ON BALANCING IN FACTORIAL EXPERIMENTS[1]

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1. Introduction and Summary. R. C. Bose [1] has considered the problem of balancing in symmetrical factorial experiments. In all the designs considered in that paper, the block size is a power of S, the number of levels of a factor. The purpose of the present paper is to consider a general class of designs, where a 'complete balance' is achieved over different effects and interactions. It is proved in this paper (Theorems 4.1 and 4.2) that if a 'complete balance' is achieved over each order of interaction, the design must be a partially balanced incomplete block design. Its parameters are found. The usual method of analysis (of a PBIB design [2]) which is not so simple, can be simplified a little for these designs (section 5), on account of the balancing of the interactions of various orders. The simplified method of analysis is illustrated by a worked out example 5.1. Finally, the problem of balancing is dealt with for asymmetrical factorial experiments also. Incidentally, it may be observed that the generalised quasifactorial designs discussed by C. R. Rao [4] are the same as found by the author, from considerations of balancing.

2. Some lemmas regarding C-matrix and orthogonal contrasts. Let there be v treatments replicated $r_1, r_2, \cdots, r_v$ times respectively, in b blocks of k plots each. Let n_{ij} be the number of times the ith treatment occurs in the jth block; ($i = 1, 2, \cdots, v$; $j = 1, 2, \cdots, b$). Then $\mathbf{N} = [n_{ij}]$ is the incidence matrix of the design. It is assumed that every n_{ij} is either zero or one. The set up assumed is that the yield of a plot in the jth block having the ith treatment is $\mu + \alpha_i + t_j + \epsilon_{ij}$ where μ is the over-all effect, α_i is the effect of the ith block, t_j is the effect of the jth treatment and ϵ_{ij} is the experimental error. ϵ_{ij}'s are assumed to be independent normal variates with zero mean and variance σ^2. Let Q_i be the adjusted treatment yield (adjusted for block effects) of the ith treatment, and $\hat{t}_i$ be a solution for t_i of the least square equations. Let $\mathbf{Q}$, $\mathbf{t}$ and $\hat{\mathbf{t}}$ denote the column vectors $(Q_1, Q_2, \cdots, Q_v)$, $(t_1, t_2, \cdots, t_v)$, and $(\hat{t}_1, \hat{t}_2, \cdots, \hat{t}_v)$ respectively.

It is well known that

$$\mathbf{Q} = \mathbf{C}\hat{\mathbf{t}} \tag{2.1}$$

and the variance-covariance matrix of Q is

$$\sigma^2\mathbf{C}. \tag{2.2}$$

where

$$\mathbf{C} = \text{diag}\,(r_1, r_2, \cdots, r_v) - \frac{1}{k}\mathbf{NN}', \tag{2.3}$$

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diag $(r_1, r_2, \cdots, r_v)$ stands for a diagonal matrix, with diagonal elements $r_1, r_2, \cdots, r_v$.

If $\mathbf{l}'\mathbf{l} = 1$, the contrast $\mathbf{l}'\mathbf{t}$ will be called a normalised contrast.

LEMMA 2.1. *Let* $\mathbf{l}_1'\mathbf{t}, \mathbf{l}_2'\mathbf{t}, \cdots, \mathbf{l}_{v-1}'\mathbf{t}$ *be* $v - 1$ *estimable normalised orthogonal contrasts* ($\mathbf{l}_i$'s *are v-vectors*), *such that*

$$V(\mathbf{l}_i'\hat{\mathbf{t}}) = \sigma^2/\theta_i \tag{2.4}$$

$$\operatorname{Cov}(\mathbf{l}_i'\hat{\mathbf{t}}, \mathbf{l}_j'\hat{\mathbf{t}}) = 0 \qquad i \neq j \tag{2.5}$$

then (i) the **C**-matrix defined in (2.3) is given by

$$\mathbf{C} = \sum_{q=1}^{v-1} \theta_q \mathbf{l}_q \mathbf{l}_q'. \tag{2.6}$$

(ii) Estimate of $\mathbf{l}_i'\mathbf{t}$ is given by

$$\mathbf{l}_i'\hat{\mathbf{t}} = \mathbf{l}_i'\mathbf{Q}/\theta_i. \tag{2.7}$$

PROOF. Let $\mathbf{E}_{mn}$ denote an $m \times n$ matrix, all the elements of which are unity and

$$\left[\mathbf{l}_1 \mid \mathbf{l}_2 \mid \cdots \mid \mathbf{l}_{v-1} \,\middle|\, \frac{1}{\sqrt{v}} \mathbf{E}_{v1}\right] = \left[\mathbf{L}_1 \,\middle|\, \frac{1}{\sqrt{v}} \mathbf{E}_{v1}\right] = \mathbf{L}, \tag{2.8}$$

then

$$\mathbf{L}\mathbf{L}' = \mathbf{I}_v = \mathbf{L}'\mathbf{L}, \tag{2.9}$$

where I_v denotes a $v \times v$ identity matrix. From (2.1) and (2.9) we have

$$\begin{aligned} \mathbf{Q} &= \mathbf{C}\mathbf{L}\mathbf{L}'\hat{\mathbf{t}}. \\ \mathbf{L}'\mathbf{Q} &= \mathbf{L}'\mathbf{C}\mathbf{L}(\mathbf{L}'\hat{\mathbf{t}}), \end{aligned} \tag{2.10}$$

but

$$\mathbf{E}_{1v}\mathbf{Q} = \mathbf{O} \quad \text{and} \quad \mathbf{E}_{1v}\mathbf{C} = \mathbf{O}; \tag{2.11}$$

hence (2.10) reduces to

$$\mathbf{L}_1'\mathbf{Q} = \mathbf{L}_1'\mathbf{C}\mathbf{L}_1(\mathbf{L}_1'\hat{\mathbf{t}}). \tag{2.12}$$

