COMBINING INTER-BLOCK AND INTRA-BLOCK INFORMATION IN BALANCED INCOMPLETE BLOCKS

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0. Introduction. When an Eisenhart Model III [1] (blocks random, error random) is assumed in a balanced incomplete block (BIB), two independent estimates of treatment differences have been exhibited by Yates [5]. A combined estimate of treatment differences has also been set forth by Yates but none of the properties of the combined estimate have been given.

It is the purpose of this paper to show that Yates' combined estimate is based on a set of minimal sufficient statistics. A form of the combined estimate is set forth in the paper which is shown to be unbiased and which is also based on a set of minimal sufficient statistics.

1. The model. The balanced incomplete block is defined as a design in which t treatments are applied to b blocks of $k < t$ cells per block with r replicates per treatment with every pair of treatments occurring in all blocks an equal number (λ) of times.

The mathematical model may be formulated as a special case of a two way classification model with unequal numbers in the cells [3].

$$y_{iqm} = \alpha + \beta_i + \tau_q^* + e_{iqm} \tag{1}$$

where $i = 1, 2, \cdots, b$; $q = 1, 2, \cdots, t$; and $m = 0, 1, \cdots, n_{iq}$.

In a BIB, n_{iq} is equal to 1 if the qth treatment is in block i and equal to 0 if the qth treatment does not occur in block i and if $n_{iq} = 0$ the corresponding observation does not exist, i.e., y_{iq0} does not exist for any i and q.

We can now deduce:

$$\sum_i n_{iq} = r; \qquad \sum_q n_{iq} = k; \qquad \sum_{iq} n_{iq} = bk = tr = n;$$
$$\sum_i n_{iq}n_{iq'} = \lambda \text{ for all } q \neq q'.$$

2. Distributional assumptions. We will assume an Eisenhart model III with the block effects (β_i) and errors (e_{iqm}) normally and independently distributed with the following properties (E will denote mathematical expectation):

$$E(e_{iqm}) = 0 \qquad E(e_{iqm} \cdot e_{rst}) = \sigma^2 \quad \text{if} \quad i = r, \quad q = s, \quad \text{and} \quad m = t$$
$$= 0 \text{ otherwise}$$
$$E(\beta_i) = 0 \qquad E(\beta_i \cdot \beta_s) = \sigma_\beta^2 \quad \text{if} \quad i = s$$
$$= 0 \text{ otherwise}$$
$$E(\beta_i e_{rst}) = 0 \text{ for all } i, r, s, \text{ and } t.$$

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3. Matrix model. (1) represents n equations and they may be written in matrix notation as,

$$Y = \alpha(J_1^n) + L'\beta + D'\tau^* + \epsilon \tag{2}$$

where β is a vector with elements β_i and τ^* a vector with elements τ_q^*, and where we will consider the n components of the vector Y ordered on the block subscript first and then the treatment subscript. The dimensions of the matrices in (2) are: $Y(n \times 1)$, $L'(n \times b)$, $\beta(b \times 1)$, $D'(n \times t)$, $\tau^*(t \times 1)$, and $\epsilon(n \times 1)$. We will let (J_m^n) denote an $n \times m$ matrix with every element equal to 1.

We will now reparameterize by constructing a $t \times t$ orthogonal matrix P^* which has the property that every element in the first column is equal to $1/\sqrt{t}$. Then rewriting (2) we have

$$Y = \alpha(J_1^n) + L'\beta + D'P^*P^{*\prime}\tau^* + \epsilon.$$

By partioning P^* into $((1/\sqrt{t})J_1^t, P)$ we will obtain

$$Y = \mu(J_1^n) + L'\beta + A'\tau + \epsilon \tag{3}$$

where $\mu = \alpha + (1/t)\sum_q \tau_q^*$, $A' = D'P$, $\tau = P'\tau^*$ where A' is $n \times (t-1)$ and τ is $(t-1) \times 1$. Each element of τ is now an estimable function (contrast) of the τ_q^*.

