A LIMIT THEOREM FOR THE PERIODOGRAM

By Salomon Bochner[1] and Tatsuo Kawata[2]

1. Introduction. Let $\mathcal{E}(t)$ be a real stationary process in the wide sense with mean 0 and let its covariance function and spectral function be $\rho(u)$, $F(x)$ respectively. We assume that $F(x)$ is absolutely continuous and has a spectral density function $p(x)$. The second-named author, [1], has discussed the periodogram

$$J(T) = \frac{1}{4\pi T}\left|\int_{-T}^{T} \mathcal{E}(t)e^{-i\xi t}\, dt\right|^2, \tag{1.1}$$

in case $\mathcal{E}(t)$ is stationary even of the fourth order, so that the expectation

$$E\mathcal{E}(t)\mathcal{E}(t+u)\mathcal{E}(t+v)\mathcal{E}(t+w) = P(u, v, w)$$

exists and is a function of u, v, w alone. It was also assumed that the function $Q(u, v, w)$, which is the difference between $P(u, v, w)$ and the corresponding fourth moment of a stationary Gaussian process, is the Fourier transform of a function and that the latter function satisfies the Lipschitz condition. Under these assumptions it has proven that (1.1) does not converge in mean to any random variable as $T \to \infty$, but that the covariance function of $J(T)$ and $J(T')$ does tend to a limit whenever T and T' both tend to infinity in a certain related manner, and the limiting value of the covariance function was determined.

The paper involved a rather troublesome manipulation of a Fourier integral, but we have found since that under somewhat different assumptions the complications can be reduced appreciably. In a separate publication, [2], a certain integral transformation was investigated on its own merit, and in the present paper an application of the somewhat modified approach will be made to the problem of the periodogram. The expression (1.1) will be replaced by a more general one, and as regards the difference function $Q(u, v, w)$ the assumptions will be modified as follows. We add expressly the requirement that $Q(u, v, w)$ shall be integrable in E_3, but the requirement that its Fourier transform shall satisfy the Lipschitz condition is being omitted entirely.

2. The Theorem. We shall consider the random variable

$$S(T) = \frac{1}{T}\left|\int_{-\infty}^{\infty} \mathcal{E}(t)M\left(\frac{t}{T}\right)e^{-i\xi t}\right|^2 dt \tag{2.1}$$

in place of (1.1). We shall call (2.1) a generalized periodogram of $\mathcal{E}(t)$.

Let us assume that

$$P(s_1, s_2, s_3) = Q(s_1, s_2, s_3) + P_G(s_1, s_2, s_3), \tag{2.2}$$

Received December 6, 1957.

[1] Research done in connection with U. S. Army Ordnance, Office of Ordnance Research, Contract No. D.A.—36-034—ORD-2001.

[2] Work done while the second author was a research fellow at Princeton University supported by the Rockefeller Foundation.

where

(2.3) $P_G(s_1, s_2, s_3) = \rho(s_1)\, \rho(s_2 - s_3) + \rho(s_2)\rho(s_3 - s_1) + \rho(s_3)\rho(s_1 - s_2).$

If $\mathcal{E}(t)$ is a stationary Gaussian process, then $Q(s_1, s_2, s_3) \equiv 0$. This assumption was set up first by Magness [3]; see also Parzen [4].

We assume further that

$$Q(s_1, s_2, s_3) \,\varepsilon\, L_1(E_3), \tag{2.4}$$

and that the Fourier transform of $Q(s_1, s_2, s_3)$ is also in $L_1(E_3)$, so that

$$q(x_1, x_2, x_3) = \int_{E_3} e^{i(s\cdot x)} Q(s_1, s_2, s_3)\, dv_s, \tag{2.5}$$

$$Q(s_1, s_2, s_3) = (2\pi)^{-3} \int_{E_3} e^{-i(s\cdot x)} q(x_1, x_2, x_3)\, dv_x, \tag{2.6}$$

where E_k denotes the whole Euclidean space of k dimension and $(s \cdot x) = s_1x_1 + s_2x_2 + s_3x_3$.

Under these conditions, we obtain the following theorem.