From (2.2) it follows that the variance-covariance matrix of $\mathbf{L}_1'\mathbf{Q}$ is

$$\mathbf{L}_1'\mathbf{C}\mathbf{L}_1 \sigma^2. \tag{2.13}$$

By hypothesis each of $\mathbf{l}_1'\mathbf{t} \cdots \mathbf{l}_{v-1}'\mathbf{t}$ is estimable, therefore $(\mathbf{L}_1'\mathbf{C}\mathbf{L}_1)$ must have rank $v - 1$. Hence its inverse exists.

$$(\mathbf{L}_1'\hat{\mathbf{t}}) = (\mathbf{L}_1'\mathbf{C}\mathbf{L}_1)^{-1}\mathbf{L}'\mathbf{Q} \tag{2.14}$$

and.

$$V(\mathbf{L}_1'\hat{\mathbf{t}}) = (\mathbf{L}_1'\mathbf{C}\mathbf{L}_1)^{-1}\sigma^2. \tag{2.15}$$

Comparing with (2.4) we have

$$(\mathbf{L}_1' \mathbf{C}\mathbf{L}_1)^{-1} = \operatorname{diag}\left(\frac{1}{\theta_1}, \frac{1}{\theta_2}, \cdots, \frac{1}{\theta_{v-1}}\right) \tag{2.16}$$

$$\mathbf{L}_1' \mathbf{C} \mathbf{L}_1 = \operatorname{diag}(\theta_1, \theta_2, \cdots, \theta_{v-1}). \tag{2.17}$$

(2.11) and (2.17) imply that $\theta_1, \theta_2, \cdots, \theta_{v-1}'0$ are canonical roots of $\mathbf{C}$, and $\mathbf{l}_1, \mathbf{l}_2, \cdots, \mathbf{l}_{v-1}, (1/\sqrt{v})\, \mathbf{E}_{v1}$ are corresponding canonical vectors. Hence $\mathbf{C}$ is given by

$$\mathbf{C} = \sum_{q=1}^{v-1} \theta_q \mathbf{l}_q \mathbf{l}_q'. \tag{2.18}$$

Also from (2.14) and (2.16) it follows

$$\mathbf{L}_1' \hat{\mathbf{t}} = \operatorname{diag}\left(\frac{1}{\theta_1}, \frac{1}{\theta_2}, \cdots, \frac{1}{\theta_{v-1}}\right) \mathbf{L}_1' \mathbf{Q}. \tag{2.19}$$

This proves (2.7).

LEMMA 2.2. *In case some of the θ's in Lemma 2.1 are equal say $\theta_1 = \theta_2 = \cdots = \theta_r = \theta$, then there will be infinitely many sets of normalised orthogonal vectors corresponding to the canonical root θ. The variance-covariance matrix of contrasts corresponding to any such set will be*

$$\frac{\sigma^2}{\theta} \mathbf{I}_r$$

and representation of $\mathbf{C}$ *as given by Lemma* (2.1) *is unique;* i.e. if $\mathbf{l}_1, \cdots, \mathbf{l}_r$; *and* $\mathbf{n}_1, \cdots, \mathbf{n}_r$ *are any two sets, then*

$$\sum_{i=1}^{r} \mathbf{l}_i \mathbf{l}_i' = \sum_{i=1}^{r} \mathbf{n}_i \mathbf{n}_i'.$$

The proof follows easily from observing that

$$[\mathbf{n}_1 \mid \mathbf{n}_2 \mid \cdots \mid \mathbf{n}_r] = [\mathbf{l}_1 \mid \mathbf{l}_2 \mid \cdots \mid \mathbf{l}_r] \cdot \mathbf{A}, \tag{2.20}$$

where $\mathbf{A}$ is an $r \times r$ orthogonal matrix.

3. Definition of 'complete balance'. In a factorial experiment with m factors $F_1, F_2, \cdots, F_m$ each at S levels, if the treatments are denoted by $(x_1 x_2, \cdots, x_m)$ where x_i is the level of ith factor $(x_i = 0, 1, 2, \cdots, S-1)$; then a contrast $\sum C_{x_1\ x_2, \cdots, x_m} (x_1 x_2, \cdots, x_m)$ (Summation is over all $x_1 x_2, \cdots, x_m$) belongs to $(q-1)$th order interaction between the factors $F_{j_1}, F_{j_2}, \cdots, F_{j_q}$, if $C_{x_1, x_2, \cdots, x_m}$ depends only on $x_{j_1}, x_{j_2}, \cdots, x_{j_q}$ and $\sum C_{x_1 x_2, \cdots, x_m}$, summed over the levels of any one of these q factors, is zero.

