ON THE LIMITING DISTRIBUTION OF THE NUMBER OF COINCIDENCES CONCERNING TELEPHONE TRAFFIC

BY L. TAKÁCS

Hungarian Academy of Sciences, Budapest

1. Introduction. Let us consider a telephone exchange. Suppose that the subscribers make calls at the instants $\tau_1, \tau_2, \cdots, \tau_n, \cdots$, where $\tau_n - \tau_{n-1}$ ($n = 1, 2, \cdots; \tau_0 = 0$) are identically distributed independent positive random variables with the distribution function $F(x)$. Put $\varphi(s) = \int_0^\infty e^{-sx}\, dF(x)$, $\alpha = \int_0^\infty x\, dF(x)$ and $\sigma^2 = \int_0^\infty (x - \alpha)^2\, dF(x)$.

Suppose that there is an infinite number of fully available channels and that each call gives rise to a connection (conversation) on one of the free channels. Denote by χ_n the duration of the holding time beginning in the instant τ_n ($n = 1, 2, \cdots$). It is assumed that χ_n ($n = 1, 2, \cdots$) are identically distributed mutually independent positive random variables, which are independent also of the random variables τ_n ($n = 1, 2, \cdots$). Suppose that $\mathbf{P}\{\chi_n \leqq x\} = 1 - e^{-\mu x}$, if $x \geqq 0$.

We say that the system is in state E_k ($k = 0, 1, 2, \cdots$) if k channels are busy. In what follows we shall deal with the determination of the distribution of the number of transitions $E_k \to E_{k+1}$ ($k = 0, 1, 2, \cdots$) occurring in the time interval $(0, t]$ and the corresponding asymptotic distribution as $t \to \infty$.

The above problem was solved earlier by the author [7] in the particular case when $\{\tau_n\}$ forms a Poisson process with density λ.

2. Notation. Denote by $\eta(t)$ the number of busy channels at the instant t and put

$$\mathbf{P}\{\eta(t) = k\} = P_k(t), \qquad (k = 0, 1, 2, \cdots).$$

Define the r-th binomial moment of $\eta(t)$ as follows:

$$B_r(t) = \sum_{k=r}^{\infty} \binom{k}{r} P_k(t), \qquad (r = 0, 1, 2, \cdots)$$

and put

$$\beta_r(s) = \int_0^\infty e^{-st} B_r(t)\, dt, \qquad (\Re(s) > 0).$$

Further denote by $\nu_t^{(k)}$ the number of transitions $E_k \to E_{k+1}$, ($k = 0, 1, 2, \cdots$), occurring in the time interval $(0, t]$. (We say that a transition $E_{-1} \to E_0$ takes place at $t = 0$.) Denote by $m_k(t)$ the expectation of the random variable $\nu_t^{(k)}$. ($m_{-1}(t) = 1$ if $t \geqq 0$ and $m_{-1}(t) = 0$ if $t < 0$.)

Finally denote by $m(t)$ the expectation of the number of calls taking place in

Received June 20, 1958; revised October 24, 1958.

the time interval $(0, t]$. Then

$$m(t) = \sum_{n=1}^{\infty} F_n(t),$$

where $F_n(t)$ denotes the n-th iterated convolution of $F(t)$ with itself. Clearly,

$$\int_0^\infty e^{-st} dm(t) = \frac{\varphi(s)}{1 - \varphi(s)}, \qquad (\Re(s) > 0).$$

3. The solution of the problem. If specifically $\{\tau_n\}$ forms a Poisson process with density λ, then $\{\eta(t)\}$ is a Markov process. In other cases $\{\eta(t)\}$ ceases to be a Markov process, but the instants τ_n always form the Markov points (or regeneration points) of the process. Accordingly, for fixed k ($k = 0, 1, 2, \cdots$), the instants of the successive transitions $E_k \to E_{k+1}$ form a recurrent (or renewal) process.

Denote by $R_k(x)$ the distribution function of the distance between two consecutive transitions $E_k \to E_{k+1}$, and by $R_k^*(x)$ the distribution function of the distance between the first transition $E_k \to E_{k+1}$ and the zero point. Knowing $R_k(x)$ and $R_k^*(x)$, the distribution function of $\nu_t^{(k)}$ can be determined easily; namely, we have

$$\mathbf{P}\{\nu_t^{(k)} > n\} = R_k^* * R_k * \cdots * R_k(t), \tag{1}$$

where the right hand side contains the n-th iterated convolution of $R_k(t)$.

