for balanced incomplete block designs [9], and of the bounds for group divisible designs [3]. Also, it is shown for any m that the factors of $| NN' |$ are nonnegative, which is obvious for balanced incomplete block designs and was shown for group divisible designs in [3].

2. The definition of a PBIB design. A PBIB design with m associate classes has been defined by Bose and Shimamoto [5] substantially as follows:

A PBIB design with m associate classes $[m \geqq 1]$ is an arrangement of v treatments (varieties) in b blocks of k experimental units (plots) each such that:

(i) Each of the v treatments is replicated r times, and no treatment appears more than once in any block.

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(ii) There exists a relationship of association between every pair of the v treatments satisfying the following conditions:

(a) Any two treatments are either first, second, $\cdots$, or mth associates, and any pair of treatments which are sth associates occur together in exactly λ_s blocks ($s = 1, 2, \cdots, m$).

(b) Each treatment has n_s sth associates.

(c) For any pair of treatments which are sth associates the number of treatments which are simultaneously jth associates of the first and uth associates of the second is p^s_{ju} and this number is independent of the pair of treatments with which we start. Furthermore $p^s_{ju} = p^s_{uj}$ ($j \neq u$; $s, j, u = 1, 2, \cdots, m$).

It is known that the following conditions are satisfied by the parameters $v, b, r, k, \lambda_1, \lambda_2, \cdots, \lambda_m, n_1, n_2, \cdots, n_m, p^s_{ju}$, $(s, j, u = 1, 2, \cdots, m)$, of the design:

$$
\begin{aligned}
&vr = bk, \\
&v - 1 = \sum_{s=1}^{m} n_s, \qquad r(k-1) = \sum_{s=1}^{m} n_s\lambda_s, \\
&\sum_{s=1}^{m} p^i_{js} = \begin{cases} n_i - 1 & \text{if } i = j \\ n_j & \text{if } i \neq j, \end{cases} \\
&n_i p^i_{ju} = n_j p^j_{iu} = n_u p^u_{ji} \qquad (i, j, u = 1, 2, \cdots, m).
\end{aligned} \tag{2.1}
$$

If $m = 2$, then clearly

$$
\begin{aligned}
&vr = bk, \\
&v - 1 = n_1 + n_2, \qquad r(k-1) = n_1\lambda_1 + n_2\lambda_2, \\
&p^1_{11} + p^1_{12} + 1 = p^2_{11} + p^2_{12} = n_1, \\
&p^1_{21} + p^1_{22} = p^2_{21} + p^2_{22} + 1 = n_2, \\
&n_1 p^1_{12} = n_2 p^2_{11} \text{ and } n_1 p^1_{22} = n_2 p^2_{12}.
\end{aligned} \tag{2.2}
$$

When $m = 2$, we shall require that $\lambda_1 \neq \lambda_2$, for if $\lambda_1 = \lambda_2$ the design becomes a balanced incomplete block design which we do not wish to consider.

3. The value of $|NN'|$ for the general case. Consider the incidence matrix N of the general PBIB design, that is,

$$
N = \begin{bmatrix} n_{11} & n_{12} & \cdots & n_{1b} \\ n_{21} & n_{22} & \cdots & n_{2b} \\ \cdots & \cdots & \cdots & \cdots \\ n_{v1} & n_{v2} & \cdots & n_{vb} \end{bmatrix}, \tag{3.1}
$$

where the rows represent treatments, the columns represent blocks, and $n_{ij} = 1$ or 0 according as the ith treatment ($i = 1, 2, \cdots, v$) does or does not occur in the jth block ($j = 1, 2, \cdots, b$). Since every treatment is replicated r times,

$$
\textstyle\sum_{j=1}^{b} n^2_{ij} = r \qquad (i = 1, 2, \cdots, v), \tag{3.2}
$$

and since every treatment must occur in λ_s blocks with each of its sth associates ($s = 1, 2, \cdots, m$), if treatments i and u are sth associates, then

$$\sum_{j=1}^{b} n_{ij} n_{uj} = \lambda_s \qquad (i \neq u;\ i, u = 1, 2, \cdots, v). \tag{3.3}$$

Hence the elements of the symmetric matrix NN' are r in the principal diagonal and λ_s's elsewhere.