THEOREM. *Let $M(\alpha)$ be bounded and integrable in $(-\infty, \infty)$ and let the Fourier transform $K(x)$ of $M(\alpha)$*

$$K(x) = \int e^{ix\alpha} M(\alpha)\, d\alpha$$

satisfy

$$K(x) = O(|x|^{-1}), \text{ as } x \to \infty. \tag{2.7}$$

Then we have, as T_1 and T_2 tend to infinity such that $T_1/T_2 \to \mu$, $\mu \neq 0$,

$$\lim \operatorname{cov} \{S(T_1), S(T_2)\} = \begin{cases} (2\pi)^2 (|C_\mu^{(1)}|^2 + |C_\mu^{(2)}|^2) p^2(0), & \text{if } \xi = 0, \\ (2\pi)^2 |C_\mu^{(2)}|^2 p^2(\xi), & \text{if } \xi \neq 0, \end{cases} \tag{2.8}$$

and

$$\lim E\{S(T_1) - S(T_2)\}^2 = \begin{cases} 2(2\pi)^2 (|C_1^{(1)}|^2 + |C_1^{(2)}|^2 - |C_\mu^{(1)}|^2 \\ - |C_\mu^{(2)}|^2) p^2(0), & \text{if } \xi = 0, \\ 2(2\pi)^2 (|C_1^{(2)}|^2 - |C_\mu^{(2)}|^2) p^2(\xi), & \text{if } (\xi) \neq 0, \end{cases} \tag{2.9}$$

provided that $p(x)$ is continuous at ξ, and the constants $C_\mu^{(j)}$ $(j = 1, 2)$ are given by

$$C_\mu^{(1)} = \mu^{\frac{1}{2}} \int_{-\infty}^{\infty} M(\alpha) M(\mu\alpha)\, d\alpha,$$

$$C_\mu^{(2)} = \mu^{\frac{1}{2}} \int_{-\infty}^{\infty} M(\alpha) \bar{M}(\mu\alpha)\, d\alpha.$$

We add a remark. If $\mu \to \infty$, or $\mu \to 0$, then $C_\mu^{(1)}$, $C_\mu^{(2)}$ converge to 0. This is easily seen from the fact that $C_\mu^{(1)} = C_{1/\mu}^{(1)}$, $C_\mu^{(2)} = \bar{C}_{1/\mu}^{(2)}$, and $|C_\mu^{(j)}| \leqq \mu^{1/2} M \int_{-\infty}^{\infty} |M(\alpha)|\, d\alpha \to 0$, $(\mu \to 0)$, M being an upper bound of $M(\alpha)$.

We also note that the theorem implies that the constant

$$|C_1^{(1)}|^2 + |C_1^{(2)}|^2 - |C_\mu^{(1)}|^2 - |C_\mu^{(2)}|^2$$

must be non-negative. This can also be established directly by verifying that it is the value of the double integral

$$\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\,|A(\alpha,\beta)|^2 + (A(\alpha,\beta))^2]\,d\alpha\,d\beta,$$

where

$$A(\alpha,\beta) = M(\alpha)\bar{M}(\beta) - \mu M(\mu\alpha)\bar{M}(\mu\beta).$$

For the proof of the theorem, we first of all state as a lemma, a theorem given in [2].

LEMMA 1. *Let $M_j(\alpha)$ $(j = 0, 1, \cdots, k)$ be bounded and integrable over $(-\infty, \infty)$ and let their Fourier transforms be*

$$K_j(x) = \int_{-\infty}^{\infty} e^{i\alpha x} M_j(\alpha)\,d\alpha, \qquad (j = 0, 1, \cdots, k).$$

Put

$$K(x_1, x_2, \cdots, x_k; T_0, T_1, \cdots, T_k) = \prod_{j=0}^{k} T_j \cdot K(T_0(x_1 + x_2 + \cdots + x_k)) \prod_{j=1}^{k} K_j(T_j x_j)$$

for any positive numbers $T_0, T_1, \cdots, T_k$. Then we have

$$\lim_{T_i\to\infty} \frac{1}{T_0}\int_{E_k} f(x_1, x_2, \cdots, x_k) K(x_1, x_2, \cdots, x_k; T_0, T_1, \cdots, T_k)\,dv_x = C_k(2\pi)^k f(0, \cdots, 0), \qquad (i = 0, 1, \cdots, k),$$

if T_j go to infinity such that $T_0/T_j \to \mu_j$ and $\mu_j \neq 0$ $(j = 1, 2, \cdots, k)$ and $f(x_1, \cdots, x_k)$ satisfies the conditions that the function $f(x_1, \cdots, x_k)$ is continuous and belongs to $L_1(E_k)$ and its Fourier transform

$$g(\alpha_1, \alpha_2, \cdots, \alpha_k) = \int_{E_k} e^{i(\alpha\cdot x)} f(x_1, \cdots, x_k)\,dv_x$$

likewise belongs to $L_1(E_k)$. C_k is

$$C_k = \int_{-\infty}^{\infty} M_0(\alpha) \prod_{j=1}^{k} M_j(-\mu_j\,\alpha)\,d\alpha.$$

