ASYMPTOTIC EXPANSIONS IN GLOBAL CENTRAL LIMIT THEOREMS

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1. Introduction. Let $\xi_1, \xi_2, \cdots$ be independent random variables having the same d.f. (distribution function) $F(x)$. We suppose that

$$\int_{-\infty}^{\infty} x\, dF(x) = 0, \qquad \int_{-\infty}^{\infty} x^2\, dF(x) = 1 \tag{1.1}$$

so that $F(x)$ has mean 0 and standard deviation 1. Let $F_n(x)$ denote the d.f. of the normalized sum

$$(\xi_1 + \xi_2 + \cdots + \xi_n)/n^{\frac{1}{2}}. \tag{1.2}$$

A special case of the central limit theorem then asserts that, for each individual x in the interval $-\infty < x < \infty$,

$$\lim_{n\to\infty} F_n(x) = \Phi(x) \tag{1.21}$$

where $\Phi(x)$ is the Gaussian d.f. defined by

$$\Phi(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{x} e^{-u^2/2}\, du. \tag{1.22}$$

It is our purpose to study the behavior as $n \to \infty$ of the constants C_n defined by

$$C_n = \int_{-\infty}^{\infty} |F_n(x) - \Phi(x)|^2\, dx. \tag{1.3}$$

For each $p > 0$, let constants $C_n(p)$ be defined by

$$C_n(p) = \int_{-\infty}^{\infty} |F_n(x) - \Phi(x)|^p\, dx \tag{1.31}$$

when these integrals exist, that is, are finite. It is known from [1] and [2] that the hypotheses (1.1) imply that if $p > \frac{1}{2}$ then the constants $C_n(p)$ exist and $\lim_{n\to\infty} C_n(p) = 0$. Beyond this, not very much is known about the constants $C_n(p)$. The moments α_k and the absolute moments β_k of $F(x)$ are defined by

$$\alpha_k = \int_{-\infty}^{\infty} x^k\, dF(x), \qquad \beta_k = \int_{-\infty}^{\infty} |x|^k\, dF(x) \tag{1.4}$$

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when these integrals exist. If β_3 exists, then an inequality of Esseen ([5], p. 78) shows that there is a constant $K(\beta_3)$, depending only upon β_3, such that

$$|F_n(x) - \Phi(x)| \leqq \frac{K(\beta_3)}{n^{\frac{1}{2}}} \frac{\log(2 + |x|)}{1 + |x|^3}. \tag{1.5}$$

This implies that if β_3 exists, then the constants $C_n(p)$ exist when $p > \frac{1}{3}$ and $C_n(p) = O(n^{-p/2})$. In particular, if β_3 exists then $C_n = O(n^{-1})$. There is a sense in which this result cannot be improved because it is shown in [2] that if $F(x)$ is the symmetric binomial d.f. satisfying (1.1), then

$$C_n = \frac{1}{n}\frac{1}{6\pi^{\frac{1}{2}}} + O\left(\frac{1}{n^2}\right). \tag{1.6}$$

The only other case in which the constants C_n have been appraised is that for which $F(x)$ is the d.f. of a random variable ξ uniformly distributed over $-a \leqq x \leqq a$; in this case (1.1) implies that $a = 3^{\frac{1}{2}}$ and it is shown in [2] that

$$C_n = \frac{1}{n^2}\frac{3}{1280\pi^{\frac{1}{2}}} + O\left(\frac{1}{n^3}\right). \tag{1.61}$$

One of our main purposes is to give conditions under which there exist constants $D_1, D_2, D_3, \cdots$ such that the expansion

$$C_n = \frac{D_1}{n} + \frac{D_2}{n^2} + \cdots + \frac{D_k}{n^k} + O\left(\frac{1}{n^{k+1}}\right) \tag{1.7}$$

is valid for each $k = 1, 2, 3, \cdots$ and to give explicit expressions for D_1 and D_2. Such results are given at the ends of sections 4 and 6. Binomial distributions are treated in section 7, and the symmetric binomial d.f. is treated more extensively in section 9.

2. Formulas for the constants C_n. Information about the constants C_n is obtainable by use of the c.f. (characteristic function) $\phi(t)$ of $F(x)$ which is defined by

$$\phi(t) = \int_{-\infty}^{\infty} e^{itx}\, dF(x). \tag{2.01}$$

It is shown in [2] that

$$\int_{-\infty}^{\infty} |F_n(x) - \Phi(x)|^2\, dx = \frac{1}{n^{\frac{1}{2}}}\frac{1}{\pi}\int_0^{\infty} |[\phi(t)]^n - e^{-nt^2/2}|^2 \frac{dt}{t^2}. \tag{2.02}$$

Hence

$$C_n = \frac{1}{n^{\frac{1}{2}}}\frac{1}{\pi}\int_0^{\infty} |[\phi(t)]^n - e^{-nt^2/2}|^2 \frac{dt}{t^2}. \tag{2.1}$$

The hypotheses (1.1) imply that, at least when $k = 0, 1, 2$,

$$\phi^{(k)}(t) = \int_{-\infty}^{\infty} (ix)^k e^{itx}\, dF(x) \tag{2.2}$$

and hence that $\phi(0) = 1$, $\phi'(0) = 0$, and $\phi''(0) = -1$. This implies that we can choose a positive constant T such that

$$|\phi(t)| < 1 \qquad (0 < t \leqq T). \tag{2.21}$$

Let constants $\delta_1, \delta_2, \cdots$ be defined by

$$\delta_n = \log n)/n^{\frac{1}{3}} \qquad (n = 1, 2, 3, \cdots). \tag{2.22}$$

For values of n so great that $0 < \delta_n < T$, we split C_n into the sum of four terms by putting

$$C_n = C_n^{(1)} + C_n^{(2)} + C_n^{(3)} + C_n^{(4)} \tag{2.3}$$

where

$$C_n^{(1)} = \frac{1}{n^{\frac{1}{2}}}\frac{1}{\pi}\int_{\delta_n}^{\infty} \{e^{-nt^2/2} - [\phi(t)]^n - [\overline{\phi(t)}]^n\}\, e^{-nt^2/2}\frac{dt}{t^2}, \tag{2.4}$$

$$C_n^{(2)} = \frac{1}{n^{\frac{1}{2}}}\frac{1}{\pi}\int_{\delta_n}^{T} |\phi(t)|^{2n}\frac{dt}{t^2}, \tag{2.5}$$

$$C_n^{(3)} = \frac{1}{n^{\frac{1}{2}}}\frac{1}{\pi}\int_{0}^{\delta_n} |[\phi(t)]^n - e^{-nt^2/2}|^2\frac{dt}{t^2}, \tag{2.6}$$

$$C_n^{(4)} = \frac{1}{n^{\frac{1}{2}}}\frac{1}{\pi}\int_{T}^{\infty} |\phi(t)|^{2n}\frac{dt}{t^2}. \tag{2.7}$$

