approaches zero ($j \neq h$). Thus variance$\{K(N)\}$ approaches zero as $N$ increases, so $K(N)$ converges stochastically to $S(t)$, as does $R_N(t)$. Therefore we have shown that $V(N)$ converges stochastically to zero as $N$ increases.

**3. Application to ranks in $k$-samples.** Define $T(i, j)$ as $F_1(X(i, j))$. Then $T(1, 1), \cdots, T(1, n_1)$ have unform distributions. Let $G_i(x)$ denote the resulting distribution function for $T(i, j)$. We assume that $G_i(x)$ allows a density function $g_i(x)$ (then $g_i(x)$ is zero outside the interval $[0, 1]$, is bounded, and has a finite number of discontinuities). Let $V_1 \leqq V_2 \leqq \cdots \leqq V_{N-n_1}$ denote the ordered values of $T(2, 1), \cdots, T(k, n_k)$, and let $V_0$ equal zero, $V_{N-n_1+1}$ equal one. Let $S_i$ denote the number of $T(1, j)$'s which lie in the interval

$$[V_{i-1}, V_i], \qquad i = 1, \cdots, N - n_1 + 1.$$

For each nonnegative integer $r$, let $Q_n(r)$ be the proportion of values among $S_1, \cdots, S_{N-n_1+1}$ which are equal to $r$. Define $g(y)$ as $\sum_{i=2}^{k} (r_i/(1 - r_1))g_i(y)$, and $\alpha$ as $(r_1/(1 - r_1))$. Define $Q(r)$ as

$$\alpha^r \int_0^1 \frac{g^2(y)}{[\alpha + g(y)]^{r+1}}\, dy.$$

Then it follows from the results above, using also the argument in [2], that $\sup_{r \geqq 0} | Q_N(r) - Q(r) |$ converges stochastically to zero as $N$ increases. This can be used to show that certain tests of the hypothesis

$$F_1(x) = F_2(x) = \cdots = F_k(x)$$

are consistent. The discussion parallels that found in [2].

## REFERENCES

[1] L. Weiss, "The stochastic convergence of a function of sample successive differences," *Ann. Math. Stat.*, Vol. 26 (1955), pp. 532–536.

[2] J. R. Blum and L. Weiss, "Consistency of certain two-sample tests," *Ann. Math. Stat.*, Vol. 28 (1957), pp. 242–246.

# CORRECTION TO "AN EXTENSION OF THE KOLMOGOROV DISTRIBUTION"

By Jerome Blackman

*Syracuse and Cornell Universities*

**1. Summary.** It has been pointed out by J. H. B. Kemperman that an error in [1] invalidates the formulas arrived at in that paper. It is the purpose of this note to supply the correct formulas for the probabilities of Theorems 1 and 2. An Appendix by Professor Kemperman is included.

Received February 18, 1957; revised August 14, 1957.

**2. Introduction.** The major error in [1] lies in the mappings on pp. 516–517. Corrections can be made for this but unfortunately the resulting formulas are more complicated than before. A smaller error appears in the statement that $N(A_{2i}) = N(B_{2i})$, but this is easily corrected. The new formulas are so much more complicated that it has not seemed worthwhile to correct Corrolaries 1 and 2 which are hereby retracted. The corrected statements of the main results follow.

THEOREM 1. *Let* $x_1, x_2, \cdots x_n, x_1', x_2', \cdots x_{nk}'$ *be a sequence of* $n(k+1)$ *independent random variables with a common continuous distribution* $F(x)$. *Let* $F_n(x)$ *and* $G_{nk}(x)$ *be empiric distributions based on the first* $n$ *and second* $kn$ *random variables respectively. Then*

$$P(-y < G_{nk}(s) - F_n(s) < x \quad \text{for all} \quad s)$$
$$= 1 - \binom{(k+1)n}{n}^{-1} \sum_{i=1}^{\infty} \{N(A_{2i-1}) + N(B_{2i-1}) - N(A_{2i}) - N(B_{2i})\},$$

*where the* $N$ *functions are given in* (1), (2), *and* (3).

THEOREM 2.

$$P(-y < F(s) - F_n(s) < x \quad \text{for all} \quad s)$$
$$= 1 - \sum_{i=1}^{\infty} \{\bar{N}(A_{2i-1}) + \bar{N}(B_{2i-1}) - \bar{N}(A_{2i}) - \bar{N}(B_{2i})\},$$

*where the* $N$ *functions are given in* (5), (6), *and* (7).