3. A lemma. For the proof of the theorem, we need one more lemma.

LEMMA 2. *Let $K_j(x)$, $(j = 1, 2)$ be a bounded function which is the Fourier transform of a bounded and integrable function $M_j(\alpha)$*

$$K_j(x) = \int_{-\infty}^{\infty} M_j(\alpha) e^{ix\alpha}\,d\alpha, \qquad j = 1, 2, \tag{3.1}$$

and let us assume that

$$K_j(x) = O(|x|^{-1}), \qquad \text{as } x \to \infty, \qquad j = 1, 2. \tag{3.2}$$

(i) If $p(x) \ \varepsilon \ L_1(-\infty, \infty)$ and continuous at $-\xi$, then

$$(T_1 T_2)^{\frac{1}{2}} \int_{-\infty}^{\infty} K_1(T_1(x + \xi))K_2(T_2(x + \xi))p(x)\, dx \tag{3.3}$$

converges to

$$2\pi \cdot C_\mu \cdot p(-\xi),$$

when $T_1, T_2 \to \infty$ *and* $T_1/T_2 \to \mu$ *and* $\mu \neq 0$, *where*

$$C_\mu = \mu^{\frac{1}{2}} \int_{-\infty}^{\infty} M_1(\beta)M_2(\mu\beta)\, d\beta.$$

(ii) *If* $p(x) \ \varepsilon \ L_1(-\infty, \infty)$, *and* $p(x)$ *continuous as* $-\xi_1$ *then*

$$(T_1 T_2)^{\frac{1}{2}} \int_{-\infty}^{\infty} K_1[T_1(x + \xi_1)]K_2[T_2(x + \xi_2)]p(x)\, dx \tag{3.4}$$

converges to zero when $T_1, T_2 \to \infty$ *such that* $T_1/T_2 \to \mu$ *and* $\mu \neq 0$ *and* $\xi_1 \neq \xi_2$.

Proof. (i) We consider the integral

$$(T_1 T_2)^{1/2} \int_{-\infty}^{\infty} K_1(T_1 x)K_2(T_2 x)\, dx, \tag{3.5}$$

which is absolutely convergent because K_1, K_2 are bounded and satisfy (3.2). By the Parseval theorem, since $K_j(x) \ \varepsilon \ L_2(-\infty, \infty)$, we have

$$\begin{aligned}(T_1 T_2)^{1/2} \int_{-\infty}^{\infty} K_1(T_1 x)K_2(T_2 x)\, dx &= \frac{2\pi}{(T_1 T_2)^{1/2}} \int_{-\infty}^{\infty} M_1\left(\frac{\alpha}{T_1}\right) M_2\left(\frac{-\alpha}{T_2}\right) d\alpha \\ &= 2\pi(T_1/T_2)^{1/2} \int_{-\infty}^{\infty} M_1(\beta)M_2\left(-\frac{T_1}{T_2}\beta\right) d\beta.\end{aligned}$$

This converges to

$$2\pi C_\mu = 2\pi\mu^{1/2} \int_{-\infty}^{\infty} M_1(\beta)M_2(-\mu\beta)\, d\beta,$$

as is easily seen from the fact that

$$\int_{-\infty}^{\infty} | M_2(a\beta) - M_2(a_0 \beta) |\, d\beta \to 0,$$

if $a \to a_0$ and $a_0 \neq 0$.

Hence it suffices to show that

$$I = (T_1 T_2)^{1/2} \int_{-\infty}^{\infty} K_1(T_1 x)K_2(T_2 x)\{p(x - \xi) - p(-\xi)\}\, dx \tag{3.6}$$

converges to zero.

We divide I into two parts:

$$I = \int_{|x|<\delta} + \int_{|x|>\delta} = I_1 + I_2,$$

where δ is taken so that $|p(x - \xi) - p(-\xi)| < \epsilon$, for $|x| < \delta$, ϵ being any assigned positive number.