Define

$$\rho_k = \int_0^\infty x \, dR_k(x) \tag{2}$$

and

$$\sigma_k^2 = \int_0^\infty (x - \rho_k)^2 \, dR_k(x). \tag{3}$$

If $\sigma_k^2 < \infty$, then we have

$$\lim_{t\to\infty} \mathbf{P}\left\{\frac{\nu_t^k - \rho_k t}{\sigma_k t^{1/2}} \leqq x\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} \, dy, \tag{4}$$

as it is well known in renewal theory. (Cf. W. Feller [1], W. L. Smith [4], and the author [5].)

If we consider other initial conditions than $\eta(0) = 0$, then we obtain similar results. In particular, the limiting distribution (4) is independent of the initial condition.

Thus, the problem is reduced to the determination of the distribution functions $R_k(x)$ and $R_k^*(x)$. We need some auxiliary theorems, which will be proved below.

4. The Palm functions. Hitherto we have not made any restrictions concerning the servicing of the calls. Now, following C. Palm [3], let us suppose that the channels are numbered by 1, 2, $\cdots$, r, $\cdots$, and that an incoming call realizes a connection through that idle channel which has the lowest serial number. This assumption does not restrict the generality since $\{\eta(t)\}$ is independent of the system of the handling of traffic. Now denote by $\tau_1^{(r)}, \tau_2^{(r)}, \cdots,$ $\tau_n^{(r)}, \cdots$ the instants of the calls which find all channels busy in the group $(1, 2, \cdots, r)$, leaving the other channels out of consideration. Obviously the time differences $\tau_{n+1}^{(r)} - \tau_n^{(r)}$ $(n = 1, 2, \cdots)$ are identically distributed independent positive random variables. Let us denote by $G_r(x)$ their common distribution function. We shall prove the following

THEOREM 1. *Define*

$$\gamma_r(s) = \int_0^\infty e^{-sx}\, dG_r(x), \qquad (r = 0, 1, 2, \cdots); \tag{5}$$

then we have

$$\gamma_r(s) = \frac{\displaystyle\sum_{j=0}^{r} \binom{r}{j} \prod_{i=0}^{j-1} \frac{1-\varphi(s+i\mu)}{\varphi(s+i\mu)}}{\displaystyle\sum_{j=0}^{r+1} \binom{r+1}{j} \prod_{i=0}^{j-1} \frac{1-\varphi(s+i\mu)}{\varphi(s+i\mu)}}, \tag{6}$$

where the empty product is 1 *and* $G_0(x) \equiv F(x)$.

PROOF. C. Palm [3] has proved that the distribution functions $G_r(x)$ $(r = 1, 2, \cdots)$ satisfy the following system of integral equations:

$$G_r(x) = G_{r-1}(x) - \int_0^x (1 - e^{-\mu y})(1 - G_r(x-y))\, dG_{r-1}(y), \qquad (r = 1,2,\cdots), \tag{7}$$

where $G_0(x) \equiv F(x)$. This can be proved easily. Let us suppose that $\tau_n^{(r)} = \tau_m^{(r-1)}$ (where $\tau_m^{(0)} = \tau_m$). Then conditionally

$$\mathbf{P}\{\tau_{n+1}^{(r)} - \tau_n^{(r)} \leqq x \mid \tau_{m+1}^{(r-1)} - \tau_m^{(r-1)} = y\} = \begin{cases} e^{-\mu y} + (1 - e^{-\mu y})G_r(x-y), & \text{if } 0 \leqq y \leqq x, \\ 0 \quad , & \text{if } y > x, \end{cases}$$

and by the theorem of total probability we have

$$\mathbf{P}\{\tau_{n+1}^{(r)} - \tau_n^{(r)} \leqq x\} = G_r(x) = \int_0^x [e^{-\mu y} + (1 - e^{-\mu y})G_r(x-y)]\, dG_{r-1}(y),$$

which proves (7).

Taking the Laplace-Stieltjes transform of (7) we obtain Palm's recurrence formula,

$$\gamma_r(s) = \frac{\gamma_{r-1}(s+\mu)}{1 - \gamma_{r-1}(s) + \gamma_{r-1}(s+\mu)}, \qquad (r = 1,2,\cdots), \tag{8}$$

where $\gamma_0(s) = \varphi(s)$.