Estimation of $C_n^{(1)}$ and $C_n^{(2)}$ offers no difficulty; as we shall see,

$$C_n^{(1)} = o(n^{-\omega}), \qquad C_n^{(2)} = o(n^{-\omega}) \tag{2.81}$$

where $o(n^{-\omega})$ denotes a quantity which is $o(n^{-k})$ for each fixed positive constant k. Since $|e^{-nt^2/2}| \leqq 1$ and $|\phi(t)| \leqq 1$, it follows from (2.4) that

$$\begin{aligned} |C_n^{(1)}| &\leqq \frac{1}{n^{\frac{3}{2}}}\frac{3}{\pi}\int_{\delta_n}^{\infty} t^{-3}\, e^{-nt^2/2}(nt)\, dt \\ &< \frac{1}{n^{\frac{3}{2}}}\frac{3}{\pi}\frac{1}{\delta_n^3}\int_{\delta_n}^{\infty} e^{-nt^2/2}(nt)\, dt = \frac{1}{n^{\frac{3}{2}}}\frac{3}{\pi}\frac{1}{\delta_n^3}\, e^{-n\delta_n^2/2} = o(n^{-\omega}). \end{aligned} \tag{2.82}$$

To estimate $C_n^{(2)}$, let $\psi(t) = |\phi(t)|^2$. Then $\psi(0) = 1$, $\psi'(0) = 0$, and $\psi''(0) = -2$. This and the fact that $\psi(t) < 1$ when $0 < t \leqq T$ imply that, for each sufficiently great n,

$$\max_{\delta_n \leqq t \leqq T} |\psi(t)| < 1 - \delta_n^2/2. \tag{2.83}$$

Hence, when n is sufficiently great,

$$\begin{aligned} |C_n^{(2)}| &< \frac{1}{n^{\frac{1}{2}}}\frac{1}{\pi}\int_{\delta_n}^{T} (1 - \delta_n^2/2)^n \delta_n^{-2}\, dt \\ &< \frac{1}{n^{\frac{1}{2}}}\frac{T}{\pi\delta_n^2}\left(1 - \frac{\delta_n}{2}\right)^n = \frac{n^{\frac{1}{2}}T}{n(\log n)^2}\left[1 - \frac{(\log n)^2}{2n}\right]^n = o(n^{-\omega}). \end{aligned} \tag{2.84}$$

This proves (2.81).

The problem of estimating C_n is therefore reduced to the problem of estimating $C_n^{(3)}$ and $C_n^{(4)}$. Instead of (2.3), we henceforth use the formula

$$C_n = o(n^{-\omega}) + C_n^{(3)} + C_n^{(4)}. \tag{2.9}$$

3. The Constants $C_n^{(3)}$. In this section we appraise the constants $C_n^{(3)}$ in terms of the Thiele [7] semi-invariants γ_k of $F(x)$. We suppose that, for some integer m for which $m \geqq 3$, the moments α_m and β_m exist. Then, as $t \to 0$,

$$\phi(t) = 1 - \frac{t^2}{2} + \sum_{k=3}^{m} \frac{(it)^k}{k!} \alpha_k + o(t^m) \tag{3.1}$$

and using the ordinary expansion of $\log(1 + x)$ in powers of x gives

$$\log \phi(t) = -\frac{t^2}{2} + \sum_{k=3}^{m} \frac{(it)^k}{k!} \gamma_k + o(t^m) \tag{3.2}$$

where

$$\begin{aligned} &\gamma_3 = \alpha_3, \qquad \gamma_4 = \alpha_4 - 3, \qquad \gamma_5 = \alpha_5 - 10\alpha_3, \\ &\gamma_6 = \alpha_6 - 15\alpha_4 - 10\alpha_3^2 + 30, \cdots . \end{aligned} \tag{3.21}$$

The constants $\gamma_3, \gamma_4, \cdots$ are the Thiele semi-invariants of $F(x)$ which are treated in the books of Cramer [3], [4] and Gnedenko and Kolmogoroff [6] and which have simplified forms here because $\alpha_0 = 1$, $\alpha_1 = 0$, and $\alpha_2 = 1$. From (3.2) we obtain, for each fixed n,

$$[\phi(t)]^n = e^{-nt^2/2} e^w \tag{3.3}$$

where w is the function of n and t defined by

$$w = n\left[\sum_{k=3}^{m} \frac{(it)^k}{k!} \gamma_k + o(t^m)\right]. \tag{3.31}$$

From (2.6) and (3.3) we find that

$$C_n^{(3)} = \frac{1}{n^{\frac{1}{2}}} \frac{1}{\pi} \int_0^{\delta_n} e^{-nt^2} \, | e^w - 1 |^2 \frac{dt}{t^2}. \tag{3.4}$$

Supposing henceforth that $0 < t < \delta_n$, we see from (2.22) that

$$0 < nt^3 < (\log n)^3/n^{\frac{1}{2}}$$

and hence from (3.31) that $w = o(1)$ as $n \to \infty$. Therefore we can use the formula

$$e^w - 1 = \sum_{k=1}^{m} \frac{w^k}{k!} + O(w^{m+1}) \tag{3.41}$$

and (3.31) to obtain a formula giving $e^w - 1$ as the sum of a finite number of terms involving n, t, and $\gamma_3, \gamma_4, \cdots, \gamma_m$. When this finite sum is written down, it is found that

$$(3.42)\qquad \left[\frac{1}{t}\,|e^{w}-1|\right]^2 = |u+iv|^2 = u^2+v^2$$

where

$$(3.43)\qquad u = \frac{\gamma_4}{24}nt^3 - \frac{\gamma_3^2}{72}n^2t^5 + \cdots,$$

$$(3.44)\qquad v = -\frac{\gamma_3}{6}nt^2 + \frac{\gamma_5}{120}nt^4 - \frac{\gamma_3\gamma_4}{144}n^2t^6 + \frac{\gamma_3^3}{1296}n^3t^8 + \cdots.$$

