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SPACINGS GENERATED BY MIXED SAMPLES

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1. Summary and introduction. Suppose $X(1, 1), X(1, 2), \cdots, X(1, n_1), X(2, 1), \cdots, X(2, n_2), \cdots, X(k, 1), \cdots, X(k, n_k)$ are independent chance variables, $X(i, j)$ having the probability density function $f_i(x)$, for $j = 1, \cdots, n_i$, $i = 1, \cdots, k$. We assume that for each i, $f_i(x)$ is bounded and has at most a finite number of discontinuities. We denote $n_1 + n_2 + \cdots + n_k$ by N, and we assume that n_i/N is equal to r_i, where r_i is a given positive number. Let $Y_1 \leqq Y_2 \leqq \cdots \leqq Y_N$ denote the ordered values of the N observations

$$X(1, 1), \cdots, X(k, n_k).$$

Define W_i as $Y_{i+1} - Y_i$ for $i = 1, \cdots, N - 1$. For any given nonnegative t, let $R_N(t)$ denote the proportion of the values $W_1, \cdots, W_{N-1}$ which are greater than t/N. Let $S(t)$ denote

$$\int_{-\infty}^{\infty} (r_1f_1(x) + r_2f_2(x) + \cdots + r_kf_k(x)) \exp \{-t[r_1f_1(x) + \cdots + r_kf_k(x)]\} \, dx$$

and $V(N)$ denote $\sup_{t \geqq 0} | R_N(t) - S(t) |$. Then it is shown that $V(N)$ converges stochastically to zero as N increases. This is a generalization of [1], where k was equal to unity. The result is applied to find the asymptotic behavior of ranks in a k-sample problem.

2. Proof of the stochastic convergence of $V(N)$. As in [1], if it can be shown that $R_N(t)$ converges stochastically to $S(t)$ for each positive t, the convergence of $V(N)$ follows. Therefore we fix a positive value for t.

We define the chance variable $Z(i, j, N)$ to be equal to unity if no observations fall in the half-open interval $[(X(i, j), X(i, j) + t/N]$, and equal to zero otherwise. We denote $1/N \sum_{i=1}^{k} \sum_{j=1}^{n_i} Z(i, j, N)$ by $K(N)$. Clearly,

$$K(N) = (1 - 1/N)R_N(t) + 1/N,$$

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so our purpose is accomplished if we show that $K(N)$ converges stochastically to $S(t)$ as N increases.

We denote $\int_{-\infty}^{x} f_i(x)\,dx$ by $F_i(x)$.

$$E\{Z(i, j, N)\} = \int_{-\infty}^{\infty} \left[1 - F_i\left(x + \frac{t}{N}\right) + F_i(x)\right]^{n_i - 1} \cdot \prod_{h \neq i} \left[1 - F_h\left(x + \frac{t}{N}\right) + F_h(x)\right]^{n_h} dF_i(x).$$

But with the exception of a finite number of points, $F_i(x + t/N) - F_i(x)$ can be written as $[f_i(x) + \epsilon_i(x, t/N)]t/N$, where $\epsilon_i(x, t/N)$ approaches zero as N increases, for each x. Since $f_i(x)$ is bounded ($i = 1, \cdots, k$), it follows easily that $E\{Z(i, j, N)\}$ approaches

$$\int_{-\infty}^{\infty} \exp\{-t[r_1 f_1(x) + \cdots + r_k f_k(x)]\}\, dF_i(x)$$

as N increases. It follows immediately that $E\{K(N)\}$ approaches $S(t)$ as N increases.