We now wish to evaluate $| NN' |$. Since for a particular design r is fixed, and in our context we wish to determine $| NN' |$ for all r, it is convenient to consider the symmetric matrix M which is obtained from NN' by replacing r with the variable z. The determinant $| M |$ may be regarded as a polynomial of the vth degree in z. We shall determine the zeros of this polynomial and thereby the factors of $| M |$. We observe that the ith row (and column) of M contains the element z in the position of the main diagonal and by Section 2 the other $v - 1$ positions of each row (and column) are occupied by n_1 λ_1's, n_2 λ_2's, $\cdots$, and n_m λ_m's. Hence if we add rows $2, 3, \cdots, v$ to the first row of $| M |$, then the elements of the first row are all

$$z + \sum_{s=1}^{m} n_s \lambda_s, \tag{3.4}$$

which we may factor out of the first row. Thus one zero of $| M |$ is

$$z_0 = -\sum_{s=1}^{m} n_s \lambda_s, \tag{3.5}$$

and therefore a factor of $| M |$ is $z - z_0$.

We next consider the problem of finding the zeros of $| M |$ from a different point of view. Let X be the column vector $[x_1, x_2, \cdots, x_v]$. Then by a well known theorem from algebra, for

$$| M | = 0, \tag{3.6}$$

it is necessary and sufficient that

$$MX = 0 \tag{3.7}$$

have a solution other than $(0, 0, \cdots, 0)$. We shall seek the v linearly independent nonnull solutions, to each of which there corresponds a zero of $| M |$.

Nair [10] and Bose [2] have shown when $v > b$ that $| A | = 0$, where A is defined by (3.12) and (3.14) below, and we shall parallel Bose's argument. If $[x_1, x_2, \cdots, x_v]$ is a nontrivial solution, then by adding the v equations in (3.7) we get

$$(z - z_0) \sum_{i=1}^{v} x_i = 0. \tag{3.8}$$

Hence for $z \neq z_0$,

$$\sum_{i=1}^{v} x_i = 0, \tag{3.9}$$

where x_i corresponds to the treatment i. The excluded solution is $X_0 = [c, c, \cdots, c]$, where c is arbitrary and X_0 corresponds to z_0.

Let us denote by $S_s(x_i)$ the sum of the variables of the sth associates of the ith treatment. Then the v equations of (3.7) may be written as

$$zx_i + \sum_{s=1}^{m} \lambda_s S_s(x_i) = 0, \qquad (i = 1, 2, \cdots, v). \tag{3.10}$$

Sum the equations (3.10) over the sth associates of the ith treatment and use the definition of a PBIB design and (3.9) to obtain

$$\begin{aligned}
&\{\lambda_1 p_{s1}^1 + \lambda_2 p_{s2}^1 + \cdots + \lambda_m p_{sm}^1 - n_s\lambda_s\} S_1(x_i) \\
&\quad + \cdots \\
&\quad + \{z + \lambda_1 p_{s1}^s + \lambda_2 p_{s2}^s + \cdots + \lambda_m p_{sm}^s - n_s\lambda_s\} S_s(x_i) \\
&\quad + \cdots \\
&\quad + \{\lambda_1 p_{s1}^m + \lambda_2 p_{s2}^m + \cdots + \lambda_m p_{sm}^m - n_s\lambda_s\} S_m(x_i) = 0
\end{aligned} \tag{3.11}$$

where $s = 1, 2, \cdots, m$.

Let us set

$$a_{su} = \lambda_1 p_{s1}^u + \lambda_2 p_{s2}^u + \cdots + \lambda_m p_{sm}^u - \lambda_s n_s, \qquad s \neq u, \tag{3.12}$$

and

$$a_{ss} = z + \lambda_1 p_{s1}^s + \lambda_2 p_{s2}^s + \cdots + \lambda_m p_{sm}^s - \lambda_s n_s.$$

Then the equations (3.11) are

$$\begin{aligned}
a_{11} S_1(x_i) + a_{12} S_2(x_i) + \cdots + a_{1m} S_m(x_i) &= 0 \\
a_{21} S_1(x_i) + a_{22} S_2(x_i) + \cdots + a_{2m} S_m(x_i) &= 0 \\
\cdots & \\
a_{m1} S_1(x_i) + a_{m2} S_2(x_i) + \cdots + a_{mm} S_m(x_i) &= 0.
\end{aligned} \tag{3.13}$$

Without loss of generality we can assume that $x_i \neq 0$. Hence $S_s(x_i)$, $(s = 1, 2, \cdots, m)$, are not all zero, since we have $x_i + \sum_{i=1}^{m} S_s(x_i) = 0$. This can happen if and only if

$$|A| = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \cdots & \cdots & \cdots \\ a_{m1} & \cdots & a_{mm} \end{bmatrix} = 0. \tag{3.14}$$

