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ADMISSIBLE ONE-SIDED TESTS FOR THE MEAN OF A RECTANGULAR DISTRIBUTION[1]

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1: Theorem. *Suppose we have a sample of $n > 1$ independent observations from a uniform distribution with unknown mean θ and known range R. Suppose we wish to test $H_0: \theta \leqq \theta_0$ against $H_1: \theta > \theta_0$. Then an essentially complete class of admissible tests is the class $\mathfrak{A}$ of all tests of the following type. Let u be the minimum observation, v the maximum. Let $g(u)$ be a nonincreasing function of u such that $g(u) = \theta_0 + \frac{1}{2}R$ for $u < \theta_0 - \frac{1}{2}R$. Accept H_0 if and only if $v < g(u)$.*

2. Discussion. The two-sided problem has been treated by Allan Birnbaum [1]. He showed that, for testing $H_0': \theta = \theta_0$ against $H_1': \theta \neq \theta_0$, an essentially complete class of admissible tests is the class of all tests of the following type. Let $v(u)$ be a nondecreasing function of u. Accept H_0 if and only if $v > v(u)$ and $\theta_0 - \frac{1}{2}R < u < v < \theta_0 + \frac{1}{2}R$.

Birnbaum [1] also noted that there is a uniformly most powerful size α test of $H_0': \theta = \theta_0$ against $H_1: \theta > \theta_0$, namely that accepting H_0' if $\theta_0 - \frac{1}{2}R < u < \theta_0 + (\frac{1}{2} - \alpha^{1/n})R$ and $v < \theta_0 + \frac{1}{2}R$. This corresponds in our notation to

$$g(u) = \begin{cases} \theta_0 + \frac{1}{2}R \text{ for } u < \theta_0 + (\frac{1}{2} - \alpha^{1/n})R, \\ \theta_0 - \frac{1}{2}R \text{ (say) otherwise.} \end{cases}$$

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In this rather simple situation, then, an essentially complete class of admissible tests of the simple hypothesis against one-sided alternatives consists of the uniformly most powerful test (just described) for each significance level, but the class of admissible tests of the composite hypothesis against one-sided alternatives is very general. The class of admissible tests of the simple hypothesis against two-sided alternatives is also very general, but quite different. It includes unions of admissible lower and upper one-sided rejection regions (if and) only if they are admissible for the simple hypothesis, and such unions form a portion "of measure zero" in the whole class.

In the following section we will prove the result stated in the first paragraph. The proof uses no general results of decision theory, such as the complete class theorem, but only direct methods of an essentially elementary constructive type. It obviously works in some slightly more general situations, which are given explicitly in [2].

3. Proof. Without loss of generality we may take $\theta_0 = 0, R = 2$. Since (u, v) is a sufficient statistic, an essentially complete class of tests is the class of all randomized tests based on (u, v). Suppose such a test is given, accepting H_1 with probability $\phi_0(u, v)$ when (u, v) is observed.

The triangle $T(\theta) = \{(u, v): \theta - 1 < u \leqq v < \theta + 1\}$ contains (u, v) with probability one if θ is the true mean. The probability of accepting H_0 using the test function ϕ is

$$E_\theta(\phi) = \iint_{T(\theta)} \phi(u, v) 2^{-n} n(n-1)(v-u)^{n-2}\, du\, dv. \tag{1}$$

If $\theta \geqq 0$, then $u > -1$ with probability one. If $\theta \leqq 0$, then $v < 1$ with probability one. Thus if (u, v) is not in $T(0)$, we know which hypothesis is correct. Accordingly, let

$$\phi_1(u, v) = \begin{cases} \phi_0(u, v) & \text{if} \quad (u, v) \in T(0), \\ 1 & \text{if} \quad u \leqq -1, \\ 0 & \text{if} \quad v \geqq 1. \end{cases} \tag{2}$$

Then ϕ_1 dominates ϕ_0, i.e. ϕ_1 is at least as good as ϕ_0 for any θ, i.e.

$$E_\theta(\phi_1) \gtreqless E_\theta(\phi_0) \quad \text{for} \quad \theta \lesseqgtr 0. \tag{3}$$

Define $f(v)$ for $-1 < v < 1$ by

$$\int_{-1}^{f(v)} (v-u)^{n-2}\, du = \int_{-1}^{v} \phi_1(u, v)(v-u)^{n-2}\, du, \quad -1 \leqq f(v) \leqq v. \tag{4}$$

Let

$$\phi_2(u, v) = \begin{cases} 1 \text{ if } u \leqq f(v), \ -1 < v < 1, \text{ or if } v \leqq -1, \\ 0 \text{ if } u > f(v), \ -1 < v < 1, \text{ or if } v \geqq 1. \end{cases} \tag{5}$$

Then, with respect to the density $2^{-n}n(n-1)(v-u)^{n-2}du$, $v-1<u<v$, ϕ_2 has the same mass as ϕ_1 on each horizontal line in the (u, v)-plane, but concentrates it as far to the left as possible. Furthermore, $\phi_2=\phi_1$ except on $T(0)$. Therefore

$$E_\theta(\phi_2) \gtreqless E_\theta(\phi_1) \quad \text{for} \quad \theta \lesseqgtr 0. \tag{6}$$

Therefore ϕ_2 dominates ϕ_1.

