ON THE IDENTIFIABILITY PROBLEM FOR FUNCTIONS OF FINITE MARKOV CHAINS

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0. Summary. A stationary sequence $\{Y_n: n = 1, 2, \cdots\}$ of random variables with D values (states) is said to be a function of a finite Markov chain if there is an integer $N \geqq D$, an $N \times N$ irreducible aperiodic Markov matrix M, a stationary Markov chain $\{X_n\}$ with transition matrix M, and a function f such that $Y_n = f(X_n)$. For any finite sequence s of states of $\{Y_n\}$, let $p(s) = P\{(Y_1, \cdots, Y_n) = s\}$. For any state ϵ, let $s\epsilon t$ be the sequence s followed by ϵ followed by the sequence t. For every state ϵ, let $n(\epsilon)$ be the largest integer n such that there are finite sequences $s_1, \cdots, s_n, t_1, \cdots, t_n$ such that the matrix $\| p(s_i\epsilon t_j): 1 \leqq i, j \leqq n \|$ is nonsingular.

If $\{Y_n\}$ is a function of a finite Markov chain, then $\sum n(\epsilon) \leqq N$. There is a finite set $\{s_1, \cdots, s_N, t_1, \cdots, t_N\}$ of finite sequences such that $p(s)$ satisfies the recurrence relations

$$p(s\epsilon t) = \sum_{f(i)=\epsilon} a_i(s) p(s_i \epsilon t), \tag{1}$$

where $a_i(s)$ either is zero for all s or else is a ratio of determinants involving only $p(s\epsilon t_k)$ and $p(s_j \epsilon t_k)$ for $f(j) = f(k) = f(i)$.

If $\{Y_n\}$ has D states and is a function of a Markov chain having N states, then the entire distribution of $\{Y_n\}$ is determined by the distribution of sequences of length $\leqq 2(N - D + 1)$. For each N and D, a function of a Markov chain is exhibited which attains this bound.

If there is a Markov chain $\{X_n\}$ with $N = \sum n(\epsilon)$ states such that $\{Y_n\}$ is a function of $\{X_n\}$, then $\{Y_n\}$ is said to be a regular function of a Markov chain. If $\{Y_n\}$ is a regular function of a Markov chain having transition matrix M, then $M = X^{-1}AX$, where A is an $N \times N$ matrix with elements $a_{ij} = a_j(s_i f(i))$—defined by (1) above. $X = \| x_{ij} \|$ is a nonsingular $N \times N$ matrix such that $x_{ij} = 0$ unless $f(i) = f(j)$, the first row of each nonzero submatrix along the diagonal consists of positive numbers, and $\sum_j x_{ij} = p(s_i f(i))$. Any $N \times N$ Markov matrix giving the same distribution for $\{Y_n\}$ can be written in this form, with the same A and with an X having the above properties. Any matrix of this form which has all elements nonnegative is a Markov matrix giving the same distribution for $\{Y_n\}$. There are $\sum\{n(\epsilon)\}^2 - N$ "unidentifiable" parameters in the matrix X, and at most $N^2 - \sum\{n(\epsilon)\}^2$ "identifiable" parameters, determined by the distribution of $\{Y_n\}$, in the matrix A.

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1. Introduction. Suppose a process is known to be a stationary irreducible aperiodic Markov chain with a finite number of states (for definitions and properties of such chains, see [2]), but for some reason the states of the process cannot be directly observed. Suppose the states of the process are partitioned into groups, and that one can identify the group from which an observation came, but not which state in the group was observed. The observable process is again stationary, with a strictly positive stationary distribution, but it is not, in general, a Markov chain. A given Markov matrix, together with the function which partitions the states into groups, uniquely determines the distribution of the observable process. However, for a given function, there is in general more than one Markov chain which gives rise to the same observable process. For this reason, even if the entire distribution of the observable process is known, the matrix of transition probabilities for the original process cannot be uniquely determined. The general problem being considered here is the question: what characteristics of the observable process are needed in order to identify the class of Markov chains which could give rise to it?