From (3.10) it is clear that (3.14) is necessary and sufficient for a nonnull solution other than X_0 of (3.7). Hence the

LEMMA 3.1. *The distinct zeros of* $|M|$ *are* z_0 *and the distinct zeros of* $|A|$, *so that*

$$|M| = (z - z_0)(z - z_1)^{\alpha_1}(z - z_2)^{\alpha_2} \cdots (z - z_t)^{\alpha_t} \tag{3.15}$$

where $z_1, z_2, \cdots, z_t$ $(t \leq m)$ *are the distinct zeros of* $|A|$ *of* (3.14), *and*

$$\sum_{u=1}^{t} \alpha_u = v - 1, \qquad (\alpha_u > 0). \tag{3.16}$$

When $z = r$, the definition of a PBIB design (Section 2) is satisfied. Further, $M = NN'$ and, by (2.1) and (3.5), $z - z_0 = rk$. Paraphrasing Lemma 3.1, we have the following theorem.

THEOREM 3.1. *For a PBIB design with m associate classes,*

$$(3.17) \qquad |NN'| = rk(r - z_1)^{\alpha_1}(r - z_2)^{\alpha_2} \cdots (r - z_t)^{\alpha_t},$$

where $z_1, z_2, \cdots, z_t$ ($t \leqq m$) are the distinct zeros of $|A|$ of (3.14) and $\sum_{u=1}^{t} \alpha_u = v - 1$, where α_u ($u = 1, 2, \cdots, t$) is a positive integer.

If the PBIB design is symmetrical, that is, $v = b$ (or equivalently, $r = k$), then

$$(3.18) \qquad |NN'| = |N|^2,$$

which must be an integral square since all of the elements of N are integers. Noting that $rk = r^2$, we obtain the following corollary.

COROLLARY 3.1.1. *For a symmetrical PBIB design with m associate classes, it is necessary that*

$$(3.19) \qquad |NN'| = r^2(r - z_1)^{\alpha_1}(r - z_2)^{\alpha_2} \cdots (r - z_t)^{\alpha_t}$$

be an integral square.

If $v > b$ (i.e., $r < k$), then as has been pointed out by Nair [10] and Bose [2], $|NN'| = 0$, so that by (3.17),

$$(3.20) \qquad rk(r - z_1)^{\alpha_1}(r - z_2)^{\alpha_2} \cdots (r - z_t)^{\alpha_t} = 0.$$

Since $r \neq 0$ and $k \neq 0$, it is necessary that r be equal to one of $z_1, z_2, \cdots, z_t$. Hence the following corollary to Theorem 3.1.

COROLLARY 3.1.2. *For a PBIB design in which $v > b$, it is necessary that*

$$|NN'| = rk(r - z_1)^{\alpha_1}(r - z_2)^{\alpha_2} \cdots (r - z_t)^{\alpha_t} = 0,$$

so that r is equal to one of $z_1, z_2, \cdots, z_t$.

In Sections 5 and 6, $\alpha_1, \alpha_2, \cdots, \alpha_t$ will be determined as functions of $z_1, z_2, \cdots, z_t$ ($t \leqq m$) for $m = 2, 3$, and 4. In the next section, for m general, we shall develop lower bounds for b, and shall show that $r - z_u$ ($u = 1, \cdots, t$) is nonnegative.

4. Lower bounds for b and the nonnegativeness of the factors of $|NN'|$. Lower bounds for the number of blocks in group divisible designs were developed in [3]. It was also shown for group divisible designs that the factors $r - z_1$ and $r - z_2$ of $|NN'|$ cannot be negative. In this section we shall extend these results to PBIB designs with m associate classes.

Since M is symmetric, there exists an orthogonal matrix C such that $C'MC$ is the diagonal matrix which has as elements the roots of the secular equation

$$(4.1) \qquad |M - yI| = 0,$$

where C' is the transpose of C and I is the identity matrix of order v [8]. But the roots of $|M| = 0$ are $z_0, z_1, \cdots, z_t$ with multiplicities $1, \alpha_1, \cdots, \alpha_t$ respectively, so that the roots of (4.1) must be $y_0 = z - z_0$, $y_1 = z - z_1$, $\cdots$, $y_t = z - z_t$ with the same respective multiplicities.