In (3.43), (3.44), and formulas which follow, the final dots represent finite sums of terms which turn out to give contributions to $C_n^{(3)}$ which are of lower orders of magnitude than the contributions of the terms which precede the dots. From (3.42), (3.43), and (3.44) we obtain

$$(3.45)\qquad \begin{aligned}\left[\frac{1}{t}\,|e^{w}-1|\right]^2 = \frac{\gamma_3^2}{36}n^2t^4 + \frac{5\gamma_4^2-8\gamma_3\gamma_5}{2880}n^2t^6 \\ + \frac{\gamma_3^2\gamma_4}{864}n^3t^8 - \frac{\gamma_3^4}{15552}n^4t^{10} + \cdots.\end{aligned}$$

From (3.4) and (3.45) we see that $C_n^{(3)}$ is a linear combination, with coefficients depending upon $\gamma_3, \gamma_4, \cdots$, of integrals of the form

$$(3.5)\qquad J_n = \frac{1}{n^{\frac{1}{2}}}\int_0^{\delta_n} e^{-nk^2}n^pt^{2q}\,dt$$

where p and q are positive integers. Putting $t = n^{-\frac{1}{2}}u$ in (3.5) and using (2.22) gives

$$(3.51)\qquad J_n = \frac{1}{n^{1+q-p}}\int_0^{\log n} e^{-u^2}u^{2q}\,du.$$

But, when n is sufficiently great,

$$(3.52)\qquad \begin{aligned}\int_{\log n}^{\infty} e^{-u^2}u^{2q}\,du &= \int_{\log n}^{\infty}(u^{2q-1}e^{-u^2/2})e^{-u^2/2}u\,du \\ &< \int_{\log n}^{\infty} e^{-u^2/2}u\,du = e^{-(\log n)^2/2} = o(n^{-\omega}).\end{aligned}$$

Hence

$$(3.53)\qquad J_n = o(n^{-\omega}) + \frac{1}{n^{1+q-p}}\int_0^{\infty} e^{-u^2}u^{2q}\,du.$$

Using a standard formula for the integral in (3.53) gives

$$(3.54)\qquad J_n = o(n^{-\omega}) + \frac{\pi^{\frac{1}{2}}}{n^{1+q-p}}\,\frac{(2q)!}{q!2^{2q+1}}.$$

Use of (3.4), (3.45), (3.5), and (3.54) gives

$$C_n^{(3)} = \frac{A_1}{n} + \frac{A_2}{n^2} + \cdots + \frac{A_k}{n^k} + O\left(\frac{1}{n^{k+1}}\right) \tag{3.6}$$

where

$$A_1 = \frac{1}{\pi^{\frac{1}{2}}} \frac{\gamma_3^2}{96}, \tag{3.61}$$

$$A_2 = \frac{1}{\pi^{\frac{1}{2}}} \left[\frac{5\gamma_4^2}{3072} - \frac{\gamma_3\gamma_5}{384} + \frac{35\gamma_3^2\gamma_4}{9216} - \frac{35\gamma_3^4}{36864}\right], \tag{3.62}$$

and each of the constants A_1, A_2, A_3, $\cdots$ depends only upon a finite number of the semi-invariants γ_3, γ_4, $\cdots$. In case the given d.f. $F(x)$ has finite moments of all positive integer orders, the integer m of this section can be chosen as great as we wish and (3.6) is then valid for each $k = 1, 2, 3, \cdots$. The expressions for A_1 and A_2 given in (3.61) and (3.62) are particulary simple in the important case in which $F(x)$ is symmetric because in this case $\gamma_3 = 0$. The complexity of the expression for A_k increases very rapidly as k increases.

4. The constants $C_n^{(4)}$; case $\limsup \mid \phi(t) \mid < 1$. In this section, we suppose that $F(x)$ is a d.f. having a c.f. $\phi(t)$ for which

$$\limsup_{t\to\infty} \mid \phi(t) \mid < 1 \tag{4.1}$$

and show that in this case

$$C_n^{(4)} = o(n^{-\omega}) \tag{4.2}$$

where, as above, $o(n^{-\omega})$ denotes a quantity which is $o(n^{-k})$ for each positive constant k.

It is known ([3], p. 26) that if a c.f. $\phi(t)$ satisfies the hypothesis (4.1), then $\mid \phi(t) \mid < 1$ when $t > 0$. Since each c.f. is everywhere continuous, the hypothesis (4.1) therefore implies that if $T > 0$, then there is a constant θ such that $0 < \theta < 1$ and $\mid \phi(t) \mid \leqq \theta$ when $t \geqq T$. The definition (2.7) of $C_n^{(4)}$ therefore implies that

$$C_n^{(4)} \leqq \int_T^{\infty} \theta^{2n} t^{-2}\, dt = T^{-1}\theta^{2n} \tag{4.3}$$

and the desired conclusion (4.2) follows.

Thus in case $F(x)$ has a nonvanishing absolutely continuous component [3, page 17 and page 25] and in other cases where (4.1) holds, we have

$$C_n = o(n^{-\omega}) + C_n^{(3)} \tag{4.4}$$

and the results of section 3 suffice for the estimation of C_n. In particular, if (4.1) holds and $F(x)$ has finite moments of all positive integer orders, then (1.7) is valid when the constants D_1, D_2, $\cdots$ are the constants A_1, A_2, $\cdots$ in (3.6).

5. The uniform distribution. Let $F(x)$ be the d.f. of a random variable ξ uniformly distributed over $-a \leqq x \leqq a$ so that $F(x) = 0$ when $x \leqq a$, $F(x) = (x + a)/2a$ when $-a \leqq x \leqq a$, and $F(x) = 1$ when $x \geqq a$. This d.f. has mean 0, and we assume that $a = 3^{\frac{1}{2}}$ so that the standard deviation is 1. In this case (4.4) holds. The moments $\alpha_1, \alpha_2, \cdots$ defined by (1.4) are

$$\alpha_k = \int_{-a}^{a} (x^k/2a)\, dx \tag{5.1}$$

so that $\alpha_k = 0$ when k is odd and $\alpha_k = a^k/(k + 1)$ when k is even. Using (3.21) gives $\gamma_3 = 0$ and $\gamma_4 = -6/5$. Using (4.4) and (3.6) then gives the result (1.61).