Functions of a finite Markov chain were studied from a different point of view by Harris [3] under the name "grouped Markov chains." For the case of a Markov matrix having all elements positive, he obtained an expression for the conditional distributions, $P\{Y_n = \epsilon \mid Y_{n-1} = u_1, \cdots, Y_{n-k} = u_k\}$, of the observable process, in terms of a finite set of continuous distributions on [0, 1] whose generating functions are determined by the originating Markov matrix. He did not, however, study the identifiability problem. Blackwell and Koopmans [1] showed that for any function of a finite Markov chain, there is a finite integer J such that the entire distribution of the function process is determined by the distribution of observable sequences of length not exceeding J, and obtained, for a Markov chain with N states, an upper bound of $2N^2 + 1$ for J. They also considered, and "almost" solved, the identifiability problem for two special cases: (a) the N states are grouped $(1, N - 1)$, and (b) $N = 4$ and the grouping is $(2, 2)$. The methods used in this paper are extensions of the method used for (b) by Blackwell and Koopmans.

Before proceeding with the investigation, it is necessary to develop some notation which will be used throughout what follows. $\{Y_n : n = 1, 2, \cdots\}$ will always be a stationary (irreducible aperiodic) sequence of random variables with a finite number, D, of states, which we assume are the integers $0, 1, \cdots, D - 1$. All elements of the stationary distribution for $\{Y_n\}$ are assumed to be positive. States of $\{Y_n\}$ will be denoted by Greek letters ϵ and μ, with or without subscripts, and letters s and t, with or without subscripts, will stand for finite sequences of states of $\{Y_n\}$. The sequence "s followed by t" will be written "st." We will have occasion to refer also to the empty sequence, $\emptyset$, and $\emptyset s$ and $s\emptyset$ will both represent the sequence s.

$M = \| m_{ij} \|$ will be an $N \times N$ irreducible aperiodic Markov matrix, and $\{m_i : 1 \leqq i \leqq N\}$ will be the (unique) stationary distribution associated with M, all $m_i > 0$. $\{X_n : n = 1, 2, \cdots\}$ will be a stationary Markov chain with M as matrix of transition probabilities. Let f be a function on $\{1, 2, \cdots, N\}$ to

$\{0, 1, \cdots, D - 1\}$ and let N_ϵ = the number of states in $f^{-1}(\epsilon)$. Let $K_\epsilon = N_0 + \cdots + N_{\epsilon-1}(K_0 = 0)$. For notational convenience, we assume that f is nondecreasing; i.e., $f^{-1}(\epsilon) = \{K_\epsilon + 1, \cdots, K_\epsilon + N_\epsilon\}$.

For any finite sequence s, and any state i of $\{X_n\}$, define

$$p(s) = P\{(Y_1, \cdots, Y_n) = s\}, \tag{2}$$

$$q_i(s) = P\{(Y_1, \cdots, Y_n) = s, X_{n+1} = i\}, \tag{3}$$

$$r_i(s) = P\{(Y_2, \cdots, Y_{n+1}) = s \mid X_1 = i\}. \tag{4}$$

It will be useful also to define, for any i, $q_i(\emptyset) = m_i$, and $r_i(\emptyset) = 1$. Then for any s, t, ϵ, and μ (including s or $t = \emptyset$), it is evident that

$$p(s\epsilon t) = \sum_{f(i)=\epsilon} q_i(s) r_i(t) \tag{5}$$

and

$$p(s\epsilon\mu t) = \sum_{f(i)=\epsilon} \sum_{f(j)=\mu} q_i(s) m_{ij} r_j(t). \tag{6}$$

These two equations will be basic in all that follows, and we shall take advantage of the notational simplification possible by restating them in the form of matrix equations. For any set of sequences $s_1, s_2, \cdots, s_n$ and any ϵ, let $Q_\epsilon(s_1, \cdots, s_n)$ be the $n \times N_\epsilon$ matrix whose (i, j)th element is $q_{K_\epsilon+j}(s_i)$. Let $R_\epsilon(s_1, \cdots, s_n)$ be the $N_\epsilon \times n$ matrix whose (i, j)th element is $r_{K_\epsilon+i}(s_j)$. The function f induces a partition of M into submatrices $M_{\epsilon\mu}$, where the (i, j)th element of $M_{\epsilon\mu}$ is $m_{K_\epsilon+i, K_\mu+j}$. Finally, let $P_\epsilon(s_1, \cdots, s_n; t_1, \cdots, t_m)$ be the $n \times m$ matrix whose (i, j)th element is $p(s_i\epsilon t_j)$. Then (5) and (6) become

$$P_\epsilon(s_1, \cdots, s_n; t_1, \cdots, t_m) = Q_\epsilon(s_1, \cdots, s_n) R_\epsilon(t_1, \cdots, t_m) \tag{7}$$

and

$$\begin{aligned} P_\epsilon(s_1, \cdots, s_n; \mu t_1, \cdots, \mu t_m) &= Q_\epsilon(s_1, \cdots, s_n) M_{\epsilon\mu} R_\mu(t_1, \cdots, t_m) \\ &= P_\mu(s_1\epsilon, \cdots, s_n\epsilon; t_1, \cdots, t_m). \end{aligned} \tag{8}$$

