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ON THE COMBINING OF INTERBLOCK AND INTRABLOCK ESTIMATES

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In a recent paper Sprott [1] has considered methods for combining interblock and intrablock estimates of variety contrasts for incomplete block designs. The intrablock estimates are derived from treatment contrasts obtained within blocks. The interblock estimates presuppose that the block effects are random, independent, and identically distributed, and they are derived from contrasts among the block averages. Under normality the intrablock estimates are independent of the interblock estimates.

Sprott compares two methods for producing combined estimates. The first method, introduced by Yates [2], is the familiar method of combining by weighting with the reciprocal of the variances, and is known to produce minimum variance when two real estimates of the same quantity are combined linearly. The second method, discussed by Rao [3] and Cochran and Cox [4], is to apply the method of maximum likelihood to the joint density function, and the resulting estimate is linear in terms of the interblock and intrablock estimates. Sprott shows that, in general, the two methods are not equivalent. The second method is direct and has considerable theoretical weight behind its use. We are left then with the implication that one of the methods is incorrect for obtaining good estimates. In a sense this is not the case. Rather, one of the methods may be *inappropriately* applied. Weighting with reciprocal variances is *appropriate* to combining real estimates but if applied to vector estimates it ignores any covariances and may not be optimum.

Suppose $x = (x_1, \cdots, x_r)$ and $y = (y_1, \cdots, y_r)$ are independent estimates of the parameter $\eta = (\eta_1, \cdots, \eta_r)$ and have nonsingular covariance matrices V and W respectively. Also suppose, for the moment, that x and y are normal. Then, the joint density function is a constant times

$$\exp\left[-\tfrac{1}{2}(x - \eta)V^{-1}(x - \eta)' - \tfrac{1}{2}(y - \eta)W^{-1}(y - \eta)'\right],$$

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which can be factored as

$$\text{(1)}\quad \begin{aligned}&\exp\{[(xV^{-1} + yW^{-1})(V^{-1} + W^{-1})^{-1} - \eta](V^{-1} + W^{-1})\\ &\qquad [(xV^{-1} + yW^{-1})(V^{-1} + W^{-1})^{-1} - \eta]'\}\\ &\qquad\qquad \times \exp\{(x - y)(V + W)^{-1}(x - y)'\}.\end{aligned}$$

Then, assuming that the covariances are known, we see that

$$\text{(2)}\qquad (xV^{-1} + yW^{-1})(V^{-1} + W^{-1})^{-1}$$

is a sufficient statistic for η. Also, from its non-singular distribution, it follows that (2) is a complete statistic. Hence, by the theorems of Lehmann and Scheffé which may be found on pp. 61–64 in [5], an unbiased estimate of η based on (2) will be unique, and it will have minimum concentration ellipsoid among *all* unbiased estimates, linear or not. Each coordinate will also have minimum variance. From (1), we see immediately that (2), as it stands, is an unbiased estimate of η, and hence has the properties above. If the assumption of normality is removed, (2) remains unbiased and has minimum variances among unbiased linear estimates.

The question arises as to when the two methods are equivalent and the more-direct first method usable in place of the second. A combined estimate based on the first method would have the form

$$\text{(3)}\qquad xD + y(I - D),$$

where D is a diagonal matrix and I is the identity matrix ($r \times r$). If (2) reduces to this form (3), then

$$\text{(4)}\qquad V^{-1}(V^{-1} + W^{-1})^{-1} = (I + W^{-1}V)^{-1}$$

is diagonal and hence VW^{-1} is diagonal: $VW^{-1} = D^*$. We have

$$\text{(5)}\qquad V = D^*W = WD^*,$$

where the second expression follows from the symmetry of covariance matrices. (5) implies that for any non-zero off-diagonal element in W the corresponding two coordinate indices have equal diagonal elements in D^*. Therefore, by permuting coordinates it follows that W can be made diagonal in blocks and that V is obtained by multiplying each block by a positive constant. Thus for the vectors x, y the rearranged coordinates fall into independent groups. A group for x and a group for y have the same covariance matrix except for a *single* scale factor. There are two extremes for this: first, the x and y have the same covariance matrix except for a single factor; second, the x and y both have independent coordinates.

Thus, in general, it is not enough to combine estimates, coordinate by coordinate. An estimate with optimum properties is obtained by weighting the *vectors* by the inverse of their covariance matrices. We can easily see that doing this for the incomplete block problem considered by Sprott will produce the estimates as obtained by the second method. For, the second method uses maximum likelihood on the joint density, and from (1) we see that this obviously produces (2).

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ABSTRACTS OF PAPERS

(Additional abstracts of papers presented at the Washington meeting of the Institute, March 7–9, 1957)

1. The K-Visit Method of Consumer Testing, GEORGE E. FERRIS, General Foods Corporation, (By Title).

When testing a pair of products for consumer preference the problem of how to treat or interpret no preference votes arises. A method is described of collecting data from a given number of consumers by repeated visits to them, or by obtaining repeated judgments from them in stores, which enables the estimation of the true proportion of those consumers in the population who have a preference for either product and of those who cannot discriminate or have no preference. For the model assumed, the maximum likelihood estimators of the above proportions are derived, their variance-covariance matrix is obtained, and a way of testing the appropriateness of the model is indicated. A decision theoretical formulation is suggested. (Received March 8, 1957; revised March 12, 1957.)

2. Factorial Treatments in Group Divisible Incomplete Block Designs, CLYDE Y. KRAMER AND RALPH A. BRADLEY, Virginia Polytechnic Institute.

Methods of incorporating factorial treatment combinations in group divisible incomplete block designs are given. The factorial treatment combinations are so matched with the basic treatments in the association matrices of the designs that the sums of squares for the factorial effects can be obtained as functions of the original treatment estimators. It is shown first how a two-factor factorial may be incorporated into group divisible incomplete block designs. Single degree of freedom contrasts are obtained for the effects in much the usual way as for factorials in complete block designs. Multifactor factorials and partial factorials are discussed, and a method of obtaining estimates and tests of significance of the effects is given. (Received March 18, 1957.)

3. Iterative Experimentation, G. E. P. BOX, Statistical Techniques Group, Princeton University.

Scientific research is usually an iterative process. The cycle: conjecture–design–experiment–analysis leads to a new cycle of conjecture–design–experiment–analysis, and so on. It is helpful to keep this picture of the experimental method in mind when considering statistical problems. Although this cycle is repeated many times during an investigation,