Bose [1] has defined balance over a particular order of interaction in symmetric factorial experiments. In general, that definition is not interpretable, e.g. when a number of levels S is not a power of a prime, or the block size is not a power of S. So a more general definition is necessary.

DEFINITION 3.1. We shall define that a 'complete balance' is achieved over a set of n normalised orthogonal contrasts $\mathbf{l}_n^1\mathbf{t}, \cdots, \mathbf{l}_n^1\mathbf{t}$ if and only if the variance-covariance matrix of their estimates is

$$\frac{\sigma^2}{\theta}\mathbf{I}_n .$$

DEFINITION 3.2. A more obvious definition of 'complete balance' over a set of vectors or contrasts represented by them is that every linear combination of these vectors giving a normalised contrast is estimated with the same variance say σ^2/θ.

THEOREM 3.1. *Two Definitions* 3.1 *and* 3.2 *are equivalent.*

We will now say that complete balance is achieved over $(q - 1)$th order of interaction; if a complete set of $\binom{m}{q}(S - 1)^q$ normalised orthogonal contrasts has variance-covariance matrix $(\sigma^2/\theta_q)\ \mathbf{I}$, or if every normalised contrast belonging to the q factor interaction is estimated with the same variance σ^2/θ_q.

4. Balanced factorial designs and PBIB. Let there be m factors each at S levels in a symmetric factorial experiment. Let $\mathbf{L}_q$ be $S^m \times \binom{m}{q}(S - 1)^q$ matrix formed by a complete set of $\binom{m}{q}(S - 1)^q$ normalised orthogonal vectors forming q factor interactions with the variance of the estimate of any normalised contrast belonging to a q factor interaction equal to σ^2/θ_q; $q = 1, 2, \cdots, m$. Further let us assume that the covariance between the estimates of any two contrasts belonging to the ith and the jth $(i \neq j)$ orders of interactions is zero.

From Lemmas 1.1 and 1.2 $\mathbf{C}$ is uniquely represented and given by

$$\mathbf{C} = \sum_{q=1}^{m} \theta_q \mathbf{L}_q \mathbf{L}_q', \tag{4.1}$$

which can also be written as

$$\mathbf{C} = \left[\sum_{q=1}^{m} \theta_q f_{ij}^q\right], \qquad i, j = 1, 2, \cdots, S^m, \tag{4.2}$$

where f_{ij}^q is the element of $\mathbf{L}_q\mathbf{L}_q'$ corresponding to ith row and jth column.

Let the ith and jth treatments be $(x_1, x_2, \cdots, x_m)$ and $(y_1, y_2, \cdots, y_m)$ respectively, and let

$$(0, 0, \cdots, 0) \text{ and } (\underbrace{0, 0, \cdots, 0}_{p \text{ times}}, \underbrace{1, 1, \cdots, 1}_{(m-p) \text{ times}})$$

be the rth and sth treatments respectively. In the ith and jth treatments suppose exactly p factors occur at the same level. Say $x_{i_1} = y_{i_1}$, $x_{i_2} = y_{i_2}$, $\cdots$, $x_{i_p} = y_{i_p}$, and rest of the x_i's are not equal to the corresponding y_i's. Now interchange the levels $x_1, x_2, \cdots, x_m$ with zeros, i.e., in any treatment if the ith factor occurs at level x_i replace it by zero and if it occurs at level zero replace it by x_i. Perform this change for all the treatments. So naturally $y_{i_1}, y_{i_2}, \cdots, y_{i_p}$ will be changed to zeros. Now in the same manner as x_i's, interchange the remaining levels y_i's with ones. After these interchanges call the i_1th factor as the first factor, i_2th factor as the second factor, $\cdots$, and lastly i_pth factor as the pth factor and the other $(m - p)$ factors as $(p + 1)$th to mth factors; and rewrite all the treatments accordingly. Then it is obvious that the ith treatment becomes $(0\ 0, \cdots, 0)$ and the jth treatment,

$$(\underbrace{0\ 0, \cdots, 0}_{p \text{ times}}, \underbrace{1\ 1, \cdots, 1}_{(m-p) \text{ times}}).$$

It is obvious that interchanges of levels or renaming the levels of any factor does not alter the order of an interaction; so also the permutation or renaming of factors. Hence the above changes will not alter the order of any interaction.