We have

$$\begin{aligned} |I_1| &\leqq \epsilon(T_1 T_2)^{1/2} \int_{|x|<\delta} |K_1(T_1 x)K_2(T_2 x)|\, dx \\ &\leqq \epsilon(T_2/T_1)^{1/2} \int_{-\infty}^{\infty} \left| K_1(u)K_2\left(\frac{T_2}{T_1} u\right)\right| du \qquad (3.7) \\ &\leqq \epsilon C \int_{-\infty}^{\infty} \frac{du}{1 + u^2}, \end{aligned}$$

for some constant C, as follows from (3.2).

Next we have

$$\begin{aligned} |I_2| \leqq\ & (T_1 T_2)^{1/2} \int_{|x|>\delta} |K_1(T_1 x)K_2(T_2 x)|\, |p(x - \xi)\, dx \\ & + |p(\xi)|\, (T_1 T_2)^{1/2} \int_{|x|>\delta} |K_1(T_1 x)K_2(T_2 x)|\, dx \\ \leqq\ & \frac{C}{(T_1 T_2)^{1/2}} \int_{|x|>\delta} \frac{|p(x - \xi)|}{x^2}\, dx + \frac{C\,|p(\xi)|}{(T_1 T_2)^{1/2}} \int_{|x|>\delta} \frac{dx}{x^2}, \end{aligned}$$

for some constant C. Hence we get

$$I_2 = o(1) \qquad (3.8)$$

as $T_1T_2 \to \infty$, and this together with (3.7) proves (i).

We shall now prove (ii). We have

$$\begin{aligned} & (T_1 T_2)^{1/2} \int_{-\infty}^{\infty} K_1(T_1(x + \xi_1))K_2(T_2(x + \xi_2))\, dx \\ &= \frac{2\pi}{(T_1 T_2)^{1/2}} \int_{-\infty}^{\infty} M_1\left(\frac{\alpha}{T_1}\right) M_2\left(\frac{-\alpha}{T_2}\right) e^{i\alpha(\xi_1-\xi_2)}\, d\alpha \qquad (3.9) \\ &= 2\pi(T_1/T_2)^{1/2} \int_{-\infty}^{\infty} M_1(\beta)M_2\left(-\frac{T_2}{T_1}\beta\right) e^{iT_1\beta(\xi_1-\xi_2)}\, d\beta \end{aligned}$$

and the difference between this and the expression

$$2\pi(T_1/T_2)^{1/2} \int_{-\infty}^{\infty} M_1(\beta)M_2(-\mu\beta)e^{iT_1\beta(\xi_1-\xi_2)}\, d\beta \qquad (3.10)$$

is in absolute value

$$\begin{aligned} &\leqq 2\pi(T_1/T_2)^{1/2} \int_{-\infty}^{\infty} |M_1(\beta)| \left| M_2\left(-\frac{T_1}{T_2}\beta\right) - M_2(-\mu\beta)\right| d\beta \\ &\leqq C \int_{-\infty}^{\infty} \left| M_2\left(-\frac{T_1}{T_2}\beta\right) - M_2(-\mu\beta)\right| d\beta. \end{aligned}$$

But this is as small as we please, for T_1, T_2 large and T_1/T_2 near to μ, provided $\mu \neq 0$.

Now (3.10) tends to zero by Riemann-Lebesgue lemma, and we conclude that (3.9) tends to zero also.

It suffices, then, to show that

$$(3.11)\quad J = (T_1 T_2)^{1/2} \int_{-\infty}^{\infty} K_1[T_1(x + \xi_1)]K_2[T_2(x + \xi_2)]\{p(x) - p(-\xi_1)\}\, dx$$

converges to zero.