6. The constants $C_n^{(4)}$; case $|\phi(t)|$ periodic. It is well known that if there is a positive value of t for which $|\phi(t)| = 1$, then $F(x)$ must be a lattice distribution and $|\phi(t)|$ must be periodic; and, conversely, if $F(x)$ is a lattice distribution, then $|\phi(t)|$ must be periodic. Throughout this section, we suppose that $F(x)$ is a d.f. for which (1.1) holds and $|\phi(t)|$ is periodic. Then $|\phi(t)|$ has a least positive period which we call $2T$.

To estimate $C_n^{(4)}$ we start with a method employed in [2] for the case in which $F(x)$ is the symmetric binomial d.f. From (2.7) we obtain

$$C_n^{(4)} = \frac{1}{n^{\frac{1}{2}}} \frac{1}{\pi} \sum_{k=1}^{\infty} \int_{2kT-T}^{2kT+T} |\phi(t)|^{2n} \frac{dt}{t^2}. \tag{6.1}$$

Since $|\phi(t)|$ has period $2T$, this implies that

$$C_n^{(4)} = \frac{1}{n^{\frac{1}{2}}} \frac{1}{\pi} \int_{-T}^{T} |\phi(t)|^{2n} S(t)\, dt \tag{6.2}$$

where

$$S(t) = \sum_{k=1}^{\infty} (2kT + t)^{-2}.$$

Since $|\phi(-t)| = |\phi(t)|$, this implies that

$$C_n^{(4)} = \frac{1}{n^{\frac{1}{2}}} \frac{1}{\pi} \int_{0}^{T} |\phi(t)|^{2n} S_1(t)\, dt \tag{6.21}$$

where $S_1(t) = S(t) + S(-t)$ and hence

$$S_1(t) = \sum_{k=1}^{\infty} [(2kT + t)^{-2} + (2kT - t)^{-2}]. \tag{6.22}$$

The function $S_1(t)$ is, as a function of a complex variable t, analytic except for simple poles at the points $\pm 2T, \pm 4T, \pm 6T, \cdots$ and, as we shall see, it is an elementary function. From (6.22) we obtain

$$S_1(t) = \frac{d}{dt} \sum_{k=1}^{\infty} [(2kT - t)^{-1} - (2kT + t)^{-1}] = \frac{d}{dt} \frac{t}{2T^2} \sum_{k=1}^{\infty} \frac{1}{k^2 - (t/2T)^2}.$$

Using the standard formula

$$\sum_{k=1}^{\infty} \frac{1}{k^2 - z^2} = \frac{1 - \pi z \cot \pi z}{2z^2}, \tag{6.31}$$

which is valid when z is not an integer, gives

$$S_1(t) = \frac{d}{dt}\left[\frac{1}{t} - \frac{\pi}{2T} \cot \frac{\pi t}{2T}\right]. \tag{6.32}$$

From this we obtain

$$S_1(t) = (\pi/2T)^2[\csc^2 x - x^{-2}] \tag{6.33}$$

where $x = \pi t/2T$. Differentiating the ordinary power series expansion of $(\cot x - x^{-1})$ gives a representation of the right side as a power series in x. Putting $x = \pi t/2T$ in this power series gives

$$S_1(t) = \left(\frac{\pi}{2T}\right)^2\left[\frac{1}{3} + \frac{1}{15}\left(\frac{\pi t}{2T}\right)^2 + \frac{2}{189}\left(\frac{\pi t}{2T}\right)^4 + \frac{1}{675}\left(\frac{\pi t}{2T}\right)^6 + \cdots\right]. \tag{6.34}$$

The numerical coefficients in (6.34) have simple expressions in terms of Bernoulli numbers, and the expansion is valid when $|t| < 2T$. Differentiating (6.22) gives

$$S_1'(t) = 4t \sum_{k=1}^{\infty} \frac{12k^2T^2 + t^2}{(4k^2T^2 - t^2)^3}. \tag{6.35}$$

This shows that $S_1'(t) > 0$ when $0 < t < T$ and hence that $S_1(t)$ is increasing when $0 \leqq t \leqq T$. With the aid of (6.34) and (6.33), we see that

$$\frac{1}{3}\left(\frac{\pi}{2T}\right)^2 = S_1(0) \leqq S_1(t) \leqq S_1(T) = \frac{\pi^2 - 4}{\pi^2}\left(\frac{\pi}{2T}\right)^2 \tag{6.36}$$

when $0 \leqq t \leqq T$. This shows that we could delete the factor $S_1(t)$ from the integrand in (6.21) without changing the order of magnitude of $C_n^{(4)}$.

We now improve the formula (6.21) by showing that

$$C_n^{(4)} = o(n^{-\omega}) + \frac{1}{n^{\frac{1}{2}}}\frac{1}{\pi}\int_0^{\delta_n} |\phi(t)|^{2n} S_1(t)\, dt \tag{6.4}$$

where $\delta_n = (\log n)/n^{\frac{1}{2}}$ as in (2.22). For this purpose, let

$$E_n = \frac{1}{n^{\frac{1}{2}}}\frac{1}{\pi}\int_{\delta_n}^{T} |\phi(t)|^{2n} S_1(t)\, dt. \tag{6.41}$$

Letting $\psi(t) = |\phi(t)|^2$ and using (6.36) gives, for some constant M,

$$E_n < M \int_{\delta_n}^{T} |\psi(t)|^n\, dt. \tag{6.42}$$

Since $\psi(t)$ is continuous over $0 \leqq t \leqq T$, $\psi(0) = 1$, $\psi'(0) = 0$, $\psi''(0) = -2$, and $0 \leqq \psi(t) < 1$ when $0 < t \leqq T$, it follows that, when n is sufficiently great,

$$\max_{\delta_n \leqq t \leqq T} |\psi(t)| = \psi(\delta_n) < 1 - \delta_n^2/2. \tag{6.43}$$

This and (6.42) show that

$$E_n < MT[1 - \delta_n^2/2]^n = o(n^{-\omega}). \tag{6.44}$$

Hence (6.21) and (6.41) imply (6.4) and (6.4) is proved.