In addition we will define N to be the $t \times b$ matrix

$$\begin{pmatrix} n_{11} & n_{21} & \cdots & n_{b1} \\ n_{12} & n_{22} & \cdots & n_{b2} \\ \vdots & \vdots & & \vdots \\ n_{1t} & n_{2t} & \cdots & n_{bt} \end{pmatrix}$$

The following relationships can be shown to hold:

$$P'P = I, \quad (J_t^1)P = \phi, \quad PP' = I - \frac{1}{t}J_t^t;$$

$$LL' = kI, \quad L'(J_1^b) = (J_1^n), \quad (J_n^1)L' = k(J_b^1);$$

$$DD' = rI, \quad D'(J_1^t) = (J_1^n), \quad (J_n^1)D' = r(J_t^1);$$

$$NN' = (r - \lambda)I + \lambda(J_t^t), \quad DL' = N, \quad (J_b^1)N' = r(J_t^1),$$

$$(J_t^1)N = k(J_b^1).$$

We will use the following definitions:

If $B_i = \sum_{qm} y_{iqm}$, then $(LY)' = (B_1 B_2 \cdots B_b) = B'$

If $T_q = \sum_i n_{iq} B_i$, then $(NLY)' = (T_1 T_2 \cdots T_t) = T'$

If $V_q = \sum_{im} y_{iqm}$, then $(DY)' = (V_1 V_2 \cdots V_t) = V'$

If $Q_q = V_q - \frac{1}{k} T_q$, then $\left[\left(D - \frac{1}{k}NL\right)Y\right]' = (Q_1 Q_2 \cdots Q_t) = Q'$

and where

$$B. = \frac{1}{b} \sum_i B_i, \qquad T. = \frac{1}{t} \sum_q T_q, \qquad V. = \frac{1}{t} \sum_q V_q.$$

$$y \cdots = \frac{1}{n} \sum_{iqm} y_{ijm}.$$

4. Minimal sufficient statistics. In this section we will exhibit a set of minimal sufficient statistics for the $t + 2$ parameters $\tau_1, \tau_2, \cdots, \tau_{t-1}, \mu, \sigma_\beta^2$, and σ^2 in (3).

We will first find the distribution of the vector Y. Y is distributed as the n-variate normal with

(a) mean equal to

$$\bar{\mu} = E(Y) = E[\mu(J_1^n) + L'\beta + A'\tau + \epsilon] = \mu(J_1^n) + A'\tau.$$

(b) and variance-covariance matrix equal to

$$\begin{aligned} \Sigma &= E(Y - EY)(Y - EY)' \\ &= E(L'\beta + \epsilon)(\epsilon' + \beta'L) \\ &= \sigma^2 I + \left(\frac{L'L}{k}\right) k\sigma_\beta^2 \end{aligned}$$

and

$$\Sigma^{-1} = wI - \left(\frac{w - w^*}{k}\right) L'L$$

where $w = \sigma^{-2}$, $w^* = (\sigma^2 + k\sigma_\beta^2)^{-1}$.

The joint density of the y_{iqm} is then

$$f(y) = \frac{1}{(2\pi)^{n/2} |\Sigma|^{1/2}} e^{-\frac{1}{2}(Y-\bar{\mu})'\Sigma^{-1}(Y-\bar{\mu})} \tag{4}$$

The quadratic form in the exponent of (4) is equal to (neglecting the $-\frac{1}{2}$)

$$[Y - \mu(J_1^n) - A'\tau]' \left[wI - \frac{(w - w^*)}{k} L'L\right] [Y - \mu(J_1^n) - A'\tau]$$

It can be shown that the quantity may be put in the following form:

$$\begin{aligned} &\frac{w^*}{k} Y'(I - D'R)'L'L(I - D'R)Y + \frac{w^*(r - \lambda)}{k} (\tau - P'RY)'(\tau - P'RY) \\ &\quad + w^*bk(y \cdots - \mu)^2 + wY'(I - D'G)'\left(I - \frac{1}{k} L'L\right)(I - D'G)Y \\ &\quad + \frac{w\lambda t}{k} (\tau - P'GY)'(\tau - P'GY), \end{aligned}$$

where we define the matrices R and G by

$$R = \frac{1}{(r - \lambda)} \left[DL'L - \frac{r}{b} (J_n^t)\right] \qquad G = \frac{k}{\lambda t}\left(D - \frac{1}{k} NL\right).$$

We are now able to define $2t + 1$ statistics, namely, $P'GY$, $P'RY$, $y\cdots$, $(1/k)Y'(I - D'R)L'L(I - D'R)Y$, and

$$Y'(I - P'G)(I - (1/k)L'L)(I - P'G)Y$$

which are seen to be sufficient for the $t + 2$ parameters in (3).