Since C is nonsingular, M and $C'MC$ have the same rank, a fact which is useful in obtaining lower bounds for b. Thus, if $z \neq z_i$, $(i = 0, 1, 2, \cdots, t)$, then

$$\text{Rank } M = v, \tag{4.2}$$

but if $z = z_i$, $(i = 0, 1, 2, \cdots, \text{or } t)$, then

$$\text{Rank } M = v - \alpha_i . \tag{4.3}$$

Now for $z = r$, the definition of a PBIB design (Section 2) is satisfied, $NN' = M$, and it is clear that

$$b \geqq \text{Rank } N \geqq \text{Rank } NN' = \text{Rank } M. \tag{4.4}$$

If $r \neq z_u$ $(u = 1, 2, \cdots, t)$, then from (4.2) and (4.4) we obtain

$$b \geqq v, \tag{4.5}$$

and if $r = z_u$, $(u = 1, 2, \cdots, \text{or } t)$, then we obtain

$$b \geqq v - \alpha_u . \tag{4.6}$$

We summarize in the following theorem.

THEOREM 4.1. *For a PBIB design with m associate classes, if $r \neq z_u$, $(u = 1, 2, \cdots, t)$, then $b \geqq v$, but if $r = z_u$, $(u = 1, 2, \cdots, \text{or } t)$, then $b \geqq v - \alpha_u$.*

If the design is resolvable (i.e., consists of r sets of b/r blocks each, b/r an integer, where a set of blocks contains every treatment once each), then these inequalities may be improved. In this case the columns of N may be arranged in r sets of b/r columns each, where a set of columns is such that 1 occurs once and only once in each row of the set. By adding the second, third, $\cdots$, and (b/r)th columns to the first column of a set we obtain a column with all 1's. Since there are r sets, it is clear that

$$b - (r - 1) \geqq \text{Rank } N. \tag{4.7}$$

Using (4.7) with (4.2), we obtain

$$b \geqq v + r - 1, \tag{4.8}$$

when $r \neq z_u$, $(u = 1, 2, \cdots, t)$, and (4.7) with (4.3) we obtain

$$b \geqq v - \alpha_u + r - 1, \tag{4.9}$$

when $r = z_u$. We summarize these results in the following theorem.

THEOREM 4.2. *For a resolvable PBIB design with m associate classes, if $r \neq z_u$, $(u = 1, 2, \cdots, t)$, then $b \geqq v + r + 1$, but if $r = z_u$, $(u = 1, 2, \cdots, \text{or } t)$, then $b \geqq v - \alpha_u + r - 1$.*

We next show that $r - z_u$, $(u = 1, 2, \cdots, t)$, is nonnegative. Let $z = r$, so that $NN' = M$, and suppose that N exists satisfying the definition of a PBIB design. Since $M = NN'$ is nonnegative, and since the transformation matrix C does not alter this property, the roots of (4.1) are nonnegative. We have proved the following theorem.

THEOREM 4.3. *For a PBIB design with m associate classes, $r \geqq z_u$, ($u = 1, 2, \cdots, t$).*

5. Partially balanced designs with two associate classes. In this section we shall treat partially balanced designs with two associate classes ($m = 2$). From (3.12) and (3.14) it is seen that

$$(5.1) \quad |A| = \begin{vmatrix} z + \lambda_1 p_{11}^1 + \lambda_2 p_{12}^1 - \lambda_1 n_1 & \lambda_1 p_{11}^2 + \lambda_2 p_{12}^2 - \lambda_1 n_1 \\ \lambda_1 p_{21}^1 + \lambda_2 p_{22}^1 - \lambda_2 n_2 & z + \lambda_1 p_{21}^2 + \lambda_2 p_{22}^2 - \lambda_2 n_2 \end{vmatrix} = 0.$$

By use of (2.2) we may express $|A|$ in terms of z, λ_1, λ_2, p_{12}^1 and p_{12}^2. After adding the second row of determinant $|A|$ to the first row, expanding, and collecting terms according to powers of z, we obtain

$$(5.2) \quad \begin{aligned} |A| = z^2 &+ [(\lambda_1 - \lambda_2)(p_{12}^2 - p_{12}^1) - (\lambda_1 + \lambda_2)]z \\ &+ [(\lambda_1 - \lambda_2)(\lambda_2 p_{12}^1 - \lambda_1 p_{12}^2) + \lambda_1\lambda_2] = 0. \end{aligned}$$

If we let

$$(5.3) \quad \gamma = p_{12}^2 - p_{12}^1, \qquad \beta = p_{12}^1 + p_{12}^2$$

and

$$\Delta = \gamma^2 + 2\beta + 1,$$

then the roots of (5.2) are

$$(5.4) \quad z_u = \tfrac{1}{2}[(\lambda_1 - \lambda_2)(-\gamma + (-)^u\sqrt{\Delta}) + (\lambda_1 + \lambda_2)], \qquad (u = 1, 2).$$

We observe that

(a) $\Delta > 0$ so that $z_1 \neq z_2$, and

(b) $z_1 < z_2$ if $\lambda_1 > \lambda_2 \geqq 0$, but $z_1 > z_2$ if $0 \leqq \lambda_1 < \lambda_2$.