**3. Corrections.** The point of departure from [1] is the middle of p. 516 where a formula for $N(\mathfrak{A}_0)$ is given. It is readily seen that upon dividing this equation by the total number of paths $\binom{(k+1)n}{n}$ one obtains Theorem 1 except for the analytical expressions for the $N$ functions. We will also use the mapping of the $A_i$ and $B_i$ classes described at the bottom of p. 516, although the conclusions drawn there about the mapping are incorrect. The error is clear if we consider the image of a path from $A_3$ under the mapping. The image will be a path which starts from the origin, reaches $2\alpha + \beta$, and then on the return to 0 stops at least once at the point $\alpha$. The class $A_3$ will be in 1:1 correspondence with the set of paths which starting at 0 reach $2\alpha + \beta$ and then on the return stop at least once at $\alpha$. Because the steps to the left are of length $k$, not every path which reaches $2\alpha + \beta$ will, at some later step, stop at $\alpha$. In Table 1 the images of the $A_i$ and $B_i$ under the mapping are given. The second column gives the points which the path must reach and the last column gives the points at which the path must stop, in order, after reaching the point described in column 2. In all cases the mapping is 1:1 between the class in the first column and the set of paths which reach the point indicated in the second column and subsequently stop in order at all the points indicated in the last column.

TABLE 1

| | | |
|---|---|---|
| $A_{2i-1}$ | $i(\alpha+\beta)-\beta$ | $(i-1)(\alpha+\beta)-\beta,\quad (i-2)(\alpha+\beta)-\beta, \cdots \alpha$ |
| $A_{2i}$ | $i(\alpha+\beta)$ | $(i-1)(\alpha+\beta),\quad (i-2)(\alpha+\beta), \cdots (\alpha+\beta)$ |
| $B_{2i-1}$ | $i(\alpha+\beta)-\alpha$ | $(i-1)(\alpha+\beta)-\alpha,\quad (i-2)(\alpha+\beta)-\alpha, \cdots \beta$ |
| $B_{2i}$ | $i(\alpha+\beta)$ | $i(\alpha+\beta)-\beta,\quad (i-1)(\alpha+\beta)-\beta, \cdots \alpha$ |

As a preliminary step consider the number of ways a path consisting of $i$ steps to the left and $ki-\alpha$ steps to the right can go from $\alpha$ to 0 without touching $\alpha$ after the first step. Let the number of these paths be $H_\alpha(i)$. While this number can be computed by elementary methods a more elegant formula has been obtained by Professor Kemperman, namely,

$$(1)\qquad H_\alpha(i) = (k+1)\sum_{0\le r<\alpha/(k+1)} \frac{(-1)^r}{(i-r)(k+1)-1}\binom{(i-r)(k+1)-1}{i-r}\binom{\alpha-1-kr}{r} - \frac{\alpha}{(k+1)i-\alpha}\binom{(k+1)i-\alpha}{i}.$$

The proof of this is contained in the appendix.

The number of ways of going from 0 to $\alpha$ after exactly $j$ steps to the left and $kj+\alpha$ steps to the right will be indicated by $J(\alpha, j)$ where

$$(2)\qquad J(\alpha, j) = \binom{(k+1)j+\alpha}{j}.$$

Combining the results of Table 1 and the definitions of $H$ and $J$ we see that

$$(3)\qquad \begin{aligned} N(A_{2i-1}) &= \sum_{j_1+\cdots+j_{i+1}=n} J(i(\alpha+\beta)-\beta, j_1)\prod_{k=2}^{i} H_{\alpha+\beta}(j_k)H_\alpha(j_{i+1}),\\ N(A_{2i}) &= \sum_{j_1+\cdots+j_{i+1}=n} J(i(\alpha+\beta), j_1)\prod_{k=2}^{i+1} H_{\alpha+\beta}(j_k),\\ N(B_{2i-1}) &= \sum_{j_1+\cdots+j_{i+1}=n} J(i(\alpha+\beta)-\alpha, j_1)\prod_{k=2}^{i} H_{\alpha+\beta}(j_k)H_\beta(j_{i+1}),\\ N(B_{2i}) &= \sum_{j_1+\cdots+j_{i+2}=n} J(i(\alpha+\beta), j_1)H_\beta(j_2)\prod_{k=3}^{i+1} H_{\alpha+\beta}(j_k)H_\alpha(j_{i+2}). \end{aligned}$$

This completes Theorem 1. The infinite series occurring in this theorem is really a finite series in view of $N(A_{2i-1}) = \cdots = N(B_{2i}) = 0$ For

$$i > nk/(\alpha+\beta).$$

To get Theorem 2 it is only necessary to take the limit as $k\to\infty$ in the various formulas given above. By Stirling's formula,

$$\binom{t(1+k)}{t} = \frac{t^t}{t!} O((1+k)^t) \quad \text{as} \quad k \to \infty. \tag{4}$$

Here and below we will use $a_k = 0(b_k)$ to mean $\lim_{k\to\infty} a_k/b_k = 1$.

