THE UNIQUENESS OF THE L_2 ASSOCIATION SCHEME[1]

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1. Summary. The L_2 association scheme for a class of partially balanced incomplete block designs determines the parameters of the second kind. This paper considers the converse problem: do these parameters imply the L_2 association scheme? Necessary conditions for the existence of such designs are also obtained.

2. Introduction. A partially balanced incomplete block design [2] with two associate classes is said to have L_2 association scheme [3], if the number of treatments is s^2, where s is a positive integer, and the treatments can be arranged in $a(s \times s)$ square such that any two treatments in the same row or the same column are first associates, whereas any two treatments not in the same row and not in the same column are second associates. The following relations are easily seen to hold in this case:

(1) The number of first associates of any treatment is $n_1 = 2s - 2$.

(2) With respect to any two treatments, θ_1 and θ_2, which are first associates, the number of treatments which are first associates of both θ_1 and θ_2 is

$$p_{11}^1(\theta_1, \theta_2) = s - 2.$$

(3) With respect to any two treatments, θ_3 and θ_4, which are second associates, the number of treatments which are first associates of both θ_3 and θ_4 is $p_{11}^2(\theta_3, \theta_4) = 2$.

We examine the converse problem, i.e., whether or not the relations (1), (2) and (3) imply that the association scheme is of the L_2 type. We show that the converse is true for $s \geqq 2$, excepting possibly $s = 4$. Necessary conditions for the existence of such designs are also obtained.

It is worthwhile to recall what is known about other partially balanced designs. It is known [1], that if in a partially balanced incomplete block design with two associate classes p_{12}^1 or $p_{12}^2 = 0$, then the design must necessarily be a group divisible design. Recently Connor [5], has shown that if in a partially balanced incomplete block design with two associate classes $v = n(n - 1)/2$, $n \geqq 9$, $n_1 = 2n - 4$, $p_{11}^1 = n - 2$, $p_{11}^2 = 4$, then the association scheme is triangular. In an unpublished thesis [8], Mesner has given corresponding results for the case of L_g designs, $g \geqq 2$. The proof presented here for L_2 is much simpler than that given by him. It is also shown that when $s = 4$, there are only two types of association schemes possible.

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3. Statement and proof of a lemma.

LEMMA. *Let the parameters of the second kind for a partially balanced incomplete block design with two associate classes with* s^2 *treatments be* $n_1 = 2s - 2$, $p_{11}^1 = s - 2$ *(and hence* $p_{12}^1 = s - 1$), $p_{11}^2 = 2$. *Then if* $s = 2, 3$ *or* $s > 4$, *and if the* 1-*associates of any treatment* θ *are* $\phi_1, \phi_2, \cdots, \phi_{s-1}, \psi_1, \psi_2, \cdots, \psi_{s-1}$, *where the set* $(\phi_2, \cdots, \phi_{s-1})$ *is the set of common* 1-*associates of both* θ *and* ϕ_1, *and the set* $(\psi_1, \cdots, \psi_{s-1})$ *is the set of* 1-*associates* θ, *which are* 2-*associates of* ϕ_1, *then any two treatments from the set* $(\phi_1, \cdots, \phi_{s-1})$ *are* 1-*associates. Similarly, any two treatments from the set* $(\psi_1, \cdots, \psi_{s-1})$ *are* 1-*associates, while any treatment* ϕ_i *is a* 2-*associate of any treatment* ψ_j, $i, j = 1, 2, \cdots, s - 1$.

PROOF. We will use the notation $(\theta, \phi) = i$ to denote that θ and ϕ are i-associates, $i = 1, 2$. We note that the Lemma is trivially true for $s = 2$. We now consider the case $s = 3$. Without loss of generality assume that the 1-associates of treatment 1 are 2, 3, 4 and 5, of which 3 is the 1-associate of 2, and 4 and 5 are 2-associates of 2. Then $(1, 3) = 1$, and 2 is the only possible common 1-associate of both, and hence 4 and 5 are both 2-associates of 3. It only remains to prove that $(4, 5) = 1$. Suppose, on the contrary, that $(4, 5) = 2$. Then among the 1-associates of 1, the treatment 4 has three 2-associates 2, 3, 5 contradicting the value $p_{12}^1\,(1, 4) = 2$. Hence we must have $(4, 5) = 1$.

Now consider the case $s > 4$. For convenience replace $\theta, \phi_1, \phi_2, \cdots, \phi_{s-1}, \psi_1, \psi_2, \cdots, \psi_{s-1}$ of the lemma by $1, 2, 3, \cdots, s, s + 1, s + 2, \cdots, (2s - 1)$, respectively.

We then have the treatments $2, 3, \cdots, s, s + 1, \cdots, (2s - 1)$ for 1-associates of 1, of which the set $T_1 = (3, 4, \cdots, s)$ is the set of common 1-associates of both 1 and 2, whereas the set $T_2 = (s + 1, s + 2, \cdots, (2s - 1))$ is the set of 1-associates of 1 and 2 associates of 2. Let α be any treatment of T_2. Then $(2, \alpha) = 2$. Since $p_{11}^2(2, \alpha) = 2$, and 1 is one of the common 1-associates of both 2 and α, therefore, α has at most one 1-associate in T_1. Since $p_{11}^1(1, \alpha) = s - 2$, α has at least $(s - 3)$ 1-associates in T_2. But T_2 contains besides α only s-2 treatments. Hence α has at most one 2-associate in T_2. Hence, we have the following two possibilities. Either (i) with respect to any treatment of T_2 every other treatment of T_2 is 1-associate, in which case any two treatments of T_2 form a 1-associate pair, or, (ii) there exists a treatment α of T_2 such that there is a treatment β of T_2 where $(\alpha, \beta) = 2$ and every other treatment of T_2 besides α and β is 1-associate of α. Put $T_2^1 = T_2 - (\alpha, \beta)$. Consider the treatment β. Since it can have at most one 2-associate in T_2 and this is α, the set T_2^1 is a set of 1-associates of β. Thus the set T_2^1 is the set of common 1-associates of both α and β where $(\alpha, \beta) = 2$. Treatment 1 is also a 1-associate of both α and β. The set T_2^1 and the treatment 1 give a set of $(s - 2)$ treatments which are 1-associates of both α and β. But $s - 2 > 2$. This contradicts the fact that $p_{11}^2(\alpha, \beta) = 2$. Thus this case is impossible. Hence we are left with case (i) only.