We have

$$(3.12)\quad \begin{aligned} J &= (T_1 T_2)^{1/2} \int_{-\infty}^{\infty} K_1(T_1 y)K_2\{T_2[y - (\xi_1 - \xi_2)]\}\{p(y - \xi_1) - p(-\xi_1)\}\, dy \\ &= (T_1 T_2)^{1/2} \int_{|y|<\delta} + (T_1 T_2)^{1/2} \int_{|y|>\delta} = J_1 + J_2, \end{aligned}$$

say. Here δ is so chosen that

$$(3.13)\quad |p(y - \xi_1) - p(-\xi_1)| < \epsilon,$$

for $|y| < \delta$ and

$$(3.14)\quad |\xi_1 - \xi_2| - \delta > c > 0,$$

for some positive constant c. Then

$$(3.15)\quad \begin{aligned} |J_1| &\leqq (T_1 T_2)^{1/2} \cdot \epsilon \int_{|y|<\delta} |K_1(T_1 y)K_2[T_2 y - T_2(\xi_1 - \xi_2)]|\, dy \\ &\leqq \epsilon(T_1 T_2)^{1/2} C \int_{|y|<\delta} \frac{dy}{T_2(|\xi_1 - \xi_2| - y)} \\ &\leqq \epsilon(T_1/T_2)^{1/2} C \cdot c \cdot \delta \leqq C\epsilon, \end{aligned}$$

for some constant C by (3.13) and (3.14).

Next we shall consider J_2. We divide J_2 further into two parts,

$$\begin{aligned} J_2 &= (T_1 T_2)^{1/2} \int_{|y|>\delta, |y-(\xi_1-\xi_2)|>\eta} + (T_1 T_2)^{1/2} \int_{|y|>\delta, |y-(\xi_1-\xi_2)|<\eta} \\ &= J_{21} + J_{22}, \end{aligned}$$

say, where $0 < \eta < \frac{1}{2} |\xi_1 - \xi_2|$. Then

$$(3.16)\quad \begin{aligned} |J_{21}| &\leqq (T_1 T_2)^{1/2} C \int_{|y|>\delta, |y-(\xi_1-\xi_2)|>\eta} \frac{1}{T_1 y} \\ &\qquad \cdot \frac{1}{T_2 |y - (\xi_1 - \xi_2)|} (|p(y - \xi_1)| + |p(-\xi_1)|)\, dy \\ &\leqq \frac{C}{(T_1 T_2)^{1/2} \delta\eta} \int_{|y|>\delta, |y-(\xi_1-\xi_2)|>\eta} \frac{|p(y - \xi_1)| + |p(-\xi_1)|}{y\, |y - (\xi_1 - \xi_2)|}\, dy, \end{aligned}$$

which converges to zero as $T_1, T_2 \to \infty$, since the integral is finite. Moreover

$$
(3.17)\qquad
\begin{aligned}
|J_{22}| &\leqq (T_1T_2)^{1/2}C\int_{(\xi_1-\xi_2)-\eta<y<(\xi_1-\xi_2)+\eta} \\
&\qquad\qquad \cdot \frac{1}{T_1 y}(|p(y-\xi_1)| + |p(-\xi_1)|)\,dy \\
&\leqq (T_2/T_1)^{1/2}\frac{2C}{|\xi_1-\xi_2|}\int_{(\xi_1-\xi_2)-\eta}^{(\xi_1-\xi_2)+\eta}(|p(y-\xi_1)| + |p(-\xi_1)|)\,dy \\
&\leqq C\int_{(\xi_1-\xi_2)-\eta}^{(\xi_1-\xi_2)+\eta}(|p(y-\xi_1)| + |p(-\xi_1)|)\,dy.
\end{aligned}
$$

Hence $\limsup_{T_1,T_2\to\infty,T_1/T_2\to\mu}$ of (3.17) is small for η small, that is

$$(3.18)\qquad \lim J_{22} = 0.$$

From (3.16), (3.18) we obtain

$$\lim J_2 = 0,$$

which together with (3.15) gives $\lim J = 0$.