To simplify the notation in the ensuing discussion we will let

$$U = P'GY \qquad X = P'RY$$

$$S^{*2} = (1/k)Y'(I - D'R)L'L(I - D'R)Y$$

$$S^2 = Y'(I - D'G)(I - (1/k)L'L)(I - D'G)Y.$$

By applying the operation defined on page 328 of [4] it can be shown that the $2t + 1$ sufficient statistics as defined above form a minimal set.

5. Distributional properties of the minimal sufficient statistics. The $t - 1 \times 1$ vector $U = (u_i)$ is distributed normally with mean τ and covariance matrix $(k/\lambda t)\sigma^2 I$, so u_i is an unbiased (intra-block) estimate of τ_i with variance

$$(k/\lambda t)\sigma^2.$$

The $t - 1$ vector $X = (x_i)$ is distributed normally with mean τ and covariance matrix $(k/(r - \lambda))(\sigma^2 + k\sigma_\beta^2)I$ so x_i is an unbiased estimate (inter-block) of τ_i with variance $(k/(r - \lambda))(\sigma^2 + k\sigma_\beta^2)$.

The scalar S^2/σ^2 is distributed as a central chi-square variate with

$$n - b - t + 1 = f$$

degrees of freedom, so S^2/f is an unbiased estimate of σ^2.

The scalar $S^{*2}/(\sigma^2 + k\sigma_\beta^2)$ is distributed as a central chi-square variate with $b - t$ degrees of freedom if $b > t$, so $S^{*2}/(b - t)$ is an unbiased estimate of $\sigma^2 + k\sigma_\beta^2$. If $b = t$, then S^{*2} is not defined.

The scalar $y\cdots$ is distributed normally with mean μ and variance

$$(\sigma^2 + k\sigma_\beta^2)/bk.$$

The statistics $u_1, u_2, \cdots, u_{t-1}, x_1, x_2, \cdots, x_{t-1}, S^2, S^{*2}, y\cdots$, can also be shown to be mutually independent.

6. The analysis of variance. To obtain estimates of the parameters from the data, we may compute the analysis of variance table as in Table 1.

In order to see that Yates' definition of the combined estimate of τ is based on the minimal sufficient statistics, we can verify that

$$A_1 = ((r - \lambda)/k)X'X, \qquad A_2 = S^{*2}, \qquad A_3 = (\lambda t/k)U'U, \qquad A_4 = S^2$$

$$A_4 = C_4; \qquad A_2 = C_2, \qquad C^* = C_1 + C_2, \qquad C^* + C_3 = A + A_3.$$

From these can be deduced the relation $C_1 = A_1 + A_3 - C_3$ or

$$C_1 = (\lambda t(r - \lambda)/rk^2)(U - X)'(U - X).$$

TABLE 1

Analysis of Variance

Source	d. f.	S. S.	S. S.	d. f.	Source
Blocks (ignoring treatments)	$b-1$	$\frac{1}{k}\sum_i (B_i - B.)^2 = A^*$	C^*	$b-1$	Blocks (eliminating trts.)
Treatment component	$t-1$	$\frac{1}{k(r-\lambda)}\sum_j (T_j - T.)^2 = A_1$	C_1	$t-1$	Treatment component
Remainder	$b-t$	$A^* - A_1 = A_2$	$\leftrightarrow C_2$	$b-t$	Remainder
Treatments (eliminating blocks)	$t-1$	$\frac{k}{\lambda t}\sum_j Q_j^2 = A_3$	C_3	$t-1$	Trts. (ignoring blocks)
Intra-block error	$n-b-t+1=f$	$TSS - A^* - A_3 = A_4$	$\leftrightarrow C_4$	$n-b-t+1=f$	Intra-block error
Total	$n-1$	$\sum_{iqm} (y_{iqm} - y \cdots)^2 = TSS$	$\leftrightarrow TSS$	$n-1$	Total

Note: $\leftrightarrow$ same in both analyses.