From (i), for every α of T_2, the $s - 2$ treatments of T_2 excepting α are the $p_{11}^1 = s - 2$ treatments which are 1-associates of both 1 and α. Hence the treatment 2 and all the treatments of T_1 are the $(s - 1)$ treatments which are 2-associates of α. Let γ be any treatment of T_1. Then $(1, \gamma) = 1$ and the $(s - 1)$ treatments of T_2 are 2-associates of γ. Thus the treatments of T_1 are all 1-associates of γ. Hence any two treatments from the set 2 and T_1 are 1-associates. This completes the proof of the lemma.

4. Statement and proof of the main theorem.

THEOREM 1. *If the parameters of the second kind for a partially balanced incomplete block design with s^2 treatments with two associate classes are given by*

$$n_1 = 2s - 2, \qquad p_{11}^1 = s - 2, \qquad p_{11}^2 = 2,$$

then the design has L_2 association scheme if $s = 2$, 3 or $s > 4$.

PROOF. The case $s = 2$ is trivial. We consider the cases $s = 3$ or $s > 4$. From the above lemma, we can write down the 1-associates of θ in the following scheme.

$$\begin{array}{l} \theta \quad \phi_1\, \phi_2 \cdots \phi_{s-1} \\ \psi_1 \\ \psi_2 \\ \vdots \\ \psi_{s-1} \end{array}$$

where any two treatments in the first row or in the first column are 1-associates, and any treatment ϕ is a 2-associate of any treatment ψ. Let δ be any 2-associate of θ. We have $p_{11}^2(\theta, \delta) = 2$. Hence δ cannot have more than two 1-associates in the set $(\phi_1, \phi_2, \cdots, \phi_{s-1})$. Similarly, it cannot have more than two 1-associates from the set $(\psi_1, \cdots, \psi_{s-1})$ and the number of 1-associates of δ from the set of ϕ_i and ψ_j is exactly 2. Suppose δ has two 1-associates ϕ_i and ϕ_j; then ϕ_i and ϕ_j have the $s - 2$ remaining treatments of the first row and δ as common 1-associates. But this makes the number $p_{11}^1(\phi_i, \phi_j) \geqq s - 1 > s - 2$ which is the value of p_{11}^1. We thus get a contradiction. Similarly, if δ has no 1-associate from the set $(\phi_1, \cdots, \phi_{s-1})$, then both these 1-associates of δ must come from the set $\psi_1, \cdots, \psi_{s-1}$, which will again give a contradiction. Thus δ has exactly one 1-associate from the set of ϕ_i's and exactly one 1-associate from the set of ψ_j's. Hence any δ, where $(\theta, \delta) = 2$, determines uniquely a pair (ϕ_i, ψ_j) such that $(\phi_i, \delta) = 1$, $(\psi_j, \delta) = 1$. Conversely we show that any pair (ϕ_i, ψ_j) uniquely determine a δ such that $(\theta, \delta) = 2$ and $(\phi_i, \delta) = (\psi_j, \delta) = 1$. For suppose there are two such δ's, say δ_1 and δ_2. Then we have the following relations.

$$(\phi_i, \psi_j) = 2$$

$$(\phi_i, \theta) = (\psi_j, \theta) = 1$$

$$(\phi_i, \delta_1) = (\psi_j, \delta_1) = (\phi_i, \delta_2) = (\psi_j, \delta_2) = 1.$$

This gives the value $p_{11}^2(\phi_i, \psi_j) = 3$ which is a contradiction. Thus the correspondence between δ and the pair (ϕ_i, ψ_j) is 1 to 1. We can, therefore, put δ

in the position determined by the column of ϕ_i and row of ψ_j. Thus the $(s - 1)^2$ positions can be uniquely filled by utilizing the $(s - 1)^2$ 2-associates of θ. We thus get the following scheme.

$$\begin{array}{llll}
\theta & \phi_1 & \phi_2 & \cdots \phi_{s-1} \\
\psi_1 & \delta_1 & \delta_2 & \cdots \delta_{s-1} \\
\psi_2 & \delta_s & \delta_{s+1} & \cdots \delta_{2(s-1)} \\
\vdots & \vdots & \vdots & \cdots \vdots \\
\psi_{s-1} & \delta_{s^2-3s+3} & \delta_{s^2-3s+4} & \cdots \delta_{(s-1)^2}
\end{array}$$

Then all the 1-associates of ϕ_i are exactly the treatments in the row and column corresponding to it. A similar result is true for any ψ_j. Now consider ψ_1. Its 1-associates are contained in the second row and first column. Among these 1-associates the elements $\psi_2, \cdots \psi_{s-1}$ are the common 1-associates of ψ_1 and θ. whereas $\delta_1, \delta_2, \cdots, \delta_{s-1}$ are the 1-associates of ψ_1 and 2-associates of θ. Hence the application of the lemma gives the result that any two treatments in the second row are 1-associates. Similarly, we get the result that any two treatments in the second column are 1-associates. A similar result is obviously true for any other row or any other column. Thus for any treatment whatsoever, all its 1-associates are obtained by taking the treatments in the row and column corresponding to that treatment. Hence any two treatments which are neither in the same row nor in the same column are 2-associates. This completes the proof of the theorem.