If we define

$$D_r(s) = \sum_{j=0}^{r} \binom{r}{j} \prod_{i=0}^{j-1} \frac{1 - \varphi(s + i\mu)}{\varphi(s + i\mu)}, \qquad (r = 0, 1, 2, \cdots), \tag{9}$$

then it is easy to see that

$$D_{r+1}(s) = D_r(s) + \frac{1 - \varphi(s)}{\varphi(s)} D_r(s + \mu), \qquad (r = 0, 1, 2, \cdots). \tag{10}$$

Further, a simple calculation shows that the function,

$$\gamma_r(s) = \frac{D_r(s)}{D_{r+1}(s)}, \qquad (r = 0, 1, 2, \cdots) \tag{11}$$

satisfies (8) and $\gamma_0(s) = \varphi(s)$. This proves (6).

5. The binomial moments $B_r(t)$. We shall prove the following:

THEOREM 2. *The binomial moments $B_r(t)$, $(r = 0, 1, 2, \cdots)$ exist for all $t \geqq 0$, and we have*

$$\beta_r(s) = \int_0^\infty e^{-st} B_r(t)\, dt = \frac{1}{s + r\mu} \prod_{i=0}^{r-1} \frac{\varphi(s + i\mu)}{1 - \varphi(s + i\mu)} \tag{12}$$

if $\Re(s) > 0$.

PROOF. Introduce the generating function

$$G(t, z) = \sum_{k=0}^{\infty} P_k(t) z^k. \tag{13}$$

$G(t, z)$ satisfies the following integral equation:

$$G(t, z) = [1 - F(t)] + \int_0^t G(t - x, z)[1 - (1 - z)e^{-\mu(t-x)}]\, dF(x). \tag{14}$$

This can be proved as follows. Define

$$f(t, u) = \begin{cases} 1, & \text{if } 0 \leqq t \leqq u, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\eta(t) = \sum_{n=1}^{\infty} f(t - \tau_n, \chi_n).$$

Now let us suppose conditionally that $\tau_1 = x$; then

$$\eta(t) = \begin{cases} f(t - x, \chi_1) + \bar{\eta}(t - x), & \text{if } x \leqq t, \\ 0, & \text{if } x > t, \end{cases}$$

where $\bar{\eta}(t - x)$ is independent of $f(t - x, \chi_1)$ and has the same distribution as $\eta(t - x)$. Here the generating function of $f(t - x, \chi_1)$ is $[1 - e^{-\mu(t-x)} + ze^{-\mu(t-x)}]$, if $0 \leqq x \leqq t$, and the generating function of $\bar{\eta}(t - x)$ is $G(t - x, z)$, if $0 \leqq x \leqq t$.

Therefore, applying the theorem of total expectation for $G(t, z) = \mathbf{E}\{z^{\eta(t)}\}$, we obtain

$$G(t, z) = [1 - F(t)] + \int_0^t G(t - x, z)[1 - e^{-\mu(t-x)} + ze^{-\mu(t-x)}]\, dF(x),$$

which proves (14).

I am indebted to R. Syski for calling my attention to the possibility of the above proof. Applying the results of R. Fortet [2] or the author [6], R. Syski showed that $G(t, z)$ satisfies the following integral equation:

$$G(t, z) = 1 - (1 - z)\int_0^t G(t - x, z)e^{-\mu(t-x)}dm(x), \tag{15}$$

where $m(t)$ denotes the expected number of the calls occurring in the time interval $(0, t]$.

Since

$$B_r(t) = \frac{1}{r!}\left(\frac{d^r\, G(t, z)}{dz^r}\right)_{z=1}, \qquad (r = 0, 1, 2, \cdots), \tag{16}$$

we obtain from (14) that

$$B_r(t) = \int_0^t B_r(t - x)\, dF(x) + \int_0^t B_{r-1}(t - x)e^{-\mu(t-x)}dF(x), \qquad (r = 1, 2, \cdots). \tag{17}$$

This is a linear integral equation of the Volterra type for the unknown $B_r(t)$. As is well known, the solution is

$$B_r(t) = \int_0^t B_{r-1}(t - x)e^{-\mu(t-x)}\, dm(x), \qquad (r = 1, 2, \cdots). \tag{18}$$

This can be obtained immediately from Syski's equation (15).