After renaming the treatments arrange them in the original order. This will mean permutation of rows of $\mathbf{L}_q$. Let the rearranged matrix be $\mathbf{M}_q$. Then the rth row of $\mathbf{M}_q$ is the ith row of $\mathbf{L}_q$ and the sth row of $\mathbf{M}_q$ is the jth row of $\mathbf{L}_q$. Let $\mathbf{L}_q\mathbf{L}_q' = [l_{ij}]$ and $\mathbf{M}_q\mathbf{M}_q' = [m_{ij}]$ $i, j = 1, 2, \cdots, s_m$. Then it is evident that

$$l_{ij} = m_{rs}. \tag{4.3}$$

It is easy to see that M_q also gives a complete set of normalised orthogonal contrasts belonging to the $(q - 1)$th order or q-factor interactions. Hence from Lemma 2.2

$$\mathbf{L}_q\mathbf{L}_q' = \mathbf{M}_q\mathbf{M}_q' \quad \text{i.e.} \quad l_{rs} = m_{rs}. \tag{4.4}$$

Hence

$$l_{ij} = l_{rs}. \tag{4.5}$$

This shows that f_{ij}^q depends only on the exact number of factors say p, which occur at the same level in both ith and jth treatments. Let us denote it by f_p^q, $p = 0, 1, \cdots, m$; $p = m$ denotes all levels equal ($i = j$) and f_m^q is a diagonal element.

Equating the two forms of $\mathbf{C}$ (2.3) and (4.2) with $v = S^m$, we obtain

$$\text{Diag}\,(r_1, r_2, \cdots, r_v) - \frac{1}{k}\mathbf{NN}' = \left[\sum_{q=1}^{m} \theta_q f_{ij}^q\right], \tag{4.6}$$

Equating the elements we get

$$\sum_{q=1}^{m} \theta_q f_{ii}^q = r_i\left(1 - \frac{1}{k}\right) \tag{4.7}$$

and

$$\sum_{q=1}^{m} \theta_q f_{ij}^q = -\frac{\lambda_{ij}}{k} \ (i = j) \tag{4.8}$$

where λ_{ij} equals number of times ith and jth treatment occur together.

Using (4.5), (4.7) and (4.8) we have

$$r_1 = r_2 = , \cdots, r_v = \frac{k}{k-1}\sum_{q=1}^{m} \theta_q f_m^q = r \quad \text{say}, \tag{4.9}$$

and if ith and jth treatments have p factors at the same level,

$$-\frac{\lambda_{ij}}{k} = \sum_{q=1}^{m} \theta_q f_p^q = -\frac{\lambda_p}{k} \quad \text{say.} \tag{4.10}$$

Now (4.9) and (4.10) imply that the design must be a partially balanced incomplete block design. The definition of P.B.I.B. was first given by Bose and Nair [2] and later generalised by Nair and Rao [3].

Parameters b, k, r, being selected to satisfy combinatorial properties of the design and $v = S^m$, pth associates of any treatment will be all the treatments which have exactly p factors at the same level as in the given treatment. Hence

$$n_p = \binom{m}{p}(S-1)^{m-p} \qquad p = 0, 1, \cdots, m-1 \tag{4.11}$$

and

$$p_{ij}^k = \sum_u \binom{k}{u}\binom{m-k}{i-u}\binom{m-k-i+u}{j-u}(S-1)^{k-u}(S-2)^{(m-k-i-j+2u)}, \tag{4.12}$$

where summation extends over all the values of u which are less than or equal to minimum of k, i, j and for which $m + 2u > k + i + j$. Parameters $\lambda_0, \lambda_1, \cdots, \lambda_{m-1}$ are given by

$$\begin{bmatrix} f_0^0 & f_0^1 & \cdots & f_0^m \\ f_1^0 & f_1^1 & \cdots & f_1^m \\ \vdots & \vdots & & \vdots \\ f_m^0 & f_m^1 & \cdots & f_m^m \end{bmatrix} \begin{bmatrix} \theta_0 \\ \theta_1 \\ \vdots \\ \theta_m \end{bmatrix} = -\frac{1}{k} \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_m \end{bmatrix} \tag{4.13}$$

where $\lambda_m = -r(k-1)$

$$f_p^0 = \frac{1}{S^m} \quad \text{for} \quad p = 0, 1, \cdots, m.$$

and θ_0 is a dummy parameter always equal to zero, introduced to simplify the inverse relation. (4.13) can be shortly written as

$$\mathbf{F}(m) \cdot \boldsymbol{\theta}(m) = -\frac{1}{k} \boldsymbol{\lambda}(m).$$

As it will be shown later in section 7 the inverse relation of (4.13) exists and can be written as

$$\boldsymbol{\theta}(m) = -\frac{1}{k} [\mathbf{F}(m)]^{-1} \boldsymbol{\lambda}(m). \tag{4.14}$$

Therefore it also follows that in every P.B.I.B. with parameters as given above 'complete balance' over each order of interaction is achieved.

Hence we have the following theorems.