5. Some known results on rational equivalence of matrices and Hilbert norm-residue symbol. Let A and B be two symmetric matrices of order n with elements in the rational field. The matrices A and B are rationally equivalent, written $A \backsim B$, if there exists a nonsingular C with elements in the same field, such that $A = C'BC$. The congruence of matrices satisfies the usual requirements of an "equals" relationship.

If A is an integral symmetric matrix of order and rank n, we can always construct an integral diagonal matrix $D = (d_1, \cdots, d_n)$, $d_i \neq 0$, $i = 1, 2, \cdots, n$, such that $D \backsim A$. The number of negative terms i, called the index of A, is an invariant of A by Sylvester's Law.

Define $d = (-1)^i\delta$, where δ is the square-free positive part of $|A|$. Then since $|B| = |C|^2 |A|$, d is another invariant of A.

Now let A be a nonsingular and symmetric integral matrix of order n. Let D_r be the leading principal minor determinant of order r and suppose that $D_r \neq 0$, $r = 1, 2, \cdots, n$. Define

$$c_p(A) = (-1, -D_n)\prod_{j=1}^{n-1} (D_j, -D_{j+1}) \tag{5.1}$$

for every odd prime p where $(m, m')_p$ is the Hilbert norm residue symbol. Then we have the following results [4], [1].

THEOREM (A). *If m and m' are integers not divisible by the odd prime p, then*

$$(m, m')_p = +1 \tag{5.2}$$

$$(m, p)_p = (p, m)_p = (m/p) \tag{5.3}$$

where (m/p) is the Legendre symbol. Moreover if $m \equiv m' \not\equiv 0 \bmod p$, then

$$(m, p)_p = (m', p)_p. \tag{5.4}$$

THEOREM (B). *For arbitrary non-zero integers m, m', n, n' and every prime p,*

$$(-m, m)_p = +1 \tag{5.5}$$

$$(m, n)_p = (n, m)_p \tag{5.6}$$

$$(mm', n)_p = (m, n)_p(m', n)_p \tag{5.7}$$

$$(mm', m - m')_p = (m, -m')_p. \tag{5.8}$$

From the above it is easy to verify that for p an odd prime and every positive integer m

$$(m, m + 1)_p = (-1, m + 1)_p \tag{5.9}$$

$$\prod_{j=1}^{m} (j, j + 1)_p = ((m + 1)!, -1)_p. \tag{5.10}$$

The fundamental Minkowski-Hasse Theorem states:

THEOREM (C). *Let A and B be two integral symmetric matrices of order and rank n. Suppose that the leading principal minor determinants of A and B are all different from zero. Then $A \backsim B$, if and only if A and B have the same invariants i, d, and c_p for every prime p including ∞.*

In the rest of this paper, "p" stands for an odd prime and will generally be omitted in writing the Hilbert norm-residue symbol.

6. Necessary conditions for the existence of symmetrical P.B.I.B. designs with $v = s^2$, $n_1 = 2s - 2$, $p_{11}^1 = s - 2$, $p_{11}^2 = 2$, when $s \geqq 3$ and $s \neq 4$. Consider the symmetrical design with parameters

$$\begin{aligned} &v = b = s^2, \quad r = k, \lambda_1, \lambda_2, \quad n_1 = 2s - 2, \quad n_2 = (s - 1)^2 \\ &p_{11}^1 = s - 2, \quad p_{12}^1 = s - 1, \quad p_{22}^1 = (s - 1)(s - 2) \\ &p_{11}^2 = 2, \quad p_{12}^2 = 2s - 4, \quad p_{22}^2 = (s - 2)^2, \quad s \geqq 3, s \neq 4. \end{aligned} \tag{6.1}$$

Then we have $r(r - 1) = 2(s - 1)\lambda_1 + (s - 1)^2\lambda_2$ or

$$r^2 = [r + (s - 1)\lambda_1] + (s - 1)[\lambda_1 + (s - 1)\lambda_2]. \tag{6.2}$$

Let $N = (n_{ij})$ be the incidence matrix of the design where

$$\begin{aligned} n_{ij} &= 1 \text{ if treatment } i \text{ occurs in block } j \\ &= 0 \text{ otherwise.} \end{aligned}$$

Then by renumbering the treatments, if necessary, and using Theorem 1, we have

$$NN' = \begin{pmatrix} A & B & \cdots & B \\ B & A & \cdots & B \\ \vdots & \vdots & \cdots & \vdots \\ B & B & \cdots & A \end{pmatrix} \tag{6.3}$$

where A is an $s \times s$ symmetric matrix with r in the main diagonal and λ_1 elsewhere and B is another $s \times s$ symmetric matrix with λ_1 in the main diagonal and λ_2 elsewhere. By a succession of elementary transformations on rows of NN' considered as a partitioned matrix and the same elementary transformation on columns of NN' and using only the rational numbers it is easy to verify that

$$NN' \sim T = \begin{bmatrix} 1\cdot 2(A-B) & 0 & \cdot\cdot & 0 & 0 \\ 0 & 2\cdot 3(A-B) & \cdot\cdot & 0 & 0 \\ \cdot & \cdot & \cdot\cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot\cdot & \cdot & \cdot \\ 0 & 0 & \cdot\cdot & (s-1)s(A-B) & 0 \\ 0 & 0 & \cdot\cdot & 0 & s(A+(s-1)B) \end{bmatrix}. \tag{6.4}$$