Our estimate of $C_n^{(4)}$ will come from (6.4). As in Section 3 we suppose that, for some integer m for which $m \geqq 3$, the moments α_m and β_m exist. The formulas (3.3) and (3.31) are then valid and, supposing henceforth that $0 < t < \delta_n$, we can use the formula

$$e^w = 1 + \sum_{k=1}^{m} \frac{w^k}{k!} + O(w^{m+1}) \tag{6.5}$$

to obtain a formula giving e^w as the sum of a finite number of terms involving n, t, and $\gamma_3, \gamma_4, \cdots, \gamma_m$. When this finite sum is written down, it is found that

$$e^w = u + iv, \qquad |e^w|^2 = u^2 + v^2 \tag{6.51}$$

where

$$u = 1 + \frac{\gamma_4}{24} nt^4 - \frac{\gamma_3^2}{72} n^2 t^6 + \cdots, \tag{6.52}$$

$$v = -\frac{\gamma_3}{6} nt^3 + \frac{\gamma_5}{120} nt^5 - \frac{\gamma_3\gamma_4}{144} n^2 t^7 + \frac{\gamma_3^3}{1296} n^3 t^9 + \cdots, \tag{6.53}$$

and remarks analogous to those following (3.44) are applicable. From (6.4), (3.3), (6.51), and (6.34), we find that

$$C_n^{(4)} = o(n^{-\omega}) + \frac{1}{n^{\frac{1}{2}}} \frac{1}{\pi} \int_0^{\delta_n} e^{-nt^2} G_n(t)\, dt \tag{6.54}$$

where

$$G_n(t) = \left(\frac{\pi}{2T}\right)^2 \left[\frac{1}{3} + \frac{\pi^2}{60T^2} t^2 + \frac{\gamma_4}{36} nt^4 + \cdots\right]. \tag{6.55}$$

Using the values in (3.54) of the integrals in (3.5) then gives

$$C_n^{(4)} = \frac{B_1}{n} + \frac{B_2^2}{n^2} + \cdots + \frac{B_k}{n_k} + O\left(\frac{1}{n^{k+1}}\right) \tag{6.6}$$

where

$$B_1 = \frac{1}{6\pi^{\frac{1}{2}}} \left(\frac{\pi}{2T}\right)^2, \tag{6.61}$$

$$B_2 = \frac{1}{60\pi^{\frac{1}{2}}} \left(\frac{\pi}{2T}\right)^4 + \frac{\gamma_4}{96\pi^{\frac{1}{2}}} \left(\frac{\pi}{2T}\right)^2, \tag{6.62}$$

and each of the constants $B_1, B_2, B_3, \cdots$ depends only upon T and a finite number of the semi-invariants $\gamma_3, \gamma_4, \cdots$. Unlike the constants $A_1, A_2, \cdots$ in (3.6), the constant B_1 in (6.6) can never be zero. In case the given d.f $F(x)$ has finite moments of all positive integer orders, the integer m can be chosen as great as we wish and (6.6) is then valid for each $k = 1, 2, 3, \cdots$.

Our results show that if $F(x)$ has finite moments of all positive orders, and if the c.f. $\phi(t)$ is such that $|\phi(t)|$ is periodic, then (1.7) is valid when $D_k = A_k + B_k$ and the constants A_k and B_k are the constants in (3.6) and (6.6).

7. The global version of the De Moivre theorem on binomial distributions. Let $0 < p < 1$ and let $F(x)$ be the binomial d.f., associated with the probability p, which has mean 0 and standard deviation 1. To simplify our formulas, we define two constants h and β related to p and to each other by the formulas

$$h = [p/(1-p)]^{\frac{1}{2}}, \qquad p = h^2/(1+h^2), \tag{7.1}$$

$$\beta = (h + h^{-1})^{-1} = [p(1-p)]^{\frac{1}{2}}. \tag{7.11}$$

A random variable ξ governed by $F(x)$ has the value $-h^{-1}$ with probability p and the value h with probability $(1 - p)$. Hence $F(x) = 0$ when $x < h^{-1}$, $F(x) = p$ when $-h^{-1} \leqq x < h$, and $F(x) = 1$ when $x \geqq h$. A classic theorem of De Moivre states that, under these conditions, (1.21) holds. In case $p = \frac{1}{2}$ and $F(x)$ is symmetric, the constant C_n in (1.3) has been estimated in [2] and the result is given in (1.6). We now treat the general case and shall show that there exists constants $D_1, D_2, \cdots$, depending only upon p, such that

$$C_n = D_1 n^{-1} + D_2 n^{-2} + \cdots + D_k n^{-k} + O(n^{-k-1}) \tag{7.2}$$

for each $k = 1, 2, 3, \cdots$. Moreover

$$D_1 = \frac{1}{24\pi^{\frac{1}{2}}} \frac{1 + \left(\frac{1}{2} - p\right)^2}{p(1-p)}. \tag{7.21}$$

It is a straightforward but tedious task to extend our work to obtain explicit formulas for D_2 and D_3.

The definition of $F(x)$ implies that $F(x)$ has finite moments of all positive integer orders and hence that (3.6) is valid. From (2.01) and the definition of $F(x)$ we obtain

$$\phi(t) = pe^{-ih^{-1}t} + (1-p)e^{iht} = e^{-ih^{-1}t}[p + (1-p)e^{i\beta^{-1}t}]. \tag{7.3}$$

While $\phi(t)$ is not necessarily periodic, we see that

$$\begin{aligned} |\phi(t)| &= |[p + (1-p)\cos\beta^{-1}t] + i[(1-p)\sin\beta^{-1}t]|^{\frac{1}{2}} \\ &= [p^2 + (1-p)^2 + 2p(1-p)\cos\beta^{-1}t]^{\frac{1}{2}} \end{aligned} \tag{7.31}$$

and hence that $|\phi(t)|$ has least period $2\pi\beta$. Therefore (6.6) is valid with $2T = 2\pi\beta$ and hence $(\pi/2T) = (2\beta)^{-1}$. Using the notation of (3.6) and (6.6), we see that (7.2) is valid when $D_k = A_k + B_k$.