4. Proof of the theorem. We now proceed to prove the theorem stated in Section 2.

We start with the computation of

$$
\begin{aligned}
ES(T_1)S(T_2) &= \frac{1}{T_1T_2}E\left|\int_{-\infty}^{\infty}\mathcal{E}(t)M\left(\frac{t}{T_1}\right)e^{-i\xi t}\,dt\right|^2\cdot\left|\int_{-\infty}^{\infty}\mathcal{E}(t)M\left(\frac{t}{T_2}\right)e^{-i\xi t}\,dt\right|^2 \\
&= \frac{1}{T_1T_2}E\int_{E_4}\mathcal{E}(t_1)\mathcal{E}(t_2)\mathcal{E}(t_3)\mathcal{E}t_4)e^{-i\xi(t_1-t_2+t_3-t_4)} \\
&\qquad\qquad \cdot M\left(\frac{t_1}{T_1}\right)\bar{M}\left(\frac{t_2}{T_1}\right)M\left(\frac{t_3}{T_2}\right)\bar{M}\left(\frac{t_4}{T_2}\right)dv_t \\
&= \frac{1}{T_1T_2}\int_{E_4}P(t_2-t_1, t_3-t_1, t_4-t_1)e^{-i\xi(t_1-t_2+t_3-t_4)} \\
&\qquad\qquad \cdot M\left(\frac{t_1}{T_1}\right)\bar{M}\left(\frac{t_2}{T_1}\right)M\left(\frac{t_3}{T_2}\right)\bar{M}\left(\frac{t_4}{T_2}\right)dv_t \\
&= \frac{1}{T_1T_2}\int_{E_4}Q(t_2-t_1, t_3-t_1, t_4-t_1)e^{-i\xi(t_1-t_2+t_3-t_4)} \\
&\qquad\qquad \cdot M\left(\frac{t_1}{T_1}\right)\bar{M}\left(\frac{t_2}{T_1}\right)M\left(\frac{t_3}{T_2}\right)\bar{M}\left(\frac{t_4}{T_2}\right)dv_t \\
&\quad + \frac{1}{T_1T_2}\int_{E_4}P_G(t_2-t_1, t_3-t_1, t_4-t_1)e^{-i\xi(t_1-t_2+t_3-t_4)} \\
&\qquad\qquad \cdot M\left(\frac{t_1}{T_1}\right)\bar{M}\left(\frac{t_2}{T_1}\right)M\left(\frac{t_3}{T_2}\right)\bar{M}\left(\frac{t_4}{T_2}\right)dv_t \\
&= S_1(T_1, T_2) + S_2(T_1, T_2).
\end{aligned}
$$

Inserting (2.6) in $S_1(T_1, T_2)$, we have

$$S_1(T_1, T_2) = (2\pi)^{-3} \frac{1}{T_1 T_2} \int_{E_4} M\left(\frac{t_1}{T_1}\right) \bar{M}\left(\frac{t_2}{T_1}\right) M\left(\frac{t_3}{T_2}\right) \bar{M}\left(\frac{t_4}{T_2}\right) \cdot e^{-i\xi(t_1-t_2+t_3-t_4)}\, dv_t$$

$$\cdot \int_{E_3} q(x_1, x_2, x_3) \exp\left[i(t_2 - t_1)x_1 + i(t_3 - t_1)x_2 + i(t_4 - t_1)x_3\right] dv_x$$

$$= (2\pi)^{-3} \frac{1}{T_1 T_2} \int_{E_3} q(x_1, x_2, x_3)\, dv_x$$

$$\cdot \int_{E_4} M\left(\frac{t_1}{T_1}\right) \bar{M}\left(\frac{t_2}{T_1}\right) M\left(\frac{t_3}{T_2}\right) \bar{M}\left(\frac{t_4}{T_2}\right) \tag{4.1}$$

$$\cdot \exp\left[i\{-t_1(x_1 + x_2 + x_3 + \xi) + t_2(x_1 + \xi) + t_3(x_2 - \xi) + t_4(x_3 + \xi)\}\right] dv_t$$

$$= (2\pi)^{-3} T_1 T_2 \int_{E_3} q(x_1, x_2, x_3) K[-T_1(x_1 + x_2 + x_3 + \xi)]$$

$$\cdot \bar{K}[-T_1(x_1 + \xi)] \cdot K[T_2(x_2 - \xi)] \bar{K}[-T_2(x_3 + \xi)]\, dv_x$$

$$= (2\pi)^{-3} T_1 T_2 \int_{E_3} q(x_1 - \xi, x_2 + \xi, x_3 - \xi)$$

$$\cdot K[-T_1(x_1 + x_2 + x_3)] \cdot \bar{K}(-T_1 x) K(T_2 x_2) \bar{K}(-T_2 x_3)\, dv_x,$$

where we denote

$$K(x) = \int_{-\infty}^{\infty} M(\alpha) e^{ix\alpha}\, d\alpha. \tag{4.2}$$

Since $M(x)$ and $q(x_1, x_2, x_3)$ satisfy the condition of Lemma 1, we obtain that (4.1) multiplied by T_2 is convergent when $T_1/T_2 \to \mu (\mu \neq 0)$. Hence (4.1) converges to zero.