Put

$$P = (r - \lambda_1) + (s - 1)(\lambda_1 - \lambda_2) \tag{6.5}$$

$$Q = r - 2\lambda_1 + \lambda_2 \tag{6.6}$$

$$\lambda = \lambda_1 - \lambda_2 , \lambda' = \lambda_1 + (s - 1)\lambda_2 . \tag{6.7}$$

Then it is easy to verify that

$$|A - B| = Q^{s-1}P \tag{6.8}$$

$$|A + (s - 1)B| = r^2p^{s-1}. \tag{6.9}$$

Hence

$$|T| = r^2(s!)^{2s}Q^{(s-1)^2}P^{2(s-1)}. \tag{6.10}$$

Since NN' is semipositive definite, so is T. Hence we have $P \geqq 0$ and $Q \geqq 0$. Further $|NN'| = |N|^2$ is a perfect square. Hence $|T|$ is a perfect square. Thus, if $P > 0$ and $Q > 0$, which means that N is nonsingular, a necessary condition for existence of the design when s is even is that Q must be a perfect square. In what follows we assume that $P > 0$ and $Q > 0$. This result can also be obtained by using the results of Connor and Clatworthy [6].

Let

$$T_1 = \begin{bmatrix} 1\cdot 2(A-B) & 0 & \cdot\cdot & 0 \\ 0 & 2\cdot 3(A-B) & \cdot\cdot & 0 \\ \cdot & \cdot & \cdot\cdot & \cdot \\ \cdot & \cdot & \cdot\cdot & \cdot \\ 0 & 0 & \cdot\cdot & (s-1)s(A-B) \end{bmatrix} \tag{6.11}$$

and

(6.12) $$T_2 = s(A + (s - 1)B).$$

Then

(6.13) $$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}.$$

Further, if R is the $(s - 1) \times (s - 1)$ diagonal matrix

(6.14) $$R = \text{diag}\,\{1\cdot 2, 2\cdot 3, \cdots, (s - 1)s\}.$$

Then

(6.15) $$T_1 = R\,x\,(A - B)$$

when x denotes the Kronecker product of the matrices. It is easily verified, using the results of section 5, that

(6.16) $$|R| = ((s - 1)!)^2 s \qquad \text{and}$$

(6.17) $$c(R) = 1.$$

We now evaluate the values of $c(A - B)$ and $c(A + (s - 1)B)$.

Following [1, p. 379] we get

$$c(A - B) = (Q, -1)^{s(s-1)/2}(PQ, \lambda)(P, Q)^s.$$

Now, since $P > 0$, $Q > 0$ and $P - Q = s\lambda \neq 0$ we get from (5.8)

$$\begin{aligned}(PQ, \lambda) &= (PQ, P - Q)(PQ, s) \\ &= (P, -1)(P, Q)(P, s)(Q, s)\end{aligned}$$

$\therefore$

(6.18) $$\begin{aligned} c(A - B) &= (Q, -1)^{s(s-1)/2}(P, Q)^{s-1}(P, -1)(P, s)(Q, s) \\ &= (P, -1)(Q, -1)^{s(s-1)/2}(-P, Q)^{s-1}(-1, Q)^{s-1}(P, s)(Q, s) \\ &= (P, -1)(Q, -1)^{(s-1)(s-2)/2}(-P, Q)^{s-1}(P, s)(Q, s).\end{aligned}$$

Again following [1, p. 379] we get $c(A + (s - 1)B) = (P, -1)^{s(s-1)/2}(P, \lambda')(r^2, \lambda')$. Since $r^2 - P = s\lambda' \neq 0$,

(6.19) $$\begin{aligned} c(A + (s - 1)B) &= (P, -1)^{s(s-1)/2}(r^2P, \lambda') \\ &= (P, -1)^{s(s-1)/2}(r^2P, r^2 - P)(r^2P, s) \\ &= (P, -1)^{s(s-1)/2}(r^2P, r^2 - P)(r^2P, s) \\ &= (P, -1)^{s(s-1)/2}(r^2, -P)(r^2, s)(P, s) \\ &= (P, -1)^{s(s-1)/2}(P, s).\end{aligned}$$

Since $T_1 = R\,x\,(A - B)$ from [9] we have

$$C(T_1) = [c(R)]^s[c(A - B)]^{s-1}(|A - B|, -1)^{(s-1)(s-2)/2}(|R|, -1)^{s(s-1)/2} (|R|, |A - B|)^{s(s-1)-1}.$$

Substituting the values obtained above we get after some simplification

$$c(T_1) = (P, -1)^{s(s-1)/2}(-P, Q)^{s-1}(P, s)^s(s, -1)^{s(s-1)/2}. \tag{6.20}$$

Similarly from [7] we have

$$\begin{aligned} c(T_2) &= c(A + (s - 1)B)(s, -1)^{s(s+1)/2}(s, |A + (s - 1)B|)^{s-1} \\ &= (P, -1)^{s(s-1)/2}(P, s)^s(s, -1)^{s(s+1)/2} \end{aligned} \tag{6.21}$$

after some simplifications. Also we have

$$|T_1| = (s - 1)!^{2s}s^sP^{s-1}Q^{(s-1)^2} \tag{6.22}$$

$$|T_2| = r^2s^sP^{s-1}. \tag{6.23}$$

Since

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$$

is the direct sum of T_1 and T_2, we have [1] $c(T) = c(T_1)c(T_2)(|T_1|, |T_2|)$. Substituting the values already found out it is easy to verify that

$$c(T) = (PQ, -1)^{s-1}. \tag{6.24}$$

Since $I \sim NN' \sim T$ and $c(I) = +1$ for all odd prime p, we must have

$$(PQ, -1)_p^{s-1} = 1 \quad \text{for all odd prime } p.$$

If s is odd the above relation is always true. For s even we have the relation $(PQ, -1) = 1$ or $(P, -1)(Q, -1) = 1$. But when s is even, a necessary condition for existence is that Q be a perfect square. Hence we get the further necessary condition for existence, i.e., $(P, -1)_p = +1$ for all odd prime p. We can thus state the following theorem.