To obtain (7.21), we use the formula $D_1 = A_1 + B_1$ where A_1 and B_1 are given by (3.61) and (6.61). Since $\gamma_3 = \alpha_3$ and

$$\alpha_3 = (-h^{-1})^3 p + h^3(1 - p) = h - h^{-1} \tag{7.4}$$

$$\alpha_3^2 = (h^2 + h^{-2} - 2) = (1 - 2p)^2/p(1 - p) \tag{7.41}$$

we see from (3.61) that

$$A_1 = (24\pi^{\frac{1}{2}})^{-1}(\tfrac{1}{2} - p)^2/p(1 - p). \tag{7.42}$$

Since

$$(\pi/2T)^2 = 1/4\beta^2 = 1/4p(1 - p) \tag{7.5}$$

we see from (6.61) that

$$B_1 = (24\pi^{\frac{1}{2}})^{-1}/p(1 - p). \tag{7.51}$$

From (7.42) and (7.51) we obtain (7.21).

When $p = \frac{1}{2}$, the d.f. $F(x)$ of this section reduces to the symmetric binomial or Bernoulli d.f. which we shall treat further in Section 9. In this case $\phi(t) = \cos t$, $|\phi(t)|$ has period π, and $2T = \pi$. With the aid of (3.21) we obtain $\gamma_3 = 0$ and $\gamma_4 = -2$. Hence (2.9), (3.6), (3.61), (3.62), (6.6), (6.61), and (6.62) give

$$C_n = \frac{1}{6\pi^{\frac{1}{2}}}\frac{1}{n} + \frac{3}{1280\pi^{\frac{1}{2}}}\frac{1}{n^2} + O\left(\frac{1}{n^3}\right). \tag{7.6}$$

or

$$C_n = .04903\ 15973n^{-1} + .00132\ 23193n^{-2} + O(n^{-3}). \tag{7.61}$$

8. An inequality for C_n. Throughout this section we suppose that $F(x)$ is a d.f. having a finite third absolute moment β_3. It is known ([5], p. 201) that there is an absolute constant E_1 such that

$$\underset{-\infty<x<\infty}{\text{l.u.b.}} |F_n(x) - \Phi(x)| \leqq E_1\beta_3 n^{-\frac{1}{2}} \qquad (n = 1, 2, 3, \cdots). \tag{8.1}$$

In the left member of (8.1) we have the distance between $F_n(x)$ and $\Phi(x)$ in the space of bounded measurable functions defined over $-\infty < x < \infty$. It is not unreasonable to conjecture that, for some constant E_2, a valid companion inequality is obtained by replacing E_1 by E_2 and replacing the left member of (8.1) by the distance $C_n^{\frac{1}{2}}$ between $F_n(x)$ and $\Phi(x)$ in the Lebesgue space $L_2(-\infty, \infty)$. While $F_n(x)$ and $\Phi(x)$ themselves cannot belong to the space L_2, we know that $F_n(x) - \Phi(x)$ belongs to the space L_2 whenever $F(x)$ has finite second moments and hence whenever $F(x)$ has finite third moments. Thus we conjecture that there is an absolute constant E_2 such that

$$C_n^{\frac{1}{2}} \leqq E_2\beta_3 n^{-\frac{1}{2}} \qquad (n = 1, 2, 3, \cdots). \tag{8.2}$$

To eliminate the fractional exponents, we write the conjecture (8.2) in the form

(8.3) $$C_n \leqq E_2^2\beta_3^2 n^{-1}.$$

Evidence that the right member of (8.3) involves β_3 and n in the correct way is obtained by examining the manner in which C_n depends upon β_3 when $F(x)$ is the binomial d.f. of Section 7. From (7.2) we obtain $C_n \sim D_1 n^{-1}$ where D_1 is defined by (7.21). From the definition of $F(x)$ in section 7, we find that

(8.4) $$\beta_3 = [p^2 + (1 - p^2)][p(1 - p)]^{-\frac{1}{2}}.$$

Squaring (8.4) and using the result in (7.21) gives

(8.5) $$D_1 = (24\pi^{\frac{1}{2}})^{-1}Q(p)\beta_3^2$$

where

(8.51) $$Q(p) = [1 + (\tfrac{1}{2} - p)^2][p^2 + (1 - p)^2]^{-2}.$$

In the range $0 < p < 1$ where p must lie, we have $5/4 < Q(p) \leqq 4$. Moreover $Q(\frac{1}{2}) = 4$. Thus for the binomial d.f. of Section 7, we have

(8.6) $$C_n \sim (24\pi^{\frac{1}{2}})^{-1}Q(p)\beta_3^2 n^{-1}$$

where $5/4 < Q(p) \leqq 4$.

While the conjecture involving (8.3) and (8.2) remains unproved, the above estimates show that if (8.3) and (8.2) are universally valid, then

(8.7) $$E_2^2 \geqq (6\pi^{\frac{1}{2}})^{-1} = .09403\ 15973$$

and

(8.71) $$E_2 \geqq (6\pi^{\frac{1}{2}})^{-\frac{1}{2}} = .30664\ 57195$$

9. The symmetric binomial or Bernoulli d.f. Let $F(x)$ be the symmetric binomial or Bernoulli d.f., this being the d.f. of Section 7 with $p = \frac{1}{2}$. This d.f. is commonly associated with problems in coin tossing. It is the purpose of this section to obtain precise information about the constants C_n defined by (1.3). We shall focus our attention upon the formula

(9.01) $$C_n = \frac{D_1}{n} + \frac{D_2}{n^2} + \frac{D_3}{n^3} + \frac{D_4}{n^4} + \frac{D_5}{n^5} + \frac{R_n}{n^6}$$

where $D_1, D_2, \cdots$ are the constants in the asymptotic expansion of C_n and the numbers $R_1, R_2, \cdots$ are determined by the formula (9.01) itself. Of course the theory of asymptotic expansions assures us that the result of neglecting R_n in the right member of (9.01) gives a good approximation to C_n when n is sufficiently great, but until the matter has been investigated we do not know whether the approximation will be good when n is 5 or 10 or 100.