By (a), $t = m = 2$ in (3.17).

Let us next determine the exponents, α_1 and α_2, of (3.17) in terms of the roots, z_1 and z_2, of $|A| = 0$. When $m = 2$,

$$(5.5) \quad |M| = (z - z_0)(z - z_1)^{\alpha_1}(z - z_2)^{\alpha_2},$$

where

$$(5.6) \quad \alpha_1 + \alpha_2 = v - 1.$$

Expanding the factors of $|M|$ and collecting the coefficients of the powers of z, we obtain from (5.5),

$$(5.7) \quad |M| = z^v - (z_0 + \alpha_1 z_1 + \alpha_2 z_2)z^{v-1} + \cdots + (-z_0)(-z_1)^{\alpha_1}(-z_2)^{\alpha_2}.$$

Again, expanding $|M|$ by its diagonal elements [1], we see that the coefficient of z^{v-1} is zero. Hence from (5.7)

$$(5.8) \quad z_1\alpha_1 + z_2\alpha_2 = -z_0.$$

Solving (5.6) and (5.8) simultaneously and using (5.4), we obtain

$$\begin{aligned} \alpha_1 &= [vz_2 + (z_0 - z_2)]/(z_2 - z_1) \\ &= [(v-1)(-\gamma + \sqrt{\Delta} + 1) - 2n_1]/2\sqrt{\Delta} \end{aligned} \tag{5.9}$$

and

$$\begin{aligned} \alpha_2 &= [vz_1 + (z_0 - z_1)]/(z_1 - z_2) \\ &= [(v-1)(\gamma + \sqrt{\Delta} + 1) - 2n_2]/2\sqrt{\Delta}. \end{aligned} \tag{5.10}$$

When $z = r$, the definition of a PBIB design is satisfied and $M = NN'$. We thus have the following theorem which is a special case of Theorem 3.1.

THEOREM 5.1. *For a PBIB design with two associate classes, it is necessary that*

$$|NN'| = rk(r - z_1)^{\alpha_1}(r - z_2)^{\alpha_2}, \qquad \alpha_1 + \alpha_2 = v - 1, \tag{5.11}$$

where z_1 and z_2 are given by (5.4), *and α_1 and α_2 are given by* (5.9) *and* (5.10). *Furthermore, z_1 and z_2 are distinct and α_1 and α_2 are positive integers.*

The positive integral condition on α_1 and α_2 ($\alpha_1 + \alpha_2 = v - 1$) is useful in showing that some sets of parameters which satisfy the necessary conditions (2.2) have no solutions. From (5.9) and (5.10) it is seen that α_1 and α_2 depend only upon the parameters n_1, n_2, p_{12}^1, and p_{12}^2 of the design. Useful computational formulas for α_1 and α_2 are provided by (5.9) and (5.10).

We now have a special case of Corollary 3.1.1.

COROLLARY 5.1.1. *For a symmetrical PBIB design with two associate classes, it is necessary that*

$$|NN'| = r^2(r - z_1)^{\alpha_1}(r - z_2)^{\alpha_2}, \qquad \alpha_1 + \alpha_2 = v - 1, \tag{5.12}$$

be an integral square.

When $v > b$, $r = z_u$, ($u = 1$ or 2), so that by (5.4), Δ is an integral square. Hence we have a special case of Corollary 3.1.2.

COROLLARY 5.1.2. *For a PBIB design with two associate classes and $v > b$, it is necessary that*

(a) $$|NN'| = rk(r - z_1)^{\alpha_1}(r - z_2)^{\alpha_2} = 0,$$

so that either $r = z_1$, or $r = z_2$, and

(b) *Δ be an integral square.*

We shall next prove corollaries for three special types of partially balanced designs with two associate classes. The first and perhaps most important of these types is known as the group divisible design which has been rather fully developed by Bose and Shimamoto [5], Bose and Connor [3], and Bose, Shrikhande, and Bhattacharya [6]. For these designs

$$\begin{aligned} v = mn, \qquad n_1 = n - 1, \qquad n_2 &= n(m-1), \\ (n-1)\lambda_1 + n(m-1)\lambda_2 &= r(k-1), \\ p_{12}^1 = 0, \qquad p_{12}^2 &= n - 1, \end{aligned} \tag{5.13}$$

where m and n are positive integers not less than 2. By (5.4), (5.9), and (5.10)

$$z_1 = -n(\lambda_1 - \lambda_2) + \lambda_1, \qquad z_2 = \lambda_1 \tag{5.14}$$

$$\alpha_1 = m - 1, \quad \text{and} \quad \alpha_2 = m(n - 1), \tag{5.15}$$

so that we have the following corollary.