THEOREM 2. *A necessary condition for existence of the symmetric partially balanced incomplete block design satisfying* (6.1).

(i) $P \geqq 0$, $Q \geqq 0$, and

(ii) *if s is even and $P \neq 0$, $Q \neq 0$, then Q must be a perfect square, and $(P, -1)_p = 1$ for all odd prime p.*

M. N. Vartak ([10]) considers a similar problem for a 3-associate class of partially balanced designs.

7. Association scheme for the case $s = 4$. Consider the partially balanced incomplete design with the following parameters

$$\begin{array}{lll} v = 16, & n_1 = 6, & n_2 = 9 \\ p_{11}^1 = 2, & p_{12}^1 = 3, & p_{22}^1 = 6 \\ p_{11}^2 = 2, & p_{12}^2 = 4, & p_{22}^2 = 4. \end{array} \tag{7.1}$$

Let $(\alpha_1, \alpha_2) = 1$ and α_3, α_4 be the common 1-associates of both α_1 and α_2. Then we have either

$$\text{case (i)} \quad (\alpha_3, \alpha_4) = 1, \quad \text{or}$$

$$\text{case (ii)} \quad (\alpha_3, \alpha_4) = 2.$$

Consider case (i). Let $\alpha_5, \alpha_6, \alpha_7$ be the remaining 1-associates of α_1; then these are obviously 2-associates of α_2, giving the following scheme:

$$\begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_5 & & & \\ \alpha_6 & & & \\ \alpha_7 . & & & \end{array}$$

Now any two treatments of the first row are 1-associates and hence $\alpha_5, \alpha_6, \alpha_7$ will be 2-associates of α_3 and α_4. Since $\alpha_2, \alpha_3, \alpha_4$ are 2-associates of α_5; α_6, α_7 must be 1-associates of α_5. Similarly α_5, α_7 are 1-associates of α_6. Hence any two treatments in the first column are 1-associates. It now follows, as in the proof of Theorem 1, that the association scheme is of L_2 type. Hence if β_1 and β_2 are any two treatments which are 1-associates, and β_3, β_4 are common 1-associates of them both, then $(\beta_3, \beta_4) = 1$, i.e., if case (i) holds for any one pair of 1-associates, it must hold for all such pairs.

We now consider case (ii). Replace treatments $\alpha_1, \alpha_2, \cdots, \alpha_7$ by $1, 2, \cdots, 7$ for sake of convenience, giving the scheme

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & & & \\ 6 & & & \\ 7 & & & \end{array}$$

Considering the pair (1, 3) and the value $p_{11}^1(1, 3) = 2$, we see that 3 has just one 1-associate from the set (5, 6, 7). Without loss of generality assume that $(3, 6) = 1$ and hence $(3, 5) = (3, 7) = 2$. Consider the pair (3, 4). Here 1 and 2 are 1-associates of both 3 and 4, accounting for the value $p_{11}^2(3, 4) = 2$. Hence since $(3, 6) = 1$, $(4, 6) = 2$. Now $(6, 2) = (6, 4) = 2$, and $(6, 3) = 1$. Hence from the values $p_{11}^1(1, 6)$ and $p_{12}^1(1, 6)$ we see that 6 has just one 1-associate and one 2-associate from the set (5, 7). Let $(6, 5) = 1$ and $(6, 7) = 2$. Now 2, 3 and 6 are 2-associates of 7; hence considering $(1, 7) = 1$ and the values $p_{12}^1(1, 7)$ and $p_{11}^1(1, 7)$, we see that 4 and 5 are 1-associates of 7. Now 1-associates of 5 are 6 and 7 accounting for the value $p_{11}^1(1, 5)$. Hence 4 must be a 2-associate of 5. We can summarize the above information in the following table, where the entry in row α and column β gives the value of (α, β), where

$\alpha \neq \beta$, and * along the main diagonal indicates that no treatment is either 1-associate or 2-associate of itself.

(7.2)

	1	2	3	4	5	6	7
1	*	1	1	1	1	1	1
2	1	*	1	1	2	2	2
3	1	1	*	2	2	1	2
4	1	1	2	*	2	2	1
5	1	2	2	2	*	1	1
6	1	2	1	2	1	*	2
7	1	2	2	1	1	2	*

Thus with respect to treatment 2, 1-associates of 1 can be exhibited in the following scheme S_1 :

$$S_1: \quad \begin{array}{llll} 1 & 2 & \underline{3 \quad 4} & \\ 5 & & & \\ \mid 6 & & & \\ \mid 7 & & & \end{array}$$

where treatments in the same row or column are 1-associates unless they are both "under"-lined in which case they are 2-associates. Treatments not in the same row or column are 2-associates, unless they are both first or both second members of "under"-lined pairs, in which case they are 1-associates.

We will adopt the convention of writing down the 1-associates of any treatment β_1 (here 1) with respect to any 1-associate treatment β_2 (here 2) in the scheme of the above type, which will bring out the association relationship amongst all the treatments involved in the scheme.