THEOREM 4.1. *Every* P.B.I.B. *design with parameters as given in* (4.11) *and* (4.12) *achieves a 'complete balance' over each order of interaction.*

THEOREM 4.2. *If in a design*

(i) *'complete balance' is obtained over each order of interaction*

(ii) *covariance between the estimates of any two contrasts belonging to different orders of interactions is zero; and*

(iii) *the number of plots is the same in every block; then the design must be a* P.B.I.B. *with parameters given above.*

COROLLARY 4.2.1. *In any design with S treatments if complete balance is achieved over all contrasts then the* **C**-*matrix is of the form given by*

$$\mathbf{C} = \theta\left(\mathbf{I}_S - \frac{1}{S}\mathbf{E}_{SS}\right) \tag{4.15}$$

COROLLARY 4.2.2. *In any design if complete balance is achieved over all contrasts and if the block size is the same for all the blocks, then the design must be a balanced incomplete block design.*

From (4.15) it follows that if $m = 1$,

$$f_0^1 = -\frac{1}{S}; \qquad f_1^1 = \frac{S-1}{S} \tag{4.16}$$

and hence

$$\mathbf{F}(1) = \frac{1}{S}\begin{bmatrix} 1 & -1 \\ 1 & S-1 \end{bmatrix}. \tag{4.17}$$

5. Analysis. Let us consider a symmetrical factorial design which is a P.B.I.B. of the type defined in section 4. Then as in (4.1)

$$\mathbf{C} = \sum_{q=1}^{m} \theta_q \mathbf{L}_q \mathbf{L}_q' \tag{5.1}$$

where θ's are given by (4.14) as

$$\boldsymbol{\theta}(m) = -\frac{1}{k}[\mathbf{F}(m)]^{-1} \cdot \boldsymbol{\lambda}(m). \tag{5.2}$$

Hence if $\mathbf{l}'\mathbf{t}$ is any normalised contrast belonging to $(q-1)$th order interaction, applying Lemma 1.1 we have

$$\mathbf{l}'\hat{\mathbf{t}} = \mathbf{l}'\mathbf{Q}/\theta_q \tag{5.3}$$

$$V(\mathbf{l}'\hat{\mathbf{t}}) = \sigma^2/\theta_q \tag{5.4}$$

and

$$\text{S.S. due to } \mathbf{l}'\mathbf{t} = \frac{(\mathbf{l}'\mathbf{Q})^2}{\theta_q}. \tag{5.5}$$

Now if T_i is the yield of the ith treatment, and $\mathbf{t}$ is a column vector $(T_1, T_2, \cdots, T_v)$ and we suppose that the experiment is a randomised block design with r replications, then

$$\mathbf{l}'\hat{\mathbf{t}} = \mathbf{l}'\mathbf{t}/r \tag{5.6}$$

(5.7) $$V(\mathbf{l}'\hat{\mathbf{t}}) = \sigma^2/r$$

and

(5.8) $$\text{S.S. due to } \mathbf{l}'\hat{\mathbf{t}} = \frac{(\mathbf{l}'\mathbf{T})^2}{r}.$$

Hence by comparing (5.3), (5.4) and (5.5) with (5.6), (5.7) and (5.8) respectively; we obtain the following procedure for analysis:

(i) calculation of $\mathbf{Q}$

(ii) calculation of sums of squares for each order of interaction separately, as if it were a randomised block experiment but using $\mathbf{Q}$ in place of $\mathbf{T}$

(iii) calculation of θ_q's by using (5.2)

(iv) correcting S.S. obtained in (ii) by θ_q's instead of by r.

If we have a quasifactorial experiment or if it is necessary for some purpose, we will require estimates of individual treatment effects and variances of elementary treatment comparisons. For that we know by (2.19),

(5.9) $$\mathbf{L}_q'\hat{\mathbf{t}} = \frac{1}{\theta_q}\mathbf{L}_q'\mathbf{Q}.$$

Hence

(5.10) $$\sum_{q=1}^{m} \mathbf{L}_q\mathbf{L}_q'\hat{\mathbf{t}} = \left[\sum_{q=1}^{m} \frac{1}{\theta_q}\mathbf{L}_q\mathbf{L}_q'\right]\mathbf{Q}.$$

Since

$$\left(\mathbf{L}_1 \mid \mathbf{L}_2 \mid \cdots \mid \mathbf{L}_m \mid \frac{1}{\sqrt{S^m}}\mathbf{E}_{s^m 1}\right)$$

is an orthogonal matrix, (5.10) simplifies to

(5.11) $$\left[\mathbf{I}_v - \frac{1}{v}\mathbf{E}_{vv}\right]\hat{\mathbf{t}} = \left[\sum_{q=1}^{m} \frac{1}{\theta_q}\mathbf{L}_q\mathbf{L}_q'\right]\mathbf{Q}.$$

where $v = s^m$. Put $\mathbf{E}_{1v}\hat{\mathbf{t}} = \mathbf{0}$ and we obtain a solution given by

(5.12) $$\hat{\mathbf{t}} = \left[\sum_{q=1}^{m} \frac{1}{\theta_q}\mathbf{L}_q\mathbf{L}_q'\right]\mathbf{Q}$$
$$\hat{\mathbf{t}} = \mathbf{MQ} \quad \text{say.}$$

Let U_i be defined as follows

(5.13) $$\mathbf{F}(m)\begin{bmatrix} 0 \\ 1/\theta_1 \\ 1/\theta_2 \\ \vdots \\ 1/\theta_m \end{bmatrix} = \begin{bmatrix} U_0 \\ U_1 \\ U_2 \\ \vdots \\ U_m \end{bmatrix}.$$

