ESTIMATES OF ERROR FOR TWO MODIFICATIONS OF THE ROBBINS-MONRO STOCHASTIC APPROXIMATION PROCESS[1]

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1. Introduction. The Robbins-Monro procedure is a process of the following form. For each number x, Y_x is a chance variable having a variance which is a bounded function of x; i.e., $E(Y_x - E(Y_x))^2 \leqq \sigma^2 < \infty$. The regression curve $y = f(x) = E(Y_x)$ is presumed to be unknown but supposed to lie below the horizontal line $y = \alpha$ for $x < \theta$ and above it for $x > \theta$, where α is specified and θ is to be estimated. Let

$$X_{n+1} = X_n - a_n(Y_{x_n} - \alpha) \qquad (n = 1, 2, \cdots), \tag{1}$$

where the a_n are specified numbers and assume that $E(X_1 - \theta)^2 = V^2 < \infty$; then under suitable conditions X_n converges to θ. Following the paper of Robbins and Monro [10] there appeared a succession of papers ([1] through [14]) in which the conditions for convergence were relaxed, the type of convergence strengthened, the asymptotic distribution of X_n found and the whole process generalized and simplified. The question of an optimal stopping rule however remains open. We assume here that the regression line lies between two straight lines with finite and positive slopes, i.e.,

$$\begin{aligned} \alpha + m(x - \theta) \leqq f(x) \leqq \alpha + M(x - \theta), \qquad \text{if } x \geqq \theta, \\ \alpha + m(x - \theta) \geqq f(x) \geqq \alpha + M(x - \theta), \qquad \text{if } x \leqq \theta, \end{aligned} \tag{2}$$

with $0 < m \leqq M < \infty$. With this condition Dvoretzky [6] showed how to choose the a_n to minimize a bound on the error $E(X_{n+1} - \theta)^2$ after a fixed number N of observations $Y_{X_1}, \cdots, Y_{X_N}$. Here we give analogous results for two modifications of the Robbins-Munro procedure: (i) instead of taking one observation at X_n one takes several and uses the average instead of Y_{x_n} in (1), i.e.,

$$X_{k+1} = X_k - a_k\left(\frac{Y_{x_k}^{(1)} + \cdots + Y_{x_k}^{(n_k)}}{n_k} - \alpha\right), \tag{3}$$

the idea being that it may cost less to take several observations at one point than the same number of observations at different points; and (ii) using (3) with $a_k = a$ ($k = 1, 2, \cdots$); the object being simplicity in performing the experiment.

Received January 11, 1957.

[1] Research under contract with the Office of Naval Research.

Clearly with the n_k as well as a_k at our disposal in (i) we can get $E(X_{p+1} - \theta)^2$, where

$$\sum_{k=1}^{p} n_k = N, \tag{4}$$

with (3) at least as small as $E(X_N - \theta)^2$ by (1). We shall see that we can't do any better with (i) either, so the only saving in (i) is in the smaller number of "set-ups" required. On the other hand we cannot expect to do better with (ii) but we shall see that under certain conditions we can do about as well, so that the increase in simplicity may be worthwhile.

The notations and assumptions introduced above will be used throughout the remainder of this paper.

The author is indebted to Professors J. Kiefer and J. Wolfowitz for introducing him to this subject.

2. The Robbins-Monro process. For the sake of completeness and comparison we give first the Robbins-Monro case already handled by Dvoretzky [6.]

THEOREM (1). *Assume that*

$$(\alpha)\quad V^2 \leqq \frac{2\sigma^2}{m(M - m)}.$$

If

$$(\beta)\quad a_n = \frac{mV^2}{\sigma^2 + m^2V^2n}, \qquad n = 1, 2, \cdots, N,$$

then with X_n given by (1),

$$(\gamma)\quad E(X_{N+1} - \theta)^2 \leqq \frac{V^2\sigma^2}{\sigma^2 + m^2V^2N}; \qquad \text{and}$$

(δ) *the constants a_n given by* (β) *are optimal in the sense that if the a_n do not satisfy* (β) *then there exist processes* (1) *satisfying* (α) *for which* (γ) *does not hold.*