Next we shall consider $S_2(T_1, T_2)$. Inserting (2.3), we obtain

$$S_2(T_1, T_2) = \frac{1}{T_1 T_2} \int_{E_4} \{\rho(t_2 - t_1)\rho(t_3 - t_4) + \rho(t_3 - t_1)\rho(t_4 - t_2)$$

$$+ \rho(t_4 - t_1)\rho(t_2 - t_3)\} \bar{M}\left(\frac{t_1}{T_1}\right) M\left(\frac{t_2}{T_1}\right) M\left(\frac{t_3}{T_2}\right) \bar{M}\left(\frac{t_4}{T_2}\right) \cdot e^{-i\xi(t_1-t_2+t_3-t_4)} \cdot dv_t \tag{4.4}$$

$$= U_1(T_1, T_2) + U_2(T_1, T_2) + U_3(T_1, T_2),$$

say, where

$$U_1(T_1, T_2) = \frac{1}{T_1 T_2} \int_{E_4} \rho(t_2 - t_1)\rho(t_3 - t_4) \cdot M\left(\frac{t_1}{T_1}\right) \bar{M}\left(\frac{t_2}{T_1}\right) M\left(\frac{t_3}{T_2}\right) \bar{M}\left(\frac{t_4}{T_2}\right) \cdot e^{-i\xi(t_1-t_2+t_3-t_4)}\, dv_t, \tag{4.5}$$

and U_2, U_3 are similar terms. By the assumptions of the theorem, we have $\rho(u) = \int_{-\infty}^{\infty} e^{iux} p(x)\,dx$, and, if we insert this into (4.5), we obtain

$$U_1(T_1, T_2) = \frac{1}{T_1 T_2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x)p(y)\,dx\,dy$$

$$\cdot \int_{E_4} M\left(\frac{t_1}{T_1}\right) e^{-it_1(x+\xi)} \bar{M}\left(\frac{t_2}{T_1}\right) e^{it_2(x+\xi)} M\left(\frac{t_3}{T_2}\right) e^{it_3(y-\xi)} \bar{M}\left(\frac{t_4}{T_2}\right) e^{-it_4(y-\xi)}\,dv_t$$

$$= T_1 T_2 \int_{-\infty}^{\infty} p(x) K[-T_1(x+\xi)] \bar{K}[-T_1(x+\xi)]\,dx$$

$$\cdot \int_{-\infty}^{\infty} p(y) K[T_2(y-\xi)] \bar{K}[T_2(y-\xi)]\,dy.$$

Since $\mathfrak{L}(t)$ is real, ρ is real too, and $p(x)$ is an even function, and hence by Lemma 2, we get

$$\lim_{T_1, T_2 \to \infty} U_1(T_1, T_2) = (2\pi)^2 C_1^2 p(\xi) p(-\xi) = (2\pi C_1)^2 p^2(\xi), \tag{4.6}$$

where

$$C_1 = \int_{-\infty}^{\infty} \bar{M}(\beta) M(\beta)\,d\beta = \int_{-\infty}^{\infty} |M(\beta)|^2\,d\beta. \tag{4.7}$$

Quite similarly

$$U_2(T_1, T_2) = T_1 T_2 \int_{-\infty}^{\infty} p(x) K[-T_1(x+\xi)] K[T_2(x-\xi)]\,dx$$

$$\cdot \int_{-\infty}^{\infty} p(y) \bar{K}[T_1(y-\xi)] \bar{K}[-T_2(y+\xi)]\,dy.$$

If $\xi = 0$, then, by (3.3),

$$U_2(T_1, T_2) \to (2\pi)^2 |C_\mu^{(1)}|^2 p^2(0), \qquad (T_1, T_2 \to \infty,\ T_1/T_2 \to \mu), \tag{4.8}$$

where

$$C_\mu^{(1)} = \mu^{1/2} \int_{-\infty}^{\infty} M(\beta) M(\mu\beta)\,d\beta. \tag{4.9}$$

If $\xi \neq 0$, then (3.4) shows

$$U_2(T_1, T_2) \to 0, \quad (T_1, T_2 \to \infty,\ T_1/T_2 \to \mu). \tag{4.10}$$