Now amongst the treatments 1, 2, $\cdots$, 7, only the treatments 1, 3, 4 are 1-associates of 2. Let the remaining 1-associates of 2 be 8, 9, 10. Then writing the row

$$2 \quad 1 \quad \underline{3 \quad 4}$$

we see that only one of the treatments 8, 9, 10 is a 1-associate of 3. Without loss of generality let $(3, 9) = 1$. Then 9 has just one 1-associate from the set (8, 10). Let $(9, 8) = 1$, and hence $(9, 10) = 2$. Hence, referring to S_1 for comparison we can write down the scheme S_2 :

$$S_2: \quad \begin{array}{llll} 2 & 1 & \underline{3 \quad 4} & \\ 8 & & & \\ \mid 9 & & & \\ \mid 10 & & & \end{array}$$

We can now indicate the relations implied by S_1 and S_2 in the following table:

(7.3)

	1	2	3	4	5	6	7	8	9	10
1	*	1	1	1	1	1	1	2	2	2
2	1	*	1	1	2	2	2	1	1	1
3	1	1	*	2	2	1	2	2	1	2
4	1	1	2	*	2	2	1	2	2	1
5	1	2	2	2	*	1	1			
6	1	2	1	2	1	*	2			
7	1	2	2	1	1	2	*			
8	2	1	2	2				*	1	1
9	2	1	1	2				1	*	2
10	2	1	2	1				1	2	*

We now consider the association relationship of any treatment from the set (5, 6, 7) with any treatment from the set (8, 9, 10).

Consider (2, 6) = 2. Treatments 1, 3 are common 1-associates of both 2 and 6, and $p_{11}^2(2, 6) = 2$. Hence the remaining 1-associates of 2, i.e., 4, 8, 9, 10 are 2-associates of 6. Thus if we combine S_1 and S_2 into a new scheme S_3 :

S_3 :

1	2	<u>3</u>	<u>4</u>
5	8		
6	9		
7	10		

we see that all the treatments of the second column are 2-associates of 6. Similarly (2, 7) = 2, and 1 and 4 are common 1-associates of both. Hence 3, 8, 9, 10 are 2-associates of 7. Hence again all the elements in the second column are 2-associates of 7. Again (1, 9) = 2; and 2, 3 are common 1-associates of both. Therefore, the remaining 1-associates of 1 are 2-associates of 9. Hence all the treatments in the first column are 2-associates of 9. Similarly they are 2-associates of 10. Now (1, 8) = 2, and 3, 4, 6, 7 are 2-associates of 8, giving $p_{12}^2(1, 8) = 4$, Hence the remaining treatment, i.e., 5 must be a 1-associate of 8.

These relations are summarized below.

$$\begin{array}{lll} (5, 8) = 1, & (5, 9) = (5, 10) = 2 \\ (6, 8) = & (6, 9) = (6, 10) = 2 \\ (7, 8) = & (7, 9) = (7, 10) = 2. \end{array} \tag{7.4}$$

A complete concise explanation of S_3 can, therefore, be given as follows: "Treatments in the same row or column are 1-associates unless they are both 'under'-lined, in which case they are 2-associates. Treatments not in the same row or column are 2-associates unless they are both first or both second members of 'under'-lined pairs, in which case they are 1-associates." We utilise this method of combining two schemes to get new relations.

Now among the treatments 1, 2, $\cdots$, 10, the treatments 1, 2, 6, 9 are 1-associates whereas 4, 5, 7, 8, 10 are 2-associates of 3. Let the remaining 1-associates of 3 be 11 and 12. The common 1-associates of 1 and 3 are 2 and 6, where $(2, 6) = 2$. Hence we write down the row

$$3 \quad 1 \quad \underline{2 \quad 6}.$$

Of the remaining treatments 9, 11, 12, we know that $(2, 9) = 1$. Hence 9 is placed in the third position in the column for 3. Again let $(9, 11) = 1$ and $(9, 12) = 2$. Then we have the scheme

$$S_4: \quad \begin{array}{cccc} 3 & 1 & \underline{2} & \underline{6} \\ 11 & & & \\ \vert\, 9 & & & \\ \vert\, 12. & & & \end{array}$$

Similarly completing the scheme for $1\ 3\ \underline{2\ 6}$ and utilizing the relations already obtained we have

$$S_5: \quad \begin{array}{cccc} 1 & 3 & \underline{2} & \underline{6} \\ 7 & & & \\ \vert\, 4 & & & \\ \vert\, 5. & & & \end{array}$$

S_4 and S_5 can be combined into

$$S_6: \quad \begin{array}{cccc} 3 & 1 & 2 & 6 \\ 11 & 7 & & \\ \vert\, 9 & \vert\, 4 & & \\ \vert\, 12 & \vert\, 5. & & \end{array}$$

From S_4, S_5, S_6 we get the following relations.

$$\begin{aligned} (1, 11) &= (1, 12) &&= 2 \\ (2, 11) &= (2, 12) &&= 2 \\ (3, 11) &= (3, 12) &&= 1 \\ (4, 11) &= (4, 12) &&= 2 \\ (5, 11) &= (5, 12) &&= 2 \\ (6, 12) &= 1,\ (6, 11) &&= 2 \\ (7, 11) &= 1,\ (7, 12) &&= 2 \\ (9, 11) &= 1,\ (9, 12) &&= 2 \\ (11, 12) &= 1. && \end{aligned} \tag{7.5}$$

Now common 1-associates of 2 and 3 are 1, 9 where $(1, 9) = 2$. Hence utilizing the previous relations we have the scheme

$$S_7: \quad \begin{array}{cccc} 3 & 2 & \underline{1} & \underline{9} \\ 12 & & & \\ \vert\, 6 & & & \\ \vert\, 11. & & & \end{array}$$

Similarly we have S_8 and then S_9 by combining S_7 and S_8.

$$S_8: \quad \begin{array}{llll} 2 & 3 & \multicolumn{2}{l}{\underline{1 \quad 9}} \\ 10 & & & \\ |4 & & & \\ |8. & & & \end{array}$$

$$S_9: \quad \begin{array}{llll} 3 & 2 & \multicolumn{2}{l}{\underline{1 \quad 9}} \\ 12 & 10 & & \\ |\ 6 & |4 & & \\ |11 & |8. & & \end{array}$$