Then as in (4.5) $U_0, U_1, \cdots, U_m$ are the elements of $\mathbf{M}$. The element in the

ith row and jth column is U_p if the ith and jth treatments have exactly p factors at the same level. Hence (5.12) simplifies to

$$\hat{t}_j = U_m Q_j + \sum_{i=1}^{m} U_i S_i(Q_j) \tag{5.14}$$

where $S_i(Q_j)$ is sum of Q_j's corresponding to the treatments which are ith associates of t_j as defined in (4.11). From solutions (5.14) it is easy to see that, if t_i and t_j are pth associates

$$V(\hat{t}_i - \hat{t}_j) = 2\sigma^2(U_m - U_p). \tag{5.15}$$

EXAMPLE 5.1. Consider example with two factors A and B each at three levels

$$V = 3^2 \quad b = 6 \quad K = 6 \quad r = 4$$

$$n_0 = n_1 = 4 \quad \lambda_0 = 3 \quad \lambda_1 = 2$$

Block No.	Treatments					
1	(1 0)	(2 0)	(0 1)	(2 1)	(0 2)	(1 2)
2	(0 0)	(1 0)	(1 1)	(2 1)	(0 2)	(2 2)
3	(0 0)	(2 0)	(0 1)	(1 1)	(1 2)	(2 2)
4	(1 0)	(2 0)	(0 1)	(1 1)	(0 2)	(2 2)
5	(0 0)	(2 0)	(1 1)	(2 1)	(1 2)	(1 2)
6	(0 0)	(0 1)	(1 0)	(2 1)	(1 2)	(2 2)

Using the formulas in section 7.

$$\mathbf{F}(2) = \frac{1}{9}\begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \\ 1 & 4 & 4 \end{bmatrix}$$

$$[\mathbf{F}(2)]^{-1} = \begin{bmatrix} 4 & 4 & 1 \\ -2 & 1 & 1 \\ 1 & -2 & 1 \end{bmatrix}$$

Apply (5.2)

$$\begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \end{bmatrix} = -\frac{1}{6}\begin{bmatrix} 4 & 4 & 1 \\ -2 & 1 & 1 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ -20 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 7/2 \end{bmatrix}$$

Let Q_{ij} denote adjusted treatment yield of (ij) and

$$Q_{.j} = \sum_{i=0}^{2} Q_{ij}$$

$$Q_{i.} = \sum_{j=0}^{2} Q_{ij}.$$

Then

$$\text{Main effect of } A = \sum_{i=0}^{2} Q_{i.}^2/4.3.$$

$$\text{Main effect of } B = \sum_{j=0}^{2} Q_{.j}^2/4.3.$$

$$\text{Interaction } AB = \frac{2}{7}\left(\sum Q_{kj}^2 - \frac{\sum Q_{.j}^2}{3} - \frac{\sum Q_{i.}^2}{3}\right).$$

Also

$$\mathbf{F}(2)\begin{bmatrix} 0 \\ 1/\theta_1 \\ 1/\theta_2 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \\ 1 & 4 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 1/4 \\ 2/7 \end{bmatrix} = \begin{bmatrix} -1/42 \\ -1/28 \\ 5/21 \end{bmatrix}.$$

Hence using (5.14)

$$\hat{t}_j = \tfrac{5}{21}Q_j - \tfrac{1}{42}S_0(Q_j) - \tfrac{1}{28}S_1(Q_j)$$

and using (5.17) we get

$$V(\hat{t}_i - \hat{t}_j) = \tfrac{3}{14}\sigma^2 \quad \text{if } t_i \text{ and } t_j \text{ are 0th associates;}$$
$$= \tfrac{17}{42}\sigma^2 \quad \text{otherwise.}$$

6. $S_1^{m_1}S_2^{m_2}, \cdots, S_h^{m_h}$ **Factorial experiment.** Some matrix operators are defined to derive certain further results.

Operator '$\times$' denotes the Kronecker product of matrices defined by

$$\mathbf{A} \times \mathbf{B} = [a_{ij}] \times \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B}, \cdots, a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B}, \cdots, a_{2n}\mathbf{B} \\ \vdots & \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B}, \cdots, a_{mn}\mathbf{B} \end{bmatrix}. \tag{6.1}$$

The operator '$\otimes$' denotes the symbolic kroneker product of suffixes defined by the following illustrations.

$$\begin{bmatrix} \lambda_0 \\ \lambda_1 \end{bmatrix} \otimes \begin{bmatrix} \lambda_0 \\ \lambda_1 \end{bmatrix} = \begin{bmatrix} \lambda_{00} \\ \lambda_{01} \\ \lambda_{10} \\ \lambda_{11} \end{bmatrix} \tag{6.2}$$

and

$$\begin{bmatrix} \theta_0 \\ \theta_1 \end{bmatrix} \otimes \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} \theta_{00} \\ \theta_{01} \\ \theta_{02} \\ \theta_{10} \\ \theta_{11} \\ \theta_{12} \end{bmatrix}. \tag{6.3}$$

THEOREM 6.1. *If in a $S_1^{m_1}S_2^{m_2}, \cdots, S_h^{m_h}$ factorial experiment*