In the first place, the numerical values of $D_1, D_2, \cdots, D_5$ can be calculated by the methods of Sections 3 and 6. The details of the calculations are quite lengthy and tedious even when full advantage is taken of the fact that the c.f. $\phi(t)$ is now the real periodic function cos t. The right side of (3.2) can be replaced by the known power series expansion of log cos t which is obtained by integrating the expansion of tan t. With the notation of (3.3), w is real and $|e^w - 1|^2$ can be replaced by $e^{2w} - 2e^w + 1$. Since $|\cos t|$ has period π, the constant $2T$ of Section 6 is π. It is found that each of the constants $D_1, D_2, \cdots$ is a rational multiple of $\pi^{-\frac{1}{2}}$ and that

$$D_1 = \frac{1}{6}\pi^{-\frac{1}{2}} = .09403\ 15972\ 57959 \tag{9.02}$$

$$D_2 = \frac{3}{1280}\pi^{-\frac{1}{2}} = .00132\ 23193\ 36440 \tag{9.03}$$

$$D_3 = \frac{-397}{258048}\pi^{-\frac{1}{2}} = -.00086\ \ 79907\ \ 01995 \tag{9.04}$$

$$D_4 = \frac{53461}{353\ \ 89440}\pi^{-\frac{1}{2}} = .00085\ \ 22920\ \ 77129 \tag{9.05}$$

$$D_5 = \frac{23\ \ 24491}{20761\ \ 80480}\pi^{-\frac{1}{2}} = .00063\ \ 16664\ \ 76919 \tag{9.06}$$

Only one of these five constants is negative, and the author has very little information about D_n when $n > 5$.

In order to obtain information about the numbers $R_1, R_2, \cdots$ in (9.01) the values of $C_1, C_2, \cdots, C_{10}$ in (9.1) were calculated by the method which is explained later in this section.

$$\begin{array}{ll} C_1 = .10244\ 13576 & \Gamma_1 = .09506\ 98844 \\ C_2 = .04706\ 47193 & \Gamma_2 = .04729\ 68250 \\ C_3 = .03147\ 89023 & \Gamma_3 = .03147\ 05293 \\ C_4 = .02358\ 02730 & \Gamma_4 = .02358\ 07083 \\ C_5 = .01885\ 37826 & \Gamma_5 = .01885\ 37765 \\ C_6 = .01570\ 53613 & \Gamma_6 = .01570\ 53651 \\ C_7 = .01345\ 79250 & \Gamma_7 = .01345\ 79258 \\ C_8 = .01177\ 31393 & \Gamma_8 = .01177\ 31395 \\ C_9 = .01046\ 32283 & \Gamma_9 = .01046\ 32284 \\ C_{10} = .00941\ 56055 & \Gamma_{10} = .00941\ 56056 \end{array} \tag{9.1}$$

The exact value of C_1 is

$$C_1 = (2/\pi e)^{\frac{1}{2}} + \psi(1) - \pi^{-\frac{1}{2}} - \tfrac{1}{2} \tag{9.11}$$

where $\psi(x)$ is the tabulated Gaussian function defined by (9.29) below, and

$$C_1 = .10244\ 13576\ 27616 \tag{9.12}$$

with uncertainty only in the last figure which should perhaps be 7. For our next step, the ten-decimal approximations given in (9.1) are scarcely adequate. With the aid of the values of $C_1, \cdots, C_{10}$ which were rounded to obtain the values in (9.1), it is possible to calculate the numbers $R_1, R_2, \cdots, R_{10}$ in (9.01). It is found that the numbers R_5, R_6, R_7, R_8, R_9, and R_{10} differ relatively little from $-.0009$ and some heuristic considerations suggest very strongly that R_n differs from $-.0009$ by less than .00018 when $n \geqq 5$. This in turn suggests that the constant Γ_n defined by

$$\Gamma_n = \frac{D_1}{n} + \frac{D_2}{n^2} + \frac{D_3}{n^3} + \frac{D_4}{n^4} + \frac{D_5}{n^5} - \frac{.0009}{n^6} \tag{9.13}$$

must be a very good approximation to C_n at least when $n \geqq 5$, and that

$$| \Gamma_n - C_n | < .00018\, n^{-6} \qquad (n \geqq 5). \tag{9.14}$$

The values of $\Gamma_1, \Gamma_2, \cdots \Gamma_{10}$ calculated from (9.13) are given in (9.1), and it is easy to see how Γ_n compares with C_n when $1 \leqq n \leqq 8$. When n is 9 or 10, rounding errors obscure the relationships. After the above results were obtained, the values of Γ_{16} and C_{16} were obtained correct to 15 decimal places. The values are

$$\Gamma_{16} = .00588\ 19417\ 80443 \tag{9.15}$$

$$C_{16} = .00588\ 19417\ 81902, \tag{9.16}$$

and the agreement is neither better nor worse than was expected. It thus appears that C_n has an exceptionally useful asymptotic expansion and that, for example, use of (9.13) gives

$$C_{100} = .00094\ 04473\ 45108 \tag{9.17}$$

where the result is correct to the full 15 decimals. It would seem to be a formidable task to obtain even a crude approximation to C_{100} by direct computation of C_{100}.

We now proceed to obtain the formulas from which the numbers $C_1, C_2, \cdots, C_{10}$ in (9.1) were calculated. Let $H_n = nC_n$ so that $C_n = H_n/n$. Since $\phi(t) = \cos t$, we find from (2.1) that

$$H_n = n^{\frac{1}{2}} \frac{1}{\pi} \int_0^\infty | e^{-nt^2/2} - \cos^n t |^2\, t^{-2}\, dt. \tag{9.2}$$

According to R. J. Walker, it is not desirable to undertake to calculate H_1, $H_2, \cdots$ by direct application of a computing machine to the right member of (9.2); it is better to use the following way of expressing H_n as a finite sum of terms which are tabulated or easily calculated. From (9.2) we obtain

$$H_n = n^{\frac{1}{2}}\pi^{-1}[2R_n - P_n - Q_{2n}] \tag{9.21}$$

where

$$(9.22) \qquad P_n = \int_0^\infty \frac{1 - e^{-nt^2}}{t^2}\, dt, \qquad Q_n = \int_0^\infty \frac{1 - \cos^n t}{t^2}\, dt$$

$$(9.23) \qquad R_n = \int_0^\infty \frac{1 - e^{-nt^2/2}\cos^n t}{t^2}\, dt.$$

We shall show that, for each $n = 1, 2, \cdots$,

$$(9.24) \qquad P_n = (n\pi)^{\frac{1}{2}},$$

$$(9.25) \qquad Q_{2n-1} = Q_{2n} = \frac{n\pi}{2^{2n}}\binom{2n}{n} = \frac{\pi}{2}\frac{3\cdot 5\cdot 7 \cdots (2n-1)}{2\cdot 4\cdot 6 \cdots (2n-2)},$$