COROLLARY 5.1.3. *For a group divisible design it is necessary that*

$$| NN' | = rk[r - \lambda_1 + n(\lambda_1 - \lambda_2)]^{m-1}[r - \lambda_1]^{m(n-1)} \tag{5.16}$$

This result was obtained by Bose and Connor [3].

The second type of partially balanced design developed in [5] is known as the triangular design. For the triangular design

$$\begin{aligned} v = n(n-1)/2, \qquad n_1 = 2(n-2), \qquad & n_2 = (n-2)(n-3)/2, \\ p_{12}^1 = n - 3, \qquad & p_{12}^2 = 2(n-4), \end{aligned} \tag{5.17}$$

where n is integral and greater than or equal to 4. From (5.4), (5.9), and (5.10) it is seen that

$$z_1 = (4 - n)\lambda_1 + (n - 3)\lambda_2, \qquad z_2 = 2\lambda_1 - \lambda_2, \tag{5.18}$$

$$\alpha_1 = n - 1, \quad \text{and} \quad \alpha_2 = n(n-3)/2, \tag{5.19}$$

so that we have the following corollary.

COROLLARY 5.1.4. *For a triangular design it is necessary that*

$$| NN' | = rk[r + (n-4)\lambda_1 - (n-3)\lambda_2]^{n-1}[r - 2\lambda_1 + \lambda_2]^{n(n-3)/2}. \tag{5.20}$$

A third type of partially balanced design with two associate classes defined in [5] is the Latin square type with i constraints. For this type of design

$$\begin{aligned} v = n^2, \qquad n_1 = i(n-1), \qquad & n_2 = (n-1)(n-i+1), \\ p_{12}^1 = (i-1)(n-i+1), \qquad & p_{12}^2 = i(n-i), \end{aligned} \tag{5.21}$$

where n and i are integers and $2 \leqq i \leqq n$. Again, from (5.4), (5.9), and (5.10) we obtain

$$z_1 = (i - n)(\lambda_1 - \lambda_2) + \lambda_2, \qquad z_2 = i(\lambda_1 - \lambda_2) + \lambda_2, \tag{5.22}$$

$$\alpha_1 = i(n-1), \quad \text{and} \quad \alpha_2 = (n-1)(n-i+1), \tag{5.23}$$

from which we have the following corollary.

COROLLARY 5.1.5. *For the Latin Square type of design with i constraints, it is necessary that*

$$\begin{aligned} | NN' | = rk[r - (i-n)(\lambda_1 - \lambda_2) - \lambda_2]^{i(n-1)} & \\ [r - i(\lambda_1 - \lambda_2) - \lambda_2]^{(n-1)(n-i+1)}&. \end{aligned} \tag{5.24}$$

Next let us consider the special case of a PBIB design with two associate classes in which $\alpha_1 = \alpha_2$. Setting the right members of (5.9) and (5.10) equal to each other and recalling that $v - 1 = n_1 + n_2$, we obtain

$$\gamma = (n_2 - n_1)/(n_1 + n_2). \tag{5.25}$$

Since n_1 and n_2 must both be positive integers it follows that

$$(5.26) \quad -1 < -n_1/(n_1 + n_2) < \gamma = (n_2 - n_1)/(n_1 + n_2) < n_2/(n_1 + n_2) < 1,$$

from which it follows that

$$(5.27) \qquad \gamma = 0, \quad \text{or} \quad p_{12}^1 = p_{12}^2,$$

since γ must also be an integer. Hence, by (5.25), and (2.2),

$$(5.28) \qquad n_1 = n_2 = (v - 1)/2.$$

Again, using (2.2) and (5.28) it is seen that v is of the form $4t + 1$, where $t = p_{12}^1 = p_{12}^2 = p_{11}^2$. Since $\alpha_1 = \alpha_2$ and $\alpha_1 + \alpha_2 = v - 1$,

$$(5.29) \qquad \alpha_1 = \alpha_2 = (v - 1)/2 = 2t.$$

From (5.3), (5.27), and (5.29)

$$(5.30) \qquad \Delta = v = 4t + 1.$$

This completes the proof of the following theorem.