Define $g(u)$ for $-1<u<1$ by

$$\int_u^{g(u)} (v-u)^{n-2}\,dv = \int_u^1 \phi_2(u, v)(v-u)^{n-2}\,dv, \quad u<g(u) \leqq 1, \tag{7}$$

if the right-hand side is positive. If the right-hand side vanishes, or if $u \geqq 1$, let $g(u)=-1$. If $u \leqq -1$, let $g(u)=1$. Let

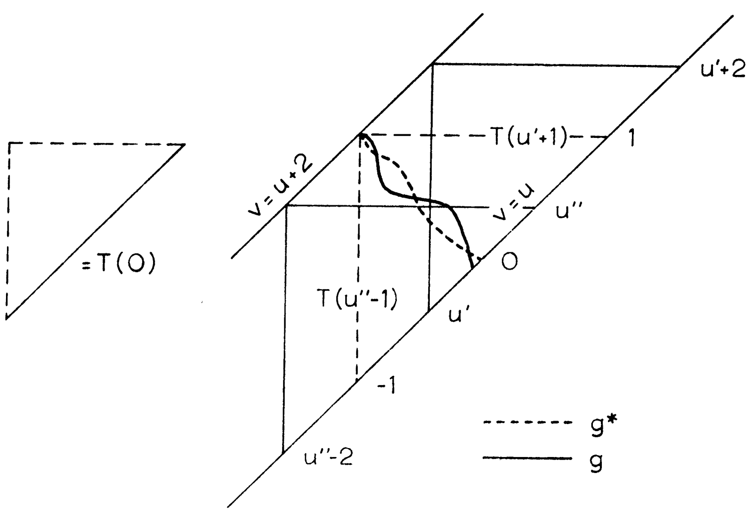
$$\phi_3(u, v) = \begin{cases} 1 & \text{if} \quad v<g(u), \\ 0 & \text{if} \quad v \geqq g(u). \end{cases} \tag{8}$$

Then, with respect to the density $2^{-n}n(n-1)(v-u)^{n-2}\,dv$, $u<v<u+1$, ϕ_3 has the same mass as ϕ_2 on each vertical line in the (u, v)-plane, but concentrates it as low as possible. Furthermore, $\phi_3=\phi_2$ except on $T(0)$. Therefore

$$E_\theta(\phi_3) \geqq E_\theta(\phi_2) \quad \text{for} \quad \theta \lesseqgtr 0. \tag{9}$$

Therefore ϕ_3 dominates ϕ_2.

By (5), $\phi_2(u, v)$ is nonincreasing in u for each v. Therefore, by (7), for $-1<u<g(u)$, $-1<g(u) \leqq 1$, and $g(u)$ is nonincreasing in u. This is the essential part of the requirement that ϕ_3 be in $\mathfrak{A}$, and $g(u)$ was defined for other values of u so that ϕ_3 actually is in $\mathfrak{A}$.



We have thus shown that any test is dominated by a test in $\mathfrak{A}$, i.e. that $\mathfrak{A}$ is essentially complete. It remains to prove admissibility. Suppose ϕ and ϕ^* are given by g and g^*. Without changing the characteristics of the tests, we may redefine g and g^* so that they are left-continuous and so that $g(u) = -1$ where $g(u) \leqq u$, and $g^*(u) = -1$ where $g^*(u) \leqq u$. Suppose there is a u' such that $g(u') > g^*(u')$. Choose u'' such that $g(u') > u'' > g^*(u')$. (See the diagram.) Let "area" be measured with respect to the density $2^{-n}n(n-1)(v-u)^{n-2}du\,dv$. By left-continuity, $g^*(u) < u$ for all u in an interval whose right endpoint is u'. Therefore either the "area" below g in $T(u'+1)$ is less than that below g^*, or the "area" below g in $T(u''-1)$ is greater than that below g^*. But the "area" below g in $T(\theta)$ is just $E_\theta(\phi)$. Thus either $E_{u'+1}(\phi) < E_{u'+1}(\phi^*)$ or $E_{u''-1}(\phi) > E_{u''-1}(\phi^*)$. But $u'+1 > 0$ and $u''-1 < 0$, so this shows ϕ doesn't dominate ϕ^*. Hence if ϕ dominates ϕ^*, $g(u') \leqq g^*(u')$ for all u'. But in this case either ϕ and ϕ^* are essentially the same or $E_\theta(\phi) < E_\theta(\phi^*)$ for sufficiently small positive θ. Therefore ϕ cannot dominate ϕ^*. Since ϕ and ϕ^* were arbitrary tests of the essentially complete class $\mathfrak{A}$, it follows that all tests in A are admissible.

This proof of admissibility is spelled out analytically in [2]. The proof of essential completeness given there uses a general property possessed by the rectangular distribution.

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A METHOD FOR SELECTING THE SIZE OF THE INITIAL SAMPLE IN STEIN'S TWO SAMPLE PROCEDURE

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1. Summary and Introduction. The use of an upper percentage point of the distribution of total sample size in conjunction with the expectation of the latter is proposed as a guide to the selection of the size of the initial sample when using some version of Stein's [5] two-sample procedure. It is a rapidly calculable function of the underlying population variance based on existing tables of the χ^2 distribution. A rule-of-thumb is proposed to be used in making the actual selection of initial sample size. It is a simple matter to investigate the nature of the percentage point for different values of the variance over a limited range;

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