7. Combining the estimators. If σ^2 and σ_β^2 are known, then the linear combination of u_i and x_i which gives the minimum variance unbiased estimate of τ_i is given by

$$\frac{u_i \operatorname{var}(x_i) + x_i \operatorname{var}(u_i)}{\operatorname{var}(x_i) + \operatorname{var}(u_i)}$$

or

$$\frac{u_i \lambda t(\sigma^2 + k\sigma_\beta^2) + x_i(r - \lambda)\sigma^2}{\lambda t(\sigma^2 + k\sigma_\beta^2) + (r - \lambda)\sigma^2} \tag{5}$$

Yates uses now C^* and C_4 from Table I to estimate σ^2 and $\sigma^2 + k\sigma_\beta^2$. That is, he uses

$$\hat{\sigma}^2 = A_4/f = C_4/f$$

$$\hat{\sigma}_\beta^2 = \left[\frac{C^*}{b-1} - \frac{C_4}{f}\right]\frac{(b-1)}{(bk-t)}$$

Letting $\hat{\sigma}_1^2$ be the estimate of $\sigma^2 + k\sigma_\beta^2$ by Yates' procedure we have

$$\hat{\sigma}_1^2 = \frac{C_4}{f} + \frac{k(b-1)}{bk-t}\left[\frac{C^*}{b-1} - \frac{C_4}{f}\right]$$

$$= \frac{k}{t(r-1)}\left[\frac{(r-\lambda)\lambda t}{rk^2}(U-X)'(U-X) + S^{*2}\right] + \frac{(k-t)}{ft(r-1)}S^2.$$

Therefore, using the form of (5) and inserting the estimates for the unknown variances we have for the estimate of τ_i

$$\bar{\tau}_i = \frac{u_i\lambda t\left[\frac{\lambda(r-\lambda)}{rk(r-1)}(U-X)'(U-X) + \frac{k}{t(r-1)}S^{*2} + \frac{k-t}{ft(r-1)}S^2\right] + x_i\frac{(r-\lambda)}{f}S^2}{\lambda t\left[\frac{\lambda(r-\lambda)}{rs(r-1)}(U-X)'(U-X) + \frac{k}{t(r-1)}S^{*2} + \frac{(k-t)}{ft(r-1)}S^2\right] + \frac{(r-\lambda)}{f}S^2} \tag{6}$$

Actually, Yates defined the estimate of τ_i to be the quantity given in (6) if $\hat{\sigma}_\beta^2 > 0$ and defined it to be equal to $(1/r)V_q$ if $\hat{\sigma}_\beta^2 \leqq 0$. Clearly Yates' estimate so defined is based on the set of minimal sufficient statistics.

8. An alternative combined estimate. If we let the estimate of τ_i be the quantity defined in (6) for all values of $\hat{\sigma}_\beta^2$, we will show that this is an unbiased estimate of τ_i. Rewriting (6) we have

$$\bar{\tau}_i = \frac{u_i\lambda t\hat{\sigma}_1^2 + x_i(r-\lambda)\hat{\sigma}^2}{\lambda t\sigma_1^2 + (r-\lambda)\sigma^2} = u_i\gamma + x_i(1-\gamma)$$

where

$$\gamma = \frac{\lambda t\hat{\sigma}_1^2}{\lambda t\hat{\sigma}_1^2 + (r-\lambda)\hat{\sigma}^2} \quad \text{and} \quad -\infty < \gamma < 1.$$

$\bar{\tau}_i$ can now be written as $x_i + (u_i - x_i)\gamma$ and since $E(x_i) = \tau_i$, $\bar{\tau}_i$ is unbiased if $E(u_i - x_i)\gamma = 0$. Letting $z_i = (u_i - x_i)$, $\bar{\tau}_i$ is a function of $t + 1$ mutually independent random variables. Denote the density of these $t + 1$ random variables by h. Since the z_i are normal with means equal to zero, h is an even function of each z_i. γ is also an even function of z_i and since

$$-\infty < z_i < \infty,$$

it follows that $E(z_i\gamma) = 0$. Therefore, $\bar{\tau}_i$ is unbiased.

The minimal set of sufficient statistics set forth in this paper are not complete since $E(u_i - x_i) = 0$, i.e., there is a non-trivial unbiased estimate of zero.

Therefore, the problem of which estimate of τ_i is "best" is not straightforward. This will be dealt with in another paper.

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