(i) *any contrast belonging to the interaction involving q_i factors at S_i levels ($i = 1, 2, \cdots, h$) is estimated with the same variance say $\sigma^2/\theta_{q_1 q_2, \cdots, q_h}$*

(ii) *the estimates of all effects and interaction are all uncorrelated and*

(iii) *the block size is a constant equal to k say; then the design must be a* PBIB *with relevant parameters and conversely.*

If any two treatments have exactly p_i factors (each at S_i level) at the same level for $i = 1, 2, \cdots, h$; they will be called $(p_1 p_2, \cdots, p_h)$th associates. Then we have

$$n_{p1, p2, \cdots, ph} = \prod_{i=1}^{h} \binom{m_i}{P_i} (S_i - 1)^{m_i - P_i} \tag{6.4}$$

and the relations between θ's and λ's are

$$\begin{aligned} &\mathbf{F}(m_1) \times \mathbf{F}(m_2) \times, \cdots, \times \mathbf{F}(m_h) \cdot \boldsymbol{\theta}(m_1) \otimes \boldsymbol{\theta}(m_2) \otimes \cdots \otimes \boldsymbol{\theta}(m_h) \\ &\qquad = -\frac{1}{k} \boldsymbol{\lambda}(m_1) \otimes \boldsymbol{\lambda}(m_2) \otimes \cdots \otimes \boldsymbol{\lambda}(m_h), \end{aligned} \tag{6.5}$$

$$\begin{aligned} &\boldsymbol{\theta}(m_1) \otimes \boldsymbol{\theta}(m_2) \otimes \cdots \otimes \boldsymbol{\theta}(m_h) \\ &\qquad = -\frac{1}{k} [\mathbf{F}(m_1)]^{-1} \times [\mathbf{F}(m_2)]^{-1} \times \cdots \times [\mathbf{F}(m_h)]^{-1} \\ &\qquad\qquad \cdot \boldsymbol{\lambda}(m_1)' \otimes \boldsymbol{\lambda}(m_2) \otimes \cdots \otimes \boldsymbol{\lambda}(m_h) \end{aligned} \tag{6.6}$$

where $\theta_{00, \cdots, 0} = 0$ and $\lambda_{m_1 m_2, \cdots, m_h} = -r(k - 1)$.

PROOF. The theorem can be proved for $h = 2$ exactly on the same lines as section 4 and relation (6.5) can be obtained by noting that the matrix representing an interaction of $(q_1 + q_2)$ factors out of $m_1 + m_2$ factors can be expressed as the Kronecker product of two matrices representing interactions of q_1 and q_2 factors, out of m_1 and m_2 factors respectively; and then using properties of the Kronecker product of matrices. And the result can be easily generalised for any value of h. (6.5) and (6.6) can be used to simplify the analysis of many asymmetrical factorial experiments. For example the design of plan 6.9 of Cochran and Cox [11] has parameters $v = 3.2^2$, $b = 6$, $r = 3$, $k = 6$ and $\lambda_{00} = 1$, $\lambda_{10} = 3$, $\lambda_{01} = 2$, $\lambda_{11} = 0$, $\lambda_{02} = 1$, $\lambda_{12} = -15$; hence θ's can be calculated as $\theta_{11} = \theta_{01} = \theta_{10} = 3$ and $\theta_{02} = 8/3$, $\theta_{12} = 5/3$ and the analysis can be performed as in section 5.

7. Evaluation of $\mathbf{F}(m)$ and $[\mathbf{F}(m)]^{-1}$. Put $m_1 = m_2 = \cdots = m_h = 1$ in (6.7) and write $\mathbf{F}(m_i)$ as $\mathbf{F}_i(1)$ to avoid ambiguity. Then (6.7) becomes

$$\begin{aligned} &\mathbf{F}_1(1) \times \mathbf{F}_2(1) \times \cdots \times \mathbf{F}_h(1) \cdot \boldsymbol{\theta}(1) \otimes \boldsymbol{\theta}(1) \otimes \cdots \otimes \boldsymbol{\theta}(1) \\ &\qquad = -\frac{1}{k} \boldsymbol{\lambda}(1) \otimes \boldsymbol{\lambda}(1) \otimes \cdots \otimes \boldsymbol{\lambda}(1). \end{aligned} \tag{7.1}$$

From (4.17) we have

$$\mathbf{F}_i(1) = \frac{1}{S_i} \begin{bmatrix} 1 & -1 \\ 1 & S_i - 1 \end{bmatrix}. \tag{7.2}$$

Hence

$$(7.3)\qquad [\mathbf{F}_i(1)]^{-1} = \begin{bmatrix} S_i - 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Hence (7.1) and its inverse relation can be written as

$$(7.4)\qquad \lambda_{d_1 d_2, \ldots, d_h} = \frac{-k}{\prod_{i=1}^{h} S_i} \sum \prod_{i=1}^{h} G_i(c_i d_i)\theta_{c_1 c_2, \cdots, c_h}$$

and

$$(7.5)\qquad \theta_{d_1 d_2, \cdots, d_h} = -\frac{1}{k} \sum \prod_{i=1}^{h} H_i(c_i d_i)\lambda_{c_1 c_2, \cdots, c_h},$$

where c_i and d_i take values 0 or 1; the summation is over all the values of $(c_1c_2, \cdots, c_h)$ and

$$\begin{aligned} G_i(11) &= S_i - 1 = H_i(0,0) \\ G_i(10) &= \quad - 1 = H_i(0,1) \\ G_i(00) = G_i(01) &= \quad\ \ 1 = H_i(10) = F_i(11). \end{aligned}$$