Using (4) and a few more applications of Stirling's formula and remembering that $\alpha = -[-xkn]$ and $\beta = -[-ykn]$, we obtain

$$\begin{aligned} H_\alpha(i) = &\left\{ \sum_{0 \le r < xn} (-1)^r \frac{(i-r)^{i-r-1}(xn-r)^r}{(i-r)!r!} - xn(i-xn)^{i-1}/i! \right\} \\ &\cdot O((1+k)^i) = \bar{H}_x(i) O((1+k)^i) \end{aligned} \tag{5}$$

where the last equality defines $\bar{H}_x(i)$. Using (4) again

$$J(\alpha, j) = \frac{1}{j!}(j+xn)^j O((1+k)^j) = \bar{J}(x,j)O((1+k)^j) \tag{6}$$

and

$$\binom{(k+1)n}{n}^{-1} = \frac{n!}{n^n} O((1+k)^{-n}).$$

Combining these results and (3) the following equations are obtained:

$$\begin{aligned}
&\lim_{k\to\infty} \binom{(k+1)n}{n}^{-1} N(A_{2i-1}) \\
&\qquad = \sum_{j_1+\cdots+j_{i+1}=n} \bar{J}(i(x+y)-y, j_1) \prod_{k=2}^{i} \bar{H}_{x+y}(j_k)\bar{H}(j_{i+1}) \\
&\qquad = \bar{N}(A_{2i-1}), \\
&\lim_{k\to\infty} \binom{(k+1)n}{n}^{-1} N(A_{2i}) \\
&\qquad = \sum_{j_1+\cdots+j_{i+1}=n} \bar{J}(i(x+y), j_1) \prod_{k=2}^{i+1} \bar{H}_{x+y}(j_k) \\
&\qquad = \bar{N}(A_{21}), \\
&\lim_{k\to\infty} \binom{(k+1)n}{n}^{-1} N(B_{2i-1}) \\
&\qquad = \sum_{j_1+\cdots+j_{i+2}=n} \bar{J}(i(x+y)-x, j_1) \prod_{k=2}^{i} \bar{H}_{x+y}(j_k)\bar{H}_y(j_{i+1}) \\
&\qquad = \bar{N}(B_{2i-1}), \\
&\lim_{k\to\infty} \binom{(k+)n}{n}^{-1} N(B_{2i}) \\
&\qquad = \prod_{j_1+\cdots+j_{i+2}=n} \bar{J}(i(x+y), j_1)\bar{H}_y(j_2) \prod_{k=3}^{i+1} \bar{H}_{x+y}(j_k)\bar{H}(j_{i+2}) \\
&\qquad = \bar{N}(B_{2i}).
\end{aligned} \tag{7}$$

This completes Theorem 2.

Attention should be drawn to a paper of Korolyuk [2] wherein the author gives different versions of the probabilities we have presented for the case $x = y$.

## REFERENCES

[1] J. Blackman, "An extension of the Kolmogorov distribution," *Ann. Math. Stat.*, Vol. 27 (1956), pp. 513–520.

[2] V. S. Korolyuk, "On the difference of the empirical distribution of two independent samples," *Izv. Akad. Nauk. SSSR*, Vol. 19 (1955), pp. 81–96.

## APPENDIX

By J. H. B. Kemperman

By a *path* of length $n$ we shall mean an ordered sequence of $n + 1$ integers $(z_0, \cdots, z_n)$, such that

$$z_i - z_{i-1} \geqq -1 \qquad (i = 1, \cdots, n).$$

For each path $\pi_n = (z_0, \cdots, z_n)$, let

$$P(\pi_n) = \prod_{i=1}^{n} p(z_i - z_{i-1}),$$

(the weight or "probability" of $\pi_n$). Here, the $p_i = p(i)$, $(i = -1, 0, +1, \cdots)$, denote given (real or complex) numbers, $p(-1) \neq 0$. Finally, let

$$e_z(n) = \sum_{\pi_n}{}' p(\pi_n),$$

the summation being extended over all the paths $\pi_n = (z_0, \cdots, z_n)$ with $z_0 = 0$, $z_n = z$, $z_i \neq z (i = 0, 1, \cdots, n - 1)$.

Theorem. *For* $n = 1, 2, \cdots,$

$$\begin{aligned} e_z(n) = -zr_z(n)/n + \sum_{j=1}^{\infty} j(j+1)p_j \\ \sum_{0<m\leqq +z} r_z(-m)r_{-j}(m+n-1)/(m+n-1). \end{aligned} \tag{8}$$

*Here, for arbitrary integers $h$ and $s$, $r_h(s)$ is defined as the coefficient of $w^{h+s}$ in the formal development*

$$(p_{-1} + p_0 w + p_1 w^2 + \cdots)^s = \sum_h r_h(s) w^{h+s};$$

*especially,* $r_h(s) = 0$ *if* $h + s < 0$.

Proof. Let $n$ and $z$ be given integers, $n \geqq 1$. For any path $(z_0, \cdots, z_n)$ with $z_0 = 0$, $z_n = z$, we have

$$z_i - z_{i-1} = z - \sum_{\substack{\nu=1 \\ \nu \neq i}}^{n} (z_\nu - z_{\nu-1}) \leqq z + n - 1,$$