S_7, S_8, S_9 give rise to the following relations.

$$(7.6) \qquad \begin{aligned} (8, 11) &= (8, 12) \qquad = 2 \\ (10, 12) &= 1, \ (10, 11) = 2. \end{aligned}$$

Now among the treatments 1, 2, $\cdots$, 12, the 1-associates of 4 are 1, 2, 7, 10. Let the remaining two 1-associates of 4 be 13 and 14. The common 1-associates of 4 and 1 are 2, 7, where $(2, 7) = 2$. Writing the row

$$4 \quad 1 \quad \underline{2 \quad 7}$$

we see that of the remaining 1-associates of 4 i.e., 13, 10, 14, the treatment 10 is 1-associate of 2. Without loss of generality assume that $(10, 13) = 1$ and $(10, 14) = 2$. Then we have the scheme

$$S_{10}: \quad \begin{array}{llll} 4 & 1 & \multicolumn{2}{l}{\underline{2 \quad 7}} \\ 13 & & & \\ |10 & & & \\ |14. & & & \end{array}$$

We have similarly

$$S_{11}: \quad \begin{array}{llll} 1 & 4 & \multicolumn{2}{l}{\underline{2 \quad 7}} \\ 6 & & & \\ |3 & & & \\ |5 & & & \end{array}$$

and by combining S_{10} and S_{11}

$$S_{12}: \quad \begin{array}{llll} 4 & 1 & \multicolumn{2}{l}{\underline{2 \quad 7}} \\ 13 & 6 & & \\ |10 & |3 & & \\ |14 & |5. & & \end{array}$$

From S_{10}, S_{11}, S_{12} we get the relations

$$
(7.7) \qquad \begin{aligned}
&(1, 13) = (1, 14) &&= 2\\
&(2, 13) = (2, 14) &&= 2\\
&(3, 13) = (3, 14) &&= 2\\
&(4, 13) = (4, 14) &&= 1\\
&(5, 13) = (5, 14) &&= 2\\
&(6, 13) = 1, \ (6, 14) &&= 2\\
&(7, 14) = 1, \ (7, 13) &&= 2\\
&(10, 13) = 1, (10, 14) &&= 2\\
&(13, 14) = 1.
\end{aligned}
$$

Again we can verify that the only possible schemes for rows
$4 \quad 2 \quad \underline{1 \quad 10}$ and $2 \quad 4 \quad \underline{1 \quad 10}$
are

$$
\begin{array}{llll}
4 & 2 & \multicolumn{2}{l}{\underline{1 \quad 10}}\\
14 & & & \\
|\,7 & & & \\
|\,13 & & &
\end{array}
$$

and

$$
\begin{array}{llll}
2 & 4 & \multicolumn{2}{l}{\underline{1 \quad 10}}\\
9 & & & \\
|\,3 & & & \\
|\,8. & & &
\end{array}
$$

Combining these into the scheme

$$
S_{13}: \quad
\begin{array}{llll}
4 & 2 & \multicolumn{2}{l}{\underline{1 \quad 10}}\\
14 & 9 & & \\
|\,7 & |\,3 & & \\
|\,13 & |\,8 & &
\end{array}
$$

we have the relations

$$
(7.8) \qquad \begin{aligned}
&(8, 13) = (8, 14) &&= 2\\
&(9, 14) = 1, (9, 13) &&= 2.
\end{aligned}
$$

Now the 1-associates of 5 amongst the treatment $1, 2, \cdots, 14$ are 1, 6, 7, 8. Hence 15 and 16 are the remaining two 1-associates of 5. The common 1-associates of 5 and 1 are 6, 7 where $(6, 7) = 2$. Now 8 is known to be 2-associate of 6 and 7. Hence 8 occupies the second position in the column for 5. Let 15 be 1-associate of 6; hence 16, 2-associate of 6. Then we have

$$\begin{array}{llll} 5 & 1 & \underline{6} & \underline{7} \\ 8 & & & \\ \vert 15 & & & \\ \vert 16. & & & \end{array}$$

We also have

$$\begin{array}{llll} 1 & 5 & \underline{6} & \underline{7} \\ 2 & & & \\ \vert 3 & & & \\ \vert 4 & & & \end{array}$$

and hence combining these two we get

$$S_{14}: \quad \begin{array}{llll} 5 & 1 & \underline{6} & \underline{7} \\ 8 & 2 & & \\ \vert 15 & \vert 3 & & \\ \vert 16 & \vert 4 & & \end{array}$$

and we get the following relations.

$$(7.9) \qquad \begin{aligned} (1, 15) &= (1, 16) = 2 \\ (2, 15) &= (2, 16) = 2 \\ (3, 15) &= (3, 16) = 2 \\ (4, 15) &= (4, 16) = 2 \\ (5, 15) &= (5, 16) = 1 \\ (6, 15) &= 1, (6, 16) = 2 \\ (7, 16) &= 1, (7, 15) = 2 \\ (8, 15) &= (8, 16) = 1 \\ (15, 16) &= 2. \end{aligned}$$

Consistent with the previous relations, it is easy to verify that we have the only possible schemes

$$S_{15}: \quad \begin{array}{llll} 5 & 6 & \underline{1} & \underline{15} \\ 16 & 12 & & \\ \vert 7 & \vert 3 & & \\ \vert 8 & \vert 13 & & \end{array}$$

giving the relations

$$(7.10) \qquad \begin{aligned} (12, 13) &= (12, 16) = 1, (12, 15) = 2 \\ (13, 15) &= 1, (13, 16) = 2 \end{aligned}$$

and

$$S_{16}: \quad \begin{array}{llll} 4 & 7 & \underline{1} & \underline{14} \\ 10 & 16 & & \\ \vert 2 & \vert 5 & & \\ \vert 13 & \vert 11 & & \end{array}$$

giving the relations

$$(10, 16) = 1$$
(7.11) $$(11, 14) = (11, 16) = 1, (11, 13) = 2$$
$$(14, 16) = 2.$$