Next we examine variance $\{K(N)\}$, which equals $N^{-2}\sum_{i=1}^{k}\sum_{j=1}^{n_i}$ variance $\{Z(i, j, N)\} + 1/N^2 \sum\sum\sum\sum_{(i,j)\neq(g,h)} \operatorname{cov}\{Z(i, j, N), Z(g, h, N)\}$. The first term in this last expression clearly approaches zero as N increases, since there are N uniformly bounded terms in the sum. We shall show that the second term also approaches zero by showing that the covariances approach zero uniformly. Since there are $N(N - 1)$ covariances, the factor $1/N^2$ guarantees the approach to zero. If $i \neq g$, $E\{Z(i, j, N) \cdot Z(g, h, N)\}$ is equal to

$$\iint_{|x-y| > \frac{t}{N}} \prod_{b \neq i, g} \left[1 - F_b\left(x + \frac{t}{N}\right) + F_b(x) - F_b\left(y + \frac{t}{N}\right) + F_b(y)\right]^{n_b}$$
$$\cdot \left[1 - F_i\left(x + \frac{t}{N}\right) + F_i(x) - F_i\left(y + \frac{t}{N}\right) + F_i(y)\right]^{n_i - 1}$$
$$\cdot \left[1 - F_g\left(x + \frac{t}{N}\right) + F_g(x) - F_g\left(y + \frac{t}{N}\right) + F_g(y)\right]^{n_g - 1} dF_i(x)\, dF_g(y).$$

By computations similar to those used on $E\{Z(i, j, N)\}$, it follows that

$$E\{Z(i, j, N) \cdot Z(g, h, N)\}$$

approaches

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{-t[r_1 f_1(x) + \cdots + r_k f_k(x)]\} \cdot \exp\{-t[r_1 f_1(y) + \cdots + r_k f_k(y)]\} \cdot dF_i(x)\, dF_g(y)$$

and from this it follows that cov $\{Z(i, j, N), Z(g, h, N)\}$ approaches zero as N increases. In the same way, it follows that

$$\operatorname{cov}\{Z(i, j, N), Z(i, h, N)\}$$

approaches zero ($j \neq h$). Thus variance$\{K(N)\}$ approaches zero as N increases, so $K(N)$ converges stochastically to $S(t)$, as does $R_N(t)$. Therefore we have shown that $V(N)$ converges stochastically to zero as N increases.

3. Application to ranks in k-samples. Define $T(i, j)$ as $F_1(X(i, j))$. Then $T(1, 1), \cdots, T(1, n_1)$ have unform distributions. Let $G_i(x)$ denote the resulting distribution function for $T(i, j)$. We assume that $G_i(x)$ allows a density function $g_i(x)$ (then $g_i(x)$ is zero outside the interval [0, 1], is bounded, and has a finite number of discontinuities). Let $V_1 \leqq V_2 \leqq \cdots \leqq V_{N-n_1}$ denote the ordered values of $T(2, 1), \cdots, T(k, n_k)$, and let V_0 equal zero, V_{N-n_1+1} equal one. Let S_i denote the number of $T(1, j)$'s which lie in the interval

$$[V_{i-1}, V_i], \qquad i = 1, \cdots, N - n_1 + 1.$$

For each nonnegative integer r, let $Q_n(r)$ be the proportion of values among $S_1, \cdots, S_{N-n_1+1}$ which are equal to r. Define $g(y)$ as $\sum_{i=2}^{k} (r_i/(1 - r_1))g_i(y)$, and α as $(r_1/(1 - r_1))$. Define $Q(r)$ as

$$\alpha^r \int_0^1 \frac{g^2(y)}{[\alpha + g(y)]^{r+1}} dy.$$

Then it follows from the results above, using also the argument in [2], that $\sup_{r \geqq 0} | Q_N(r) - Q(r) |$ converges stochastically to zero as N increases. This can be used to show that certain tests of the hypothesis

$$F_1(x) = F_2(x) = \cdots = F_k(x)$$

are consistent. The discussion parallels that found in [2].

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CORRECTION TO "AN EXTENSION OF THE KOLMOGOROV DISTRIBUTION"

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1. Summary. It has been pointed out by J. H. B. Kemperman that an error in [1] invalidates the formulas arrived at in that paper. It is the purpose of this note to supply the correct formulas for the probabilities of Theorems 1 and 2. An Appendix by Professor Kemperman is included.

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