General Remarks. Consider the case where there are r instead of 2 response classes. Then it is convenient to regard the r probabilities $p_1, \cdots, p_r$ as a normalized column vector, $\mathbf{p}$. With t possible events, there are t corresponding linear operators, which can be represented by t $r \times r$ stochastic matrices, $M_1, \cdots, M_i, \cdots, M_t$. Then, the value of the vector $\mathbf{p}$ at the $k + 1$st trial, after the occurrence of event E_i, is given by $M_i \mathbf{p}_k$ where $\mathbf{p}_k$ is the value of the vector at the kth trial. Under the assumption of combining classes, T_i may be written as $M_i = \alpha_i I + (1 - \alpha_i)\Lambda_i$ where I is the $r \times r$ identity matrix, and Λ_i is an $r \times r$ matrix in which all columns are identical, and the r entries are denoted by $\lambda_1^{(i)}, \cdots, \lambda_r^{(i)}$. It is then readily shown that the commutator of M_i and M_j is the vector: $\boldsymbol{\mu} = (1 - \alpha_i)(1 - \alpha_j)(\Lambda_i - \Lambda_j)^*$. The last term $(\Lambda_i - \Lambda_j)^*$ is any of the r identical column vectors of the matrix $(\Lambda_i - \Lambda_j)$. It is now necessary to find f such that $f(M_i\mathbf{p}) = T_i f(\mathbf{p})$ and such that $T_i T_j f(\mathbf{p}) = T_j T_i f(\mathbf{p})$, where $f(\mathbf{p})$ denotes the column vector with elements $f(p_1), \cdots, f(p_r)$. The theorem goes through as before, these conditions being satisfied if and only if f is periodic with $f(\mathbf{p}) = f(\mathbf{p} + \boldsymbol{\mu})$, where $\boldsymbol{\mu}$ is the commutator vector defined above. The determination of conditions under which f has an inverse is a somewhat deeper question. For the present, it is sufficient to remark that if the qth component of $\mathbf{p}_k$ is bounded by A_q and B_q for some $q \leqq r$ and f is monotone in $[A_q, B_q]$, then f has an inverse in that region, and the values of this qth component on successive trials can be used to estimate the parameters.

Returning to the case of $r = 2$ and $t = 2$, it appears that for a given Q_1 and Q_2 half the commutator $\mu/2$, gives a measure of the largest set of values of p on which it is possible to find a 1–1 mapping f such that the induced transformations T_1 and T_2 commute. At the same time, μ also gives a measure of the fraction of the interval [0, 1] on which the commutativity of Q_1 and Q_2 fails to hold.

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ADDENDA TO "INTRA BLOCK ANALYSIS FOR FACTORIALS IN TWO-ASSOCIATE CLASS GROUP DIVISIBLE DESIGNS"[1]

By Ralph Allan Bradley and Clyde Young Kramer

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1. Nair and Rao [1] in a very fundamental paper discussed confounding in asymmetrical (asymmetrical in the factor levels) factorial experiments. They gave a general formulation of the combinatorial set-up for balanced confounded designs, assuming their existence, of asymmetrical factorial experiments and

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showed how to construct some optimum designs for two-factor experiments with some extensions to three and four factors.

Requirements for balanced confounded designs of asymmetrical factorials were set forth. Using their notation, we let $(i_1, \cdots, i_m)$ be the treatment combination with the i_tth level of factor F_t, $t = 1, \cdots, m$, F_t having s_t levels. There are $v = \prod_t s_t$ treatment combinations to be arranged in b blocks of k experimental units with no treatment combination on two units of the same block. Requirements for balanced confounding were:

(i) Every treatment combination is replicated r times.

(ii) The treatments $(i_1, \cdots, i_m)$ and $(j_1, \cdots, j_m)$ occur together in $\lambda_{k_1, \cdots, k_m}$ blocks where $k_t = 0$ or 1 as $i_t = j_t$ or $i_t \neq j_t$.

Nair and Rao discussed two-factor experiments in detail showing the estimation of treatment differences, efficiency and amount of information, and tests of significance.

2. Nair [2] in a short paper in 1953 showed that the earlier work of Bose and Connor [3] on group divisible, partially balanced, incomplete block designs with two associate classes could be regarded as a special case of the analysis for confounded asymmetrical factorial experiments with two factors. Also, he showed that designs constructed by Nair and Rao correspond to designs of the semi-regular class of group divisible designs typed by Bose and Shimamoto [4].

3. Kramer and Bradley [5], using group divisible designs catalogued by Bose, Clatworthy, and Shrikhande [6], showed how factorial treatment combinations may be used in these designs and presented the straight-forward least squares derivation of the intra-block analysis for such experiments. This essentially completes the cycle. The discussion of confounding in asymmetrical factorials is the most general of the papers; the factors could be regarded as pseudo-factors to derive the analysis for non-factorial treatments in the two-associate class group divisible designs. Finally, the treatments in the group divisible designs were replaced by factorial treatment combinations to produce confounded asymmetrical factorials.

4. Analyses for the basic two-factor factorial in [5] could have been based on the work of Nair and Rao [1] and Nair [2]. The association of notation (the Bradley-Kramer notation followed by that of Nair and Rao), where notations differed, is as follows:

$$m, s_2 ; n, s_1 ; \lambda_1 , \lambda_{10} ; \lambda_2 , \lambda_{01} = \lambda_{11} ; \qquad (\lambda_1 + rk - r)/k, p_{11} \doteq p_{1\cdot} ;$$

$$mn\lambda_2/k, p_{\cdot 1} ; \qquad Q_{ij} , Q(i, j);$$

$$t_{ij} , t(i, j); \quad A\text{-factor}, F_2\text{-factor}; \quad \text{and } C\text{-factor}, F_1\text{-factor}.$$

The association of notations leads to equivalences of results. In the order as before, Table 1 corresponds to Table 2, variances of effects in (5.22) and (5.23) with (3.23) and (3.22), and efficiencies (5.27), (5.28), and (5.29) with those indicated on the bottom of page 113 of [1].

5. We are indebted to K. R. Nair for drawing these matters to our attention.

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ACKNOWLEDGMENT OF PRIORITY

By John S. White

It has been called to my attention that the results in my note 'A t-test for the serial correlation coefficient' (Ann. Math. Stat., Dec. 1957) duplicate results obtained by M. H. Quenouille in 'Approximate tests of correlation in times-series 3' (Proc. Cambridge Phil. Soc., Vol. 45, part 3, 1949). I wish to acknowledge the priority of Prof. Quenouille's results which were inadvertently overlooked.

CORRECTION TO "ON THE POWER OF CERTAIN TESTS FOR INDEPENDENCE IN BIVARIATE POPULATIONS"

By H. S. Konijn

p. 304, line 13: like the left-hand side, the right-hand side is a function of n^*.

p. 305: beginning with the word "exists" Theorem 1.2 should read the same as Theorem 1.1, except that the exponent changes from $1/h$ to $1/hp^*$.

p. 306, line 1: change "of" to "at".

p. 309, line 3: insert "if ρ exists," preceding the expression for ER_n.

p. 309, last line of section 1: for $ER_n = 0$ read $ER_n \rightarrow 0$.

p. 309, line 8 of section 2: change "consist merely of" to "contain", and "or" to "plus".

p. 309, line 3 from below: change Λ to $\Lambda - \{\lambda^0\}$.