and

$$(9.26) \qquad R_n = S_n + T_n$$

where

$$(9.27) \qquad S_n = \frac{(2n\pi)^{\frac{1}{2}}}{2^{n+1}}\sum_{k=0}^{n}\binom{n}{k} e^{-(n-2k)^2/2n},$$

$$(9.28) \qquad T_n = \frac{\pi}{2^{n+1}}\sum_{k=0}^{n}\binom{n}{k} |n-2k|\, \psi\,(|n-2k|\, n^{-\frac{1}{2}}),$$

and $\psi(x)$ is the thoroughly tabulated Gaussian function

$$(9.29) \qquad \psi(x) = (2\pi)^{-\frac{1}{2}}\int_{-x}^{x} e^{-t^2/2}\, dt.$$

In (9.27) and (9.28), $\sum_{k=0}^{n}$ can be replaced by $2\sum_{k\leqq n/2}^{*}$ where the star on the $\sum$ signifies that when n is even the term for which $k = n/2$ is to be divided by 2. The numbers H_n are calculated from (9.21) with the aid of (9.24), (9.25), and (9.26). We shall omit these calculations and hence it remains only for us to establish (9.24), (9.25), and (9.26).

Starting with (9.22) and using standard integral formulas gives

$$(9.3) \quad P_n = \int_0^\infty dt \int_0^n e^{-xt^2}\, dx = \int_0^n dx \int_0^\infty e^{-xt^2}\, dt = \pi^{\frac{1}{2}}2^{-1}\int_0^n x^{-\frac{1}{2}}\, dx = (n\,\pi)^{\frac{1}{2}}$$

and (9.24) is established.

We now establish (9.25) by a method which exhibits material we shall use to establish the more complicated formula (9.26). Using the Euler formula for $\cos t$ and the binomial formula we obtain, when t is real,

$$(9.4) \qquad \begin{aligned}\cos^n t &= 2^{-n}(e^{it} + e^{-it})^n \\ &= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} e^{i(n-2k)t} = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos(n-2k)t\end{aligned}$$

and hence

$$(9.41) \qquad 1 - \cos^n t = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}[1 - \cos(n-2k)t].$$

Putting this in the second of the formulas (9.22) and using a standard integral formula gives

$$Q_n = \frac{\pi}{2^{n+1}} \sum_{k=0}^{n} \binom{n}{k} \mid n - 2k \mid . \tag{9.42}$$

While other proofs of (9.25) may be more elegant, we dismiss the matter with the remark that it is not difficult to use (9.42) to prove (9.25) by induction.

To establish (9.26) we suppose that n is a fixed positive integer, put

$$G(x) = \int_0^\infty [1 - e^{-xt^2/2} \cos^n t] t^{-2} \, dt, \tag{9.5}$$

and observe that $G(n) = R_n$ and $G(0) = Q_n$. Differentiating (9.5) gives

$$G'(x) = \frac{1}{2} \int_0^\infty e^{-xt^2/2} \cos^n t \, dt. \tag{9.51}$$

Use of (9.4) gives

$$G'(x) = \frac{1}{2^{n+1}} \sum_{k=0}^{n} \binom{n}{k} \int_0^\infty e^{-xt^2/2} \cos (n - 2k)t \, dt \tag{9.52}$$

and use of a standard integral formula then gives

$$G'(x) = \frac{(2\pi)^{\frac{1}{2}}}{2^{n+2}} \sum_{k=0}^{n} \binom{n}{k} x^{-\frac{1}{2}} e^{-(n-2k)^2/2x}. \tag{9.53}$$

Defining $I(m)$ by the formula

$$I(m) = \int_0^n x^{-\frac{1}{2}} e^{-m^2/2x} \, dx, \tag{9.54}$$

we use (9.53) and the fact that $G(n) = R_n$ and $G(0) = Q_n$ to obtain

$$R_n = Q_n + \frac{(2\pi)^{\frac{1}{2}}}{2^{n+2}} \sum_{k=0}^{n} \binom{n}{k} I(n - 2k). \tag{9.6}$$

Our next step is to obtain a better formula for $I(m)$. Suppose first that $m \neq 0$. A change of the variable of integration in (9.54) gives

$$I(m) = 2 \mid m \mid \int_{|m| n^{-\frac{1}{2}}}^\infty t^{-2} e^{-t^2/2} \, dt. \tag{9.61}$$

Using the well known formula

$$\int_a^\infty e^{-t^2/2} \, dt = a^{-1} e^{-a^2/2} - \int_a^\infty t^{-2} e^{-t^2/2} \, dt, \tag{9.62}$$

which is easily derived by intergration by parts, gives

$$I(m) = 2n^{\frac{1}{2}} e^{-m^2/2n} - 2 \mid m \mid \int_{|m| n^{-\frac{1}{2}}}^\infty e^{-t^2/2} \, dt. \tag{9.63}$$

In case m = 0, an easy evaluation of the right members of (9.54) and (9.63) shows that (9.63) is still valid. Substituting (9.63) in (9.6) and using (9.42) gives (9.26). This completes the derivations of the formulas used to obtain numerical values of $C_1, \cdots, C_{10}$ and C_{16}. The tables of the exponential and probability functions put out by the U.S. National Bureau of Standards were used.

While our work does not actually prove the result, it indicates very strongly that the sequence $C_1, C_2, C_3, \cdots$ converges monotonically to zero and hence that, in the mean square sense, each one of the distribution functions $F_1(x)$, $F_2(x)$, $F_3(x), \cdots$ is more nearly Gaussian than its predecessors. There was a time when the author rather expected that the sequence $H_1, H_2, H_3, \cdots$ defined by $H_n = nC_n$ would also be monotone, but it turns out that this is not so. In fact

$$\begin{aligned} H_1 &= .10244\ 136 \qquad & H_3 &= .09443\ 671 \\ H_2 &= .09412\ 944 \qquad & H_4 &= .09432\ 109 \end{aligned} \tag{9.7}$$

As a check upon the value of H_2 and upon the relative values of H_1, H_2, and H_4, the author, at that time, calculated H_1, H_2, and H_4 by a method completely independent of the calculations of this section and of the theories upon which they are based. By use of the distribution functions $F_1(x)$, $F_2(x)$, $F_4(x)$ and the formula (1.3) itself, the constants C_1, C_2, and C_4 were calculated by use of the Simpson parabolic formula for approximate evaluation of integrals. The resulting values of H_1, H_2, and H_4 were found to agree to 6 decimal places with the values in (9.7).

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