THEOREM 5.2. *If in a PBIB design with two associate classes $\alpha_1 = \alpha_2$, then*

(a) $$p_{12}^1 = p_{12}^2 = t,$$

(b) $$\alpha_1 = \alpha_2 = n_1 = n_2 = (v - 1)/2 = 2t,$$

and

(c) $$v = \Delta = 4t + 1,$$

where t is a nonnegative integer defined by (a).

It is known that any integral square must be of the form $4p$ or $4p + 1$, p a nonnegative integer. However, $\alpha_1 = \alpha_2$ does not imply that Δ is an integral square. In fact, designs having solutions exist with $\alpha_1 = \alpha_2$ and Δ not an integral square while others having $\alpha_1 = \alpha_2$ and Δ an integral square also have solutions.

Let us next consider a partially balanced design with two associate classes in which v is odd and Δ is not an integral square. Let η be defined by

$$(5.31) \qquad \eta = [(v - 1)(1 - \gamma) - 2n_1]/2\sqrt{\Delta}.$$

Whether v is odd or even (5.9) may be expressed in the form

$$(5.32) \qquad \alpha_1 = (v - 1)/2 + \eta.$$

Since v is odd, $(v - 1)/2$ is integral, and, since α_1 must also be integral, η must be integral. Since Δ is not an integral square, the only way η can be integral is for η to be equal to zero, that is,

$$(5.33) \qquad (v - 1)(1 - \gamma) - 2n_1 = 0.$$

Using $v - 1 = n_1 + n_2$ in (5.33) it is seen that (5.25), (5.26), (5.27), (5.28), and (5.29) follow. Hence the following theorem.

THEOREM 5.3. *If in a PBIB design with two associate classes v is odd and Δ is not an integral square, then it is necessary that*

(a) $$p^1_{12} = p^2_{12} = p^2_{11} = t,$$

(b) $$n_1 = n_2 = \alpha_1 = \alpha_2 = (v - 1)/2 = 2t$$

and

(c) $$v = \Delta = 4t + 1,$$

where t is a nonnegative integer defined by (a).

Now if v is odd and Δ is an integral square (5.32) holds and η must be an integer. Thus, we have the following theorem.

THEOREM 5.4. *If in a PBIB design with two associate classes v is odd and Δ is an integral square, then it is necessary that η be an integer, where Δ is defined by* (5.3) *and η is defined by* (5.31).

Finally let us consider the case of a PBIB design with two associate classes in which v is even. Then (5.9) can be written in the form

$$\alpha_1 = \tfrac{1}{2}[v - 1 + 2\eta]. \tag{5.34}$$

Since α_1 must be integral, $v - 1 + 2\eta$ must be an even integer. But $v - 1$ is odd, and so 2η must be an odd integer. Since v, γ, and n_1 must all be integral, it is seen from (5.31) that Δ must be an integral square. This proves the following theorem.

THEOREM 5.5. *If in a PBIB design with two associate classes v is even, then it is necessary that*

(a) *Δ be an integral square, and*

(b) *2η be an odd (positive or negative) integer,*

where Δ and η are defined by (5.3) *and* (5.31) *respectively.*

6. Partially balanced designs with three and four associate classes. In this section we shall obtain expressions for $\alpha_1, \alpha_2, \cdots, \alpha_t$, $(t = 3, 4)$, in terms of the roots $z_1, z_2, \cdots, z_t$ of $|A| = 0$ when the z_i, $(i = 1, 2, \cdots, t)$, are all different. First, we shall discuss the case of partially balanced designs with three associate classes ($m = 3$).

When $m = 3$ we obtain from Lemma 3.1

$$|M| = (z - z_0)(z - z_1)^{\alpha_1}(z - z_2)^{\alpha_2}(z - z_3)^{\alpha_3} \tag{6.1}$$

wherein the z_i, $(i = 1, 2, 3)$, are the distinct zeros of $|A|$ and

$$\textstyle\sum_{i=1}^{3} \alpha_i = v - 1, \qquad (\alpha_i > 0). \tag{6.2}$$

Expanding each factor of $|M|$ of (6.1) and collecting coefficients of powers of z gives

$$\begin{aligned} |M| = z^v &+ \left[- z_0 - \sum_{i=1}^{3} \alpha_i z_i \right] z^{v-1} \\ &+ \left[z_0 \sum_{i=1}^{3} \alpha_i z_i + \tfrac{1}{2} \sum_{i=1}^{3} \alpha_i(\alpha_i - 1) z_i^2 + \sum_{\substack{i,j=1 \\ i<j}}^{3} \alpha_i \alpha_j z_i z_j \right] z^{v-2} \\ &+ \cdots (-)^v z_0 z_1 z_2 z_3 . \end{aligned} \tag{6.3}$$