Taking the Laplace transform of (18), we obtain the following functional equation:

$$\beta_r(s) = \frac{\varphi(s)}{1 - \varphi(s)}\, \beta_{r-1}(s + \mu). \tag{19}$$

Since $B_0(t) \equiv 1$, consequently $\beta_0(s) = 1/s$, and applying repeatedly formula (19) we finally obtain

$$\beta_r(s) = \frac{1}{s + r\mu} \prod_{i=0}^{r-1} \frac{\varphi(s + i\mu)}{1 - \varphi(s + i\mu)},$$

as was to be proved.

It is to be remarked that there exists a constant C so that

$$B_r(t) < \frac{C^r}{r!}, \qquad (r = 0, 1, 2, \cdots), \tag{20}$$

for all $t \geqq 0$. This can be proved by virtue of (18).

Remark. Since

$$B_r(t) = \sum_{k=r}^{\infty} \binom{k}{r} P_k(t) \tag{21}$$

and $B_r(t) < C^r / r!$, $(r = 0, 1, 2, \cdots)$, we obtain easily that

$$P_k(t) = \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} B_r(t). \tag{22}$$

Hence, specifically,

$$\int_0^{\infty} e^{-st} P_k(t)\, dt = \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} \frac{1}{s + r\mu} \sum_{i=0}^{r-1} \frac{\varphi(s + i\mu)}{1 - \varphi(s + i\mu)}. \tag{23}$$

6. The transitions $E_k \to E_{k+1}$.

THEOREM 3. *If $m_k(t)$ denotes the expectation of the number of transitions $E_k \to E_{k+1}$ occurring in the time interval $(0, t]$, then we have*

$$\int_0^{\infty} e^{-st}\, dm_k(t) = \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} \prod_{i=0}^{r} \frac{\varphi(s + i\mu)}{1 - \varphi(s + i\mu)}, \quad (k = 0, 1, 2, \cdots). \tag{24}$$

PROOF. Applying the theorem of total probability we can write

$$P_k(t) = \sum_{j=k}^{\infty} \binom{j}{k} \int_0^t e^{-k\mu(t-u)}[1 - e^{-\mu(t-u)}]^{j-k}[1 - F(t - u)]\, dm_{j-1}(u). \tag{25}$$

This follows from the fact that the event that there is a state E_k at the instant t can occur in several mutually exclusive ways: the last transition in the time interval $(0, t]$ is $E_{j-1} \to E_j$ $(j = k, k + 1, \cdots)$ and this transition takes place at the instant $u(0 \leq u \leq t)$, and in the time interval $(u, t]$ there does not occur any new call, but $j - k$ conversations terminate.

Hence,

$$B_r(t) = \sum_{k=r}^{\infty} \binom{k}{r} P_k(t) = \sum_{j=r}^{\infty} \binom{j}{r} \int_0^t e^{-r\mu(t-u)}[1 - F(t - u)]\, dm_{j-1}(u), \tag{26}$$

where $\binom{k}{r}\binom{j}{k} = \binom{j}{r}\binom{j-r}{k-r}$ has been used. Forming the Laplace transform of (26), we have

$$\beta_r(s) = \frac{1 - \varphi(s + r\mu)}{s + r\mu} \sum_{j=r}^{\infty} \binom{j}{r} \int_0^{\infty} e^{-st}\, dm_{j-1}(t).$$

Now by the aid of (12) we obtain

$$\sum_{j=r}^{\infty} \binom{j}{r} \int_0^{\infty} e^{-st}\, dm_{j-1}(t) = \frac{1}{1 - \varphi(s + r\mu)} \prod_{i=0}^{r-1} \frac{\varphi(s + i\mu)}{1 - \varphi(s + i\mu)}.$$

Multiplying both sides of this formula by $(-1)^{r-l}\binom{r}{l}$ and summing over $r =$

$l, l+1, \cdots$ we obtain

$$\int_0^\infty e^{-st}\, dm_{l-1}(t) = \sum_{r=l}^{\infty} (-1)^{r-l}\binom{r}{l}\left[\prod_{i=0}^{r-1}\frac{\varphi(s+i\mu)}{1-\varphi(s+i\mu)} + \prod_{i=0}^{r}\frac{\varphi(s+i\mu)}{1-\varphi(s+i\mu)}\right]. \tag{27}$$

If we write $l = k+1$, then

$$\int_0^\infty e^{-st}\, dm_k(t) = \sum_{r=k}^{\infty}(-1)^{r-k}\binom{r}{k}\prod_{i=0}^{r}\frac{\varphi(s+i\mu)}{1-\varphi(s+i\mu)}, \tag{28}$$

which was to be proved.