Now put $S_1 = S_2 = \cdots = S_h = S$ in (7.4) and $\theta_{c_1 c_2, \ldots, c_h} = \theta_q$ where $q =$ number of ones in $(c_1c_2, \cdots, c_h)$; on simplifying the coefficient of θ_q on the right side of (7.4) is given by

$$(7.6)\qquad \sum{}' \prod_{i=1}^{h} G_i(c_i d_i)$$

where $\sum'$ is summation for those values of $(c_1c_2, \cdots, c_h)$ which have exactly q ones and $h - q$ zeros. Now if the number of ones in $(d_1d_2, \cdots, d_h)$ is p, then it is easy to prove that,

$$(7.7)\qquad \sum{}' \prod_{i=1}^{h} G_i(c_i d_i) = \sum_i{}^{*} \binom{p}{i}\binom{h-p}{q-i}(-1)^{q-1}(S-1)^i$$

where $\sum_i^*$ is summation over all the values of i such that

$$\max(0, p + q - h) \leqq i \leqq \min(p, q).$$

Hence if there is balance over each order of interaction, $\lambda_{d_1 d_2, \ldots, d_h}$ depends only on the exact number of factors (say p) which occur at the same level. This must be so, as it was proved in section 4. Now writing $\lambda_{d_1 d_2, \ldots, d_h}$ as λ_p (7.4) becomes

$$(7.8)\qquad \lambda_p = \frac{-k}{S^h} \sum_{q=0}^{m} \sum_i{}^{*} \binom{p}{i}\binom{m-p}{q-i}(-1)^{q-1}(S-1)^i\theta_q.$$

Comparing (7.8) and (4.13) with $m = h$ we obtain

$$(7.9)\qquad f_p^q = \frac{1}{S^m} \sum_i{}^{*} \binom{p}{i}\binom{m-p}{q-i}(-1)^{q-1}(S-1)^i.$$

Working similiarly with (7.5) we obtain

$$(7.10)\quad \theta_q = -\frac{1}{K}\sum_{P=0}^{m}\sum_j{}^{*}\binom{m-q}{j}\binom{q}{m-p-j}(-1)^{m-p-j}(S-1)^j\lambda_P,$$

where $\sum_j^*$ is summation over all the values of j such that

$$\max\ (0,\ m-p-q) \leq j \leq \min\ (m-p, m-q).$$

Hence the inverse relation of (4.13) exists and is given by (7.10). If g_P^q is an element in the $(p+1)$th row and $(q+1)$th column of $[\mathbf{F}(m)]^{-1}$ then on comparing (7.10) and (4.14), we have

$$(7.11)\qquad g_p^q = \sum_j{}^{*}\binom{m-p}{j}\binom{p}{m-q-j}(-1)^{m-q-j}(S-1)^j.$$

Equations (7.9) and (7.11) are not convenient for writing down the matrices $\mathbf{F}(m)$ and $[\mathbf{F}(m)]^{-1}$. But the following relations, easily derivable from them will enable us to write out these matrices easily, along with a check.

$$(7.12)\qquad g_0^q = \binom{m}{q}(S-1)^{m-q}$$

$$(7.13)\qquad g_p^0 = (-1)^p\ (S-1)^{m-p}$$

$$(7.14)\qquad g_p^m = 1$$

$$(7.15)\qquad g_m^q = \binom{m}{q}(-1)^{m-q}$$

$$(7.16)\qquad g_{p-1}^{q-1} = g_p^{q-1} + g_{p-1}^q + (S-1)g_p^q$$

$$(7.17)\qquad g_p^q = S^m \cdot f_{m-p}^{m-q}.$$

8. Remarks. It should be noted that a general class of quasifactorial designs as defined by C. R. Rao [4] has the same parameters as given in (7.4). Hence the variance of a treatment contrast for any design belonging to that class can beo btained from (7.5).

Two factor designs in the above class form an important group. Their analysis can be done by using (7.4) and (7.5) with $h = 2$ and the method given in section 5. It will yield the same expressions as given by C. R. Rao and K. R. Nair in [10]. They are, therefore, not reproduced here.

Secondly construction of PBIB designs with parameters as required in the above designs is considered by M. N. Vartak [5] D. A. Sprott [6] and C. R. Rao [4].

Furthermore in the above design if $\lambda_{00} = \lambda_{01}$ or λ_{10} then $\theta_{11} = \theta_{01}$ or θ_{10} and the design becomes a group divisible PBIB.

All the designs mentioned in this paper can be successfully used by introducing Pseudo-factors. The method of introducing Pseudo-factors is discussed by Kramer and Bradley [12] for factorial experiments in group divisible PBIB.

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