REMARK 1. If the condition (α) is not satisfied it is still not difficult to find the optimal a_n's from the derivation below (see Dvoretzky [6]). In this case the estimate of error is not so neat; if we take instead $a_n = 2/[M + (2n - 1)m]$ it is not hard to verify that

$$E(x_{N+1} - \theta)^2 \leqq \frac{(M - m)^2V^2 + 4N\sigma^2}{[M + (2N - 1)m]^2}. \tag{5}$$

PROOF. Let $Y_{X_n} = f(X_n) + \epsilon_n$, where $E(\epsilon_n \mid x_1, \cdots, x_n) = 0$, $E(\epsilon_n^2 \mid x_1, \cdots, x_n) \leqq \sigma^2$. Then from (1)

$$\begin{aligned} X_{n+1} - \theta &= X_n - \theta - a_n(f(X_n) - \alpha) - a_n\epsilon_n \\ &= [1 - a_ng(X_n)](X_n - \theta) - a_n\epsilon_n, \end{aligned}$$

where $f(x) - \alpha = g(x)(x - \theta)$ and in virtue of (2), $g(x)$ satisfies: $0 < m \leqq g(x) \leqq M$. $E(X_{n+1} - \theta)^2 \leqq E[(1 - a_ng(X_n))^2(X_n - \theta)^2] + a_n^2\sigma^2$. Now

$$1 - a_ng(X_n) \leqq 1 - a_nm$$

and since from (α) and (β) $a_n \leqq a_1 \leqq 2/(M + m)$ we also have $a_n g(X_n) - 1 \leqq a_n M - 1 \leqq 1 - a_n m$. Hence

$$E(X_{n+1} - \theta)^2 \leqq (1 - a_n m)^2 E(X_n - \theta)^2 + a_n^2 \sigma^2, \tag{6}$$

or using (β) and then iterating from $n = N$ downward

$$\begin{aligned} E(X_{N+1} - \theta)^2 &\leqq \frac{[\sigma^2 + (N-1)m^2V^2]^2 E(X_N - \theta)^2 + \sigma^2 m^2 V^4}{(\sigma^2 + Nm^2V^2)^2} \\ &\leqq \frac{[\sigma^2 + (N-2)m^2V^2]^2 E(X_{N-1} - \theta)^2 + 2\sigma^2 m^2 V^4}{(\sigma^2 + Nm^2V^2)^2} \leqq \cdots \\ &\leqq \frac{V^2\sigma^2}{\sigma^2 + m^2V^2N}. \end{aligned}$$

To verify (δ) let $g(x) = m$ and var $\epsilon_n = \sigma^2$. Then (6) becomes an equality and the unique minimizing a_n's are given by (β); to see this let $E(X_n - \theta)^2 = B_n$ and note that from (6), (with $n = N$), having fixed $a_1, \cdots, a_{N-1}$, the value of B_N is fixed and $B_{N+1} = (1 - a_N m)^2 B_N + a_N^2 \sigma^2$. The minimizing value of a_N satisfies $a_N = mB_N / (\sigma^2 + m^2 B_N)$, and the minimum value of B_{N+1} satisfies $B_{N+1} = (\sigma^2 B_N) / (\sigma^2 + m^2 B_N)$ or

$$\frac{1}{B_{N+1}} = \frac{1}{B_N} + \frac{m^2}{\sigma^2} = \cdots = \frac{1}{B_1} + \frac{Nm^2}{\sigma^2}.$$

Hence

$$B_{N+1} \geqq \frac{V^2\sigma^2}{\sigma^2 + m^2V^2N}$$

with equality holding if and only if the a_n satisfy (β).

REMARK 2. It is of some interest to compare the error for the Robbins-Monro procedure

$$E(X_{N+1} - \theta)^2 \leqq \frac{\sigma^2}{\frac{\sigma^2}{V^2} + m^2N}$$

with the error of the maximum likelihood estimator of the special situation when the regression is known to be linear with known slope m with errors normal $(0, \sigma^2)$. If we take N values for $x: X_1, X_2, \cdots X_N$ by a procedure which is independent of θ, and observe $Y_n = f(X_n) + \epsilon_n = \alpha + m(X_n - \theta) + \epsilon_n$, the joint density function of $X_1, \cdots, X_n, Y_1, \cdots, Y_n$ is

$$\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^N e^{-\frac{1}{2\sigma^2}\Sigma(y_n - \alpha - m(x_n - \theta))^2} \, dF(x_1, \cdots, x_n),$$

where σ is known and θ is to be estimated. The maximum likelihood estimator is $\hat{\theta} = \bar{X} - (1/m)(\bar{Y} - \alpha)$ which is distributed normally $(\theta, \sigma^2 / m^2N)$. Now $\sigma^2 / [(\sigma^2/V^2) + m^2N] < \sigma^2/m^2N$ so that as long as $V^2 < \infty$ the Robbins Monro estimator is better. The condition $V^2 < \infty$ is guaranteed, e.g., by taking $X_1 = 0$.

3. The averaged Robbins-Monro process.

THEOREM 2. *Let*

(i) $$\frac{2\sigma^2}{m(M-m)V^2} = Q \geqq 1,$$

and let $n_1, n_2, \cdots, n_p$ *be positive integers whose sum is* N *and such that*

(ii) $$n_1 \leqq Q, \qquad n_2 \leqq Q + \frac{2mn_1}{M-m},$$

$$n_3 \leqq Q + \frac{2m(n_1+n_2)}{M-m}, \cdots, \qquad n_p \leqq Q + \frac{2m(n_1 + \cdots + n_{p-1})}{M-m}.$$

If

(iii) $$a_k = \frac{mn_kV^2}{\sigma^2 + m^2V^2(n_1+n_2+\cdots+n_k)} \qquad (k = 1, 2, \cdots, p),$$

and the X_k *are given by Eq.* (3), *then*

(iv) $$E[X_{p+1} - \theta]^2 \leqq \frac{\sigma^2V^2}{\sigma^2 + m^2V^2N}, \qquad \textit{and}$$

(v) *the constants* a_k *are optimal in the sense that if the* a_k *do not satisfy* (iii) *then there exists a process* (3) *satisfying* (i) *and* (ii) *for which* (iv) *does not hold.*

REMARK 3. Again if (i) is not satisfied it is not hard to find the optimal a_k's from the derivation below. Of course we can always achieve the estimate (5), with the a_k's chosen as in Remark 1.

PROOF. From Eq. (3)

$$X_{k+1} - \theta = [1 - a_k g(X_k)](X_k - \theta) - \frac{a_k}{n_k}(\epsilon_k^{(1)} + \cdots + \epsilon_k^{(n_k)}),$$

$$E(X_{k+1} - \theta)^2 \leqq E\{[1 - a_k g(X_k)]^2(X_k - \theta)^2\} + \frac{a_k^2\sigma^2}{n_k};$$

again $1 - a_kg(X_k) \leqq 1 - a_km$ and $a_k \leqq 2/(M+m)$ so that

(7) $$E(X_{k+1} - \theta)^2 \leqq (1 - a_km)^2E(X_k - \theta)^2 + \frac{a_k^2\sigma^2}{n_k}.$$

Using (iii) and iterating from $k = p$ down, one finds (iv). Again with $g(x) = m$ and var $(\epsilon_j^{(k)}) = \sigma^2$, inequality (7) (with $k = p$) becomes an equality which is minimized when $a_p = (mB_pn_p) / (\sigma^2 + m^2B_pn_p)$ and

$$B_{p+1} = \frac{B_p\sigma^2}{\sigma^2 + m^2B_pn_p} \qquad \text{or}$$

$$\frac{1}{B_{p+1}} = \frac{1}{B_p} + \frac{m^2n_p}{\sigma^2} = \cdots = \frac{1}{B_1} + \frac{m^2(n_1 + \cdots + n_p)}{\sigma^2};$$

i.e., $B_{p+1} \geqq (\sigma^2V^2) / (\sigma^2 + m^2V^2N)$, and the equality is achieved if and only if the a_k are given by (iii).

4. A fixed value for the a_k. Here X_k is given by (3) with

$$a_k = a \ (k = 1, 2, \cdots, p).$$

THEOREM 3. *Let*

(a) $$\frac{2\sigma^2}{m(M - m)V^2} = Q \geqq \frac{1}{1 + \frac{1}{N}\left[\binom{N}{2}\left(\frac{2m}{M - m}\right) + \cdots + \binom{N}{N}\left(\frac{2m}{M - m}\right)^{N-1}\right]}.$$

Let p be any integer satisfying

(b) $$\frac{\log\left(1 + \frac{V^2m^2N}{\sigma^2}\right)}{\log\left(\frac{M + m}{M - m}\right)} \leqq p \leqq N.$$

It follows easily from (a) *that at least one such p exists.*

If

(c) $$a = \frac{1 - \left(\frac{\sigma^2}{\sigma^2 + V^2m^2N}\right)^{\frac{1}{p}}}{m},$$

and if the equations

(d) $$n_k = (1 - ma)^{p-k}\frac{(\sigma^2 + V^2m^2N)a}{V^2m} \qquad (k = 1, 2, \cdots, p)$$

define $n_1, \cdots, n_p$ as integers, then

(e) $$E(X_{p+1} - \theta)^2 \leqq \frac{V^2\sigma^2}{\sigma^2 + m^2V^2N}, \qquad \textit{and}$$

(f) *for a fixed value of p satisfying* (b) *this choice of $a, n_1, \cdots, n_p$ is optimal in the sense that if* (d) *defines integers but n_k are chosen satisfying* (4) *but not condition* (d), *or if a does not satisfy* (c) *then there exists a process* (3) *with the a_k's equal, for which* (a) *and* (b) *are satisfied but* (e) *does not hold.*

REMARK 4. The condition (a) here is less stringent than the corresponding conditions (α), (i) in the preceding theorems; but the assumption that (d) defines integers is of course unpleasant; if (d) does not give integers, but one chooses the n_k as the nearest integers to them then one would expect that the estimate (e) would not be very much in error, especially if N is large. We have not done the computations for that case.

PROOF.

$$(X_{k+1} - \theta) = (1 - ag(X_k))(X_k - \theta) - a\left[\frac{\epsilon_k^{(1)} + \cdots + \epsilon_k^{(n_k)}}{n_k}\right],$$

$$E(X_{k+1} - \theta)^2 \leqq E(1 - ag(X_k))^2(X_k - \theta)^2 + \frac{a^2\sigma^2}{n_k}.$$

Again $1 - ag(X_k) \leqq 1 - ma$ and from (a) and (b) it follows that

$$a \leqq 2/(M + m)$$

so that $ag(X_k) - 1 \leqq aM - 1 \leqq 1 - ma$. Hence

$$E(X_{k+1} - \theta)^2 \leqq (1 - ma)^2 E(X_k - \theta)^2 + (a^2\sigma^2) / n_k .$$

Iterating we get

$$\begin{aligned} E(X_{p+1} - \theta)^2 &\leqq (1 - ma)^2 \left[(1 - ma)^2 E(X_{p-1} - \theta)^2 + \frac{a^2\sigma^2}{n_{p-1}} \right] + \frac{a^2\sigma^2}{n_p} \\ &\leqq \cdots \\ &\leqq (1 - ma)^{2p} V^2 + a^2\sigma^2 \left[\frac{1}{n_p} + \frac{(1 - ma)^2}{n_{p-1}} + \frac{(1 - ma)^4}{n_{p-2}} \right. \\ &\qquad \left. + \cdots + \frac{(1 - ma)^{2p-2}}{n_1} \right]. \end{aligned} \tag{8}$$

Using (c), (d), we now get (e). The argument for (f) follows as before; namely, if $g(x) = m$, var $(\epsilon_j^{(i)}) = \sigma^2$, then the equality in (8) holds, and the unique minimum, subject to the constraint (4), occurs when a, n_k satisfy (c), (d). To see this let $r = 1 - ma$ so that

$$\begin{aligned} E(X_{p+1} - \theta)^2 &= r^{2p} V^2 + \frac{(1 - r)^2 \sigma^2}{m^2} \\ &\quad \left(\frac{1}{N - n_1 - n_2 - \cdots - n_{p-1}} + \frac{r^2}{n_{p-1}} + \cdots + \frac{r^{2p-2}}{n_1} \right). \end{aligned} \tag{9}$$

From

$$\frac{\partial E(X_{p+1} - \theta)^2}{\partial n_k} = 0, \qquad k = 1, 2, \cdots, p - 1,$$

we get

$$n_k = r^{p-k} \left(N - \sum_{i=1}^{p-1} n_i \right), \qquad k = 1, 2, \cdots, p - 1. \tag{10}$$

Hence

$$\sum_{k=1}^{p-1} n_k = \frac{(r - r^p) \left(N - \sum_{k=1}^{p-1} n_k \right)}{1 - r}$$

and

$$N - \sum_{k=1}^{p-1} n_k = \frac{(1 - r)N}{(1 - r^p)}. \tag{11}$$

Using (11) in (10) we get $n_k = [r^{p-k}(1 - r)N] / (1 - r^p)$, $k = 1, 2, \cdots, p - 1$. Using this in (9) we get

$$E(X_{p+1} - \theta)^2 = r^{2p}V^2 + \frac{(1 - r^p)(1 - r)\sigma^2}{m^2N} \sum_{k=1}^{p} r^{k-1}$$

$$= r^{2p}V^2 + \frac{(1 - r^p)^2\sigma^2}{Nm^2}.$$

This has its minimum when $r^p = \sigma^2 / (Nm^2V^2 + \sigma^2)$. The corresponding values of a, n_k are given by (c), (d), and so $E(X_{p+1} - \theta)^2 \geqq (\sigma^2V^2) / (\sigma^2 + Nm^2V^2)$ with equality holding if and only if (c) and (d) are satisfied.

REMARK 5. The estimate of error in each theorem ((γ), (iv), (e)) is the same. In Theorem 2 it is independent of how the N observations are partitioned amongst $n_1, \cdots, n_p$ as long as (ii) is satisfied. In Theorem 3 this partition affects the estimate while the choice of p (subject to condition (a)) does not. If it costs less to take several observations at the same point than the same number of observations at various points then one will keep p as small as possible in each case.

Of course with these estimates of error we can at once give confidence intervals for θ, via Tchebycheff's inequality.

REMARK 6. Similar estimates can be made for a process taking place in a Hilbert space; for example suppose we wish to find a solution θ to the equation $K(\theta) = \alpha$ where K is an operator satisfying $||K(x) - \alpha||^2 \leqq C||x - \theta||^2$, $C < \infty$, $(K(x) - K(\theta), x - \theta) \geqq c||x - \theta||^2$, $c > 0$. (For example, a positive definite continuous linear operator has this property; cf. Blum [2]). If $X_{n+1} = X_n - a_n(K(X_n) - \alpha + \epsilon_n)$, where ϵ_n is a vector; and if $\mathcal{E}$ is an additive and homogeneous function such that $\mathcal{E}(\epsilon_n, g(X_n)) = 0$ for any vector function $g(X)$ and $\mathcal{E}(||\epsilon_n||^2) \leqq \sigma^2$, then

$$\mathcal{E}||X_{n+1} - \theta||^2 \leqq (1 - 2a_nc + a_n^2C)\mathcal{E}||X_n - \theta||^2 + a_n^2\sigma^2.$$

The optimal a_n's and best estimates of error are now obtained from the recursions $a_n = (cB_n) / (CB_n + \sigma^2)$, $B_{n+1} = (1 - 2a_nc + a_n^2C)B_n + a_n^2\sigma^2$. The modified procedures may be treated similarly; now, however, it turns out that the estimate of error in case (i) does depend on the partition $(n_1, \cdots, n_p)$ of N.

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