Finally we have

$$U_3(T_1, T_2) = T_1 T_2 \int_{-\infty}^{\infty} p(x) K[-T_1(x+\xi)] \bar{K}[-T_2(x+\xi)]\,dx$$

$$\cdot \int_{-\infty}^{\infty} p(y) \bar{K}[-T_1(y+\xi)] K[-T_2(y+\xi)]\,dy,$$

and

$$U_3(T_1, T_2) \to (2\pi)^2 |C_\mu^{(2)}|^2 p^2(\xi), \qquad \text{for every } \xi, \tag{4.11}$$

where

$$C_\mu^{(2)} = \mu^{1/2} \int_{-\infty}^{\infty} M(\beta)\bar{M}(\mu\beta)\, d\beta. \tag{4.12}$$

Inserting (4.7) (4.8) (4.10) and (4.11) into (4.4), we get: If $\xi \neq 0$

$$S_2(T_1, T_2) \to (2\pi)^2\, (C_1^2 + |C_\mu^{(2)}|^2)p^2(\xi) \tag{4.13}$$

as T_1, $T_2 \to \infty$, $T_1/T_2 \to \mu(\neq 0)$, and if $\xi = 0$

$$S_2(T_1, T_2) \to (2\pi)^2(C_1^2 + |C_\mu^{(1)}|^2 + |C_\mu^{(2)}|^2). \tag{4.14}$$

Hence we get

$$ES(T_1)S(T_2) \to (2\pi)^2(C_1^2 + |C_\mu^{(2)}|^2)p^2(\xi), \qquad \text{if } \xi \neq 0, \mu \neq 0, \tag{4.15}$$

$$E(T_1)S(T_2) \to (2\pi)^2(C_1^2 + |C_\mu^{(1)}|^2 + |C_\mu^{(2)}|^2)p^2(0), \qquad \text{if } \xi = 0, \mu \neq 0. \tag{4.16}$$

We also have

$$\begin{aligned} ES(T) &= \frac{1}{T}\int_{E_2} E\mathfrak{L}(t_1)\mathfrak{L}(t_2)M\left(\frac{t_1}{T}\right)\bar{M}\left(\frac{t_2}{T}\right)e^{-\xi(t_1-t_2)}\, dv_t \\ &= T\int_{-\infty}^{\infty} p(x)\,|\,K[T(x-\xi)]\,|^2\, dx, \end{aligned}$$

and by Lemma 2 this converges to $2\pi C_1 p(\xi)$. Thus we find that

$$\operatorname{cov}\{S(T_1), S(T_2)\} \equiv ES(T_1)S(T_2) - ES(T_1)\cdot ES(T_2)$$

converges to

$$(2\pi)^2|C_\mu^{(2)}|^2p^2(\xi), \qquad \text{if } \xi \neq 0,$$

and to

$$(2\pi)^2(|C_\mu^{(1)}|^2 + |C_\mu^{(2)}|^2)p^2(0), \qquad \text{if } \xi = 0,$$

when T_1, T_2 increase indefinitely such as $T_1/T_2 \to \mu(\mu \neq 0)$.

Especially var $S(T)$ converges to

$$(2\pi)^2|C_\mu^{(2)}|^2p^2(\xi), \qquad \text{if } \xi = 0,$$

and to

$$((2\pi)^2(|C_\mu^{(1)}|^2 + |C_\mu^{(2)}|^2)p^2(0), \qquad \text{if } \xi = 0.$$

Also, we easily find that

$$E\,|S(T_1) - S(T_2)|^2$$

converges to

$$2(2\pi)^2(|C_1^{(2)}|^2 - |C_\mu^{(2)}|^2)p^2(\xi), \qquad \text{if } \xi \neq 0$$

and to

$$2(2\pi)^2(|C_1^{(1)}|^2 + |C_1^{(2)}|^2 - |C_\mu^{(1)}|^2 - |C_\mu^{(2)}|^2)p^2(0), \qquad \text{if } \xi = 0.$$

Hence the theorem is proved.

REFERENCES

[1] S. Bochner and T. Kawata, "Special integral transformation in Euclidean space," to be published.
[2] T. Kawata, "Some convergence theorems for the stationary stochastic process," to be published.
[3] T. A. Magness, "Special response of quadratic device to non-Gaussian noise," *J. Appl. Phys.*, Vol. 25 (1954).
[4] E. Parzen, "On consistent estimates of the spectrum of a stationary process," *Ann. Math. Stat.*, Vol. 28 (1957).