7. The distributions $R_k(x)$ and $R_k^*(x)$.

THEOREM 4. *We have*

$$\int_0^\infty e^{-sx}\, dR_k^*(x) = \left[\sum_{j=0}^{k+1}\binom{k+1}{j}\prod_{i=0}^{j-1}\frac{1-\varphi(s+i\mu)}{\varphi(s+i\mu)}\right]^{-1} \tag{29}$$

and

$$\int_0^\infty e^{-sx}\, dR_k(x) = 1 - \left\{\left[\sum_{j=0}^{k+1}\binom{k+1}{j}\prod_{i=0}^{j-1}\frac{1-\varphi(s+i\mu)}{\varphi(s+i\mu)}\right] \cdot \left[\sum_{r=k}^{\infty}(-1)^{r-k}\binom{r}{k}\prod_{i=0}^{r}\frac{\varphi(s+i\mu)}{1-\varphi(s+i\mu)}\right]\right\}^{-1}, \tag{30}$$

if $\Re(s) \geqq 0$.

PROOF. Denote by $G_0(x), G_1(x), \cdots, G_k(x)$ the distribution functions of the distances between the successive transitions $E_{-1}\to E_0$, $E_0 \to E_1$, $E_1\to E_2$, $\cdots$, $E_k \to E_{k+1}$, respectively. It is easy to see that the distribution functions $G_r(x)$ $(r = 0, 1, \cdots, k)$ are just Palm's distribution functions defined by (6). Now clearly

$$R_k^*(x) = G_0 * G_1 * \cdots * G_k(x), \tag{31}$$

and thus,

$$\int_0^\infty e^{-sx}\, dR_k^*(x) = \gamma_0(s)\gamma_1(s)\cdots\gamma_k(s), \tag{32}$$

where $\gamma_r(s)$ $(r = 0, 1, 2, \cdots)$ is defined by (6). This proves (29).

On the other hand, if

$$\Psi_k(s) = \int_0^\infty e^{-sx}\, dR_k(x), \tag{33}$$

then we have

$$\int_0^\infty e^{-st}\, dm_k(t) = \frac{\gamma_0(s)\gamma_1(s)\cdots\gamma_k(s)}{1-\Psi_k(s)}. \tag{34}$$

For, as is well known in renewal theory, we have

$$m_k(t) = R_k^*(t) + R_k^* * R_k(t) + R_k^* * R_k * R_k(t) + \cdots$$

Taking into consideration (6) and (24), we can determine $\Psi_k(s)$ from (34), and thus we get (30).

THEOREM 5. *We have*

$$\rho_k = \frac{\alpha}{\sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} C_r} \tag{35}$$

and

$$\sigma_k^2 = 2\alpha\rho_k \sum_{j=1}^{k+1} \binom{k+1}{j} C_{j-1} - \rho_k^2 \Bigg[1 - \frac{\sigma^2 - \alpha^2}{\alpha^2} \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} C_r - \frac{2}{\alpha} \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} C_r \sum_{i=1}^{r} \frac{\varphi'(i\mu)}{\varphi(i\mu)[1-\varphi(i\mu)]} \Bigg], \tag{36}$$

where

$$C_r = \prod_{i=1}^{r} \frac{1 - \varphi(i\mu)}{\varphi(i\mu)}, \qquad (r = 0, 1, 2, \cdots). \tag{37}$$

PROOF. Since

$$\sum_{j=0}^{k+1} \binom{k+1}{j} \prod_{i=0}^{j-1} \frac{1 - \varphi(s + i\mu)}{\varphi(s + i\mu)} = 1 + s\alpha \sum_{j=1}^{k+1} \binom{k+1}{j} C_{j-1} + o(s)$$

and

$$\sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} \prod_{i=0}^{r} \frac{\varphi(s + i\mu)}{1 - \varphi(s + i\mu)} = \frac{1}{s\alpha} \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} C_r$$

$$+ \frac{\sigma^2 - \alpha^2}{2\alpha^2} \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} C_r + \frac{1}{\alpha} \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} C_r \sum_{i=1}^{r} \frac{\varphi'(i\mu)}{\varphi(i\mu)[1-\varphi(i\mu)]} + o(s),$$

as $s \to 0$, we obtain easily that

$$\Psi_k(s) = 1 - \rho_k s + \frac{\sigma_k^2 + \rho_k^2}{2} s^2 + o(s^2),$$

as $s \to 0$, where ρ_k and σ_k^2 are defined by (35) and (36) respectively. This proves the theorem.

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