A Markov chain is characterized by the property that the conditional probability of the sequence $s\epsilon\mu$, given $s\epsilon$, is independent of s. In terms of the functions $p(s)$, this is $p(s\epsilon\mu)/p(s\epsilon) = p(\epsilon\mu)/p(\epsilon)$. In fact, for any sequences s_1, t_1, s_2, t_2, we have

$$\begin{vmatrix} p(s_1 \epsilon t_1) p(s_1 \epsilon t_2) \\ p(s_2 \epsilon t_1) p(s_2 \epsilon t_2) \end{vmatrix} = 0.$$

In still other words, the largest square matrix of the form $\| p(s_i\epsilon t_j) \|$ which is nonsingular is one by one. It is this property which we shall generalize to functions of a Markov chain.

For each state ϵ of a stationary sequence $\{Y_n\}$ of random variables, let $n(\epsilon)$ be the largest integer n such that there are finite sequences $s_1, \cdots s_n, t_1, \cdots, t_n$

of states of $\{Y_n\}$ such that the matrix $\| p(s_i \epsilon t_j):1 \leqq i, j \leqq n \|$ is nonsingular. (If no such largest integer exists, let $n(\epsilon) = \infty$.)

LEMMA 1. *If $\{Y_n\}$ is a function of a finite Markov chain, then $\sum n(\epsilon) \leqq N$.*

PROOF. By Equation (7), for any set $\{s_i\,;\, t_i:1 \leqq i \leqq N_\epsilon + 1\}$ of finite sequences of states of $\{Y_n\}$,

$$
\begin{aligned}
&P_\epsilon(s_i\,;\, t_j:1 \leqq i, j \leqq N_\epsilon + 1) \\
&\qquad = Q_\epsilon(s_i:1 \leqq i \leqq N_\epsilon + 1)R_\epsilon(t_j:1 \leqq j \leqq N_\epsilon + 1),
\end{aligned} \tag{9}
$$

and each of the matrices on the right-hand side has rank at most N_ϵ; so the product is singular. Thus $n(\epsilon)$ cannot be larger than N_ϵ. Therefore

$$\sum n(\epsilon) \leqq \sum N_\epsilon = N.$$

It is an interesting conjecture that $\sum n(\epsilon) < \infty$ is a necessary and sufficient condition that a stationary (irreducible aperiodic) sequence be a function of a finite Markov chain. We shall later see evidence which seems to support this conjecture, but the writer has not been able to complete a proof (or disproof) of it.

2. Regular Functions. The set of all $N \times N$ irreducible aperiodic Markov matrices may be thought of as a subset of Euclidean $N(N - 1)$ dimensional space. For a given function f, the set for which $\sum n(\epsilon) < N$ is a set of dimension less than $N(N - 1)$, having Lebesgue measure zero in the set of all $N \times N$ Markov matrices. In this sense, the case where $\sum n(\epsilon) = N$ is the most important case to investigate. For this reason, and for others to be mentioned later, we shall say that $\{Y_n\}$ is a *regular* function of a Markov chain if there is a representation of $\{Y_n\}$ as a function of a finite Markov chain having N states, and $\sum n(\epsilon) = N$. In the remainder of this section, unless otherwise stated, we shall assume that $\{Y_n\}$ is a regular function of a Markov chain. Some of the results of this section are true also for the case $\sum n(\epsilon) < N$, and these will be pointed out in the next section.

Let $s_1, \cdots, s_N, t_1, \cdots, t_N$ be a set of sequences such that for each ϵ, $P_\epsilon(s_i\,;\, t_j:f(i) = f(j) = \epsilon)$ is nonsingular. Then for each ϵ, the rows of $Q_\epsilon(s_i:f(i) = \epsilon)$ form a basis for Euclidean N_ϵ-space. In order to obtain a basis which is associated with sequences of minimum length, it is convenient to order the set of finite sequences (including $\varnothing$, considered as a sequence of length zero) in such a way that s follows t if length $(s) >$ length (t), and (say) numerical order for sequences of the same length. Then we may start with $Q_\epsilon(\varnothing)$ and proceed to consider each sequence in order until we have found a basis. In this manner we obtain a set $\{s_i^*:1 \leqq i \leqq N\}$ with the property that for every s,

$$Q_\epsilon(s) = \sum_{f(i