Again, expanding $|M|$ of (6.1) by its diagonal elements [1], it is seen that

$$|M| = z^v - \frac{v}{2}\left(\sum_{i=1}^{3} n_i\lambda_i^2\right)z^{v-2} + \cdots + |M_0| \tag{6.4}$$

where M_0 is obtained from M by replacing z by zero.

Equating the coefficients of z^{v-1} in (6.3) and (6.4), we obtain

$$\sum_{i=1}^{3} \alpha_i z_i = -z_0, \tag{6.5}$$

while equating the coefficients of z^{v-2} gives

$$2z_0 \sum_{i=1}^{3} \alpha_i z_i + \sum_{i=1}^{3} \alpha_i(\alpha_i - 1)z_i^2 + 2 \sum_{\substack{i,j=1\\ i<j}}^{3} \alpha_i\alpha_j z_i z_j = -v \sum_{i=1}^{3} n_i\lambda_i^2. \tag{6.6}$$

Now

$$\left(\sum_{i=1}^{3} \alpha_i z_i\right)^2 = \sum_{i=1}^{3} \alpha_i^2 z_i^2 + 2 \sum_{\substack{i,j=1\\ i<j}}^{3} \alpha_i\alpha_j z_i z_j. \tag{6.7}$$

By use of (6.5), (6.6), and (6.7) we obtain

$$\sum_{i=1}^{3} \alpha_i z_i^2 = -z_0^2 + v \sum_{i=1}^{3} n_i\lambda_i^2. \tag{6.8}$$

Thus (6.2), (6.5), and (6.8) comprise a system of three nonhomogeneous linear equations in unknowns α_1, α_2, and α_3,

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 &= k_1 \\ z_1\alpha_1 + z_2\alpha_2 + z_3\alpha_3 &= k_2 \\ z_1^2\alpha_1 + z_2^2\alpha_2 + z_3^2\alpha_3 &= k_3, \end{aligned} \tag{6.9}$$

wherein the coefficient matrix is the Vandermonde matrix

$$A_3 = \begin{bmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ z_1^2 & z_2^2 & z_3^2 \end{bmatrix}, \tag{6.10}$$

whose determinant is

$$|A_3| = (z_2 - z_1)(z_3 - z_1)(z_3 - z_2), \tag{6.11}$$

and

$$k_1 = v - 1, \qquad k_2 = -z_0, \qquad k_3 = -z_0^2 + v\textstyle\sum_{i=1}^{3} n_i\lambda_i^2. \tag{6.12}$$

If z_1, z_2, and z_3 are all distinct, then (6.9) has a unique solution. In fact, we obtain

$$\alpha_1 = \frac{v\left[z_2 z_3 + \sum_{i=1}^{3} n_i\lambda_i^2\right] - (z_0 - z_2)(z_0 - z_3)}{(z_2 - z_1)(z_3 - z_1)}, \tag{6.13}$$

and from (6.9) it is clear that the corresponding expressions for α_2 and α_3 can be obtained from (6.13) by cyclically permuting the indices 1, 2, and 3.

For partially balanced designs with four associate classes the above procedure leads to

$$\alpha_j \prod_{\substack{i=1\\ i\neq j}}^{4} (z_i - z_j) = v\left[\prod_{\substack{i=1\\ i\neq j}}^{4} z_i + \left(\sum_{\substack{i=1\\ i\neq j}}^{4} z_i\right)\left(\sum_{i=1}^{4} n_i\lambda_i^2\right) - K\right] + \prod_{\substack{i=1\\ i\neq j}}^{4} (z_0 - z_i), \tag{6.14}$$

$$j = 1, 2, 3, 4,$$

where

$$K = -\tfrac{1}{2}\sum_{i=1}^{4} n_i \sum_{j,k=1}^{4} p_{jk}^i \Delta_{jk}^i,$$

$$\Delta_{jk}^i = \begin{bmatrix} 0 & \lambda_i & \lambda_j \\ \lambda_i & 0 & \lambda_k \\ \lambda_j & \lambda_k & 0 \end{bmatrix},$$

and z_1, z_2, z_3, and z_4 are the distinct roots of $|A| = 0$.

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