Now counting the 1-associates and 2-associates of 12 in the previous relations we get

(7.12) $$(12, 14) = 2.$$

Similarly counting the 1-associates and 2-associates of 9 in the previous relations we see that the 1-associates of 9 are 2, 3, 8, 11, 14 and either 15 or 16. Now $(7, 9) = 2$ and 1-associates of 7 are 1, 4, 5, 11, 14 and 16. Hence from the value $p_{11}^2(7, 9) = 2$ it is easv to see that

(7.13) $$(9, 15) = 1$$
$$(9, 16) = 2.$$

Again counting the 1-associates and 2-associates of 10 in the previous relations we see that

(7.14) $$(10, 15) = 2.$$

Similarly we can verify that

(7.15) $$(11, 15) = 2$$
$$(14, 15) = 1.$$

The relations (7.2) to (7.15) give the following table of 1-associates.

Treatment	1-associates
1	2, 3, 4, 5, 6, 7
2	1, 3, 4, 8, 9, 10
3	1, 2, 6, 9, 11, 12
4	1, 2, 7, 10, 13, 14
5	1, 6, 7, 8, 15, 16
6	1, 3, 5, 12, 13, 15
7	1, 4, 5, 11, 14, 16
8	2, 5, 9, 10, 15, 16
9	2, 3, 8, 11, 14, 15
10	2, 4, 8, 12, 13, 16
11	3, 7, 9, 12, 14, 16
12	3, 6, 10, 11, 13, 16
13	4, 6, 10, 12, 14, 15
14	4, 7, 9, 13, 11, 15
15	5, 6, 8, 9, 13, 14
16	5, 7, 8, 10, 11, 12

It is obvious that the association scheme for this case is unique and that the two common 1-associates of any two treatments, which are 1-associates, must be 2-associate, for otherwise the association scheme is of L_2 type from case (i). Mesner [8] has shown that for $s = 4$, if we interchange the first and second associates in L_3 we get a design with parameters (7.1). The association scheme for case (ii) must therefore be of the same type as obtained from Mesner's result.

We now give an example due to Mesner [8], to show that there actually exists a design for $s = 4$, which has the association scheme of case (ii) described above. Consider the following Latin Square

$$\begin{matrix} A & B & C & D \\ B & C & D & A \\ C & D & A & B \\ D & A & B & C \end{matrix}$$

which has the property that there exists no Latin Square of side 4 which is orthogonal to it. Superimposing the above Latin Square on the square array

$$\begin{matrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{matrix}$$

and forming blocks corresponding to the rows, columns, and letters of the Latin square we get the following twelve blocks for 16 treatments: (1, 2, 3, 4), (5, 6, 7, 8), (9, 10, 11, 12), (13, 14, 15, 16), (1, 5, 9, 13), (2, 6, 10, 14), (3, 7, 11, 15), (4, 8, 12, 16), (1, 8, 11, 14), (2, 5, 12, 15), (3, 6, 9, 16), (4, 7, 10, 13). Any two treatments either do not occur together in any block (in which case they are 1-associates), or they occur together exactly in one block (in which case they are 2-associates). It is easily verified that the design is a partially balanced design with two associate classes with $v = 16$, $b = 12$, $r = 3$, $k = 4$, $n_1 = 6$, $n_2 = 9$, $\lambda_1 = 0$, $\lambda_2 = 1$, $p_{11}^1 = 2$, $p_{12}^1 = 3$, $p_{11}^2 = 2$. It is easy to see that $(1, 6) = 1$, and their common 1-associates are 12 and 15 where $(12, 15) = 2$. Hence the association scheme of this design is not of L_2 type. It must, therefore, correspond to the association scheme of case (ii).

REFERENCES

[1] R. C. Bose and W. S. Connor, "Combinatorial properties of group divisible incomplete block designs," *Ann. Math. Stat.*, Vol. 23 (1952), pp. 367–383.

[2] R. C. Bose and K. R. Nair, "Partially balanced incomplete block designs," *Sankhyā*, Vol. 4 (1939), pp. 337–372.

[3] R. C. Bose and T. Shimamoto, "Classification and analysis of partially balanced incomplete block designs with two associate classes," *J. Amer. Stat. Assn.*, Vol. 47 (1952), pp. 151–184.

[4] R. H. Bruck and H. J. Ryser, "Nonexistence of certain finite projective planes," *Canadian Jour. Math.*, Vol. 1 (1949), pp. 88–93.

[5] W. S. Connor, "The uniqueness of the triangular association scheme," *Ann. Math. Stat.*, Vol. 29 (1958), pp. 262–266.
[6] W. S. Connor and W. H. Clatworthy, "Some theorems for partially balanced designs," *Ann. Math. Stat.*, Vol. 25 (1954), pp. 100–112.
[7] B. W. Jones, *The Arithmetic Theory of Quadratic Forms*, John Wiley and Sons, New York, 1950.
[8] D. M. Mesner, "An investigation of certain combinatorial properties of partially balanced incomplete block designs and association schemes, with a detailed study of designs of Latin square and related types," unpublished thesis, Michigan State University, 1956.
[9] M. N. Vartak, "On the Hasse-Minkowski invariant of the Kronecker product of matrices," *Canadian Jour. Math.*, Vol. 10 (1958), pp. 66–72.
[10] Manohar Narhar Vartak, "The non-existence of certain PBIB designs," to be published in *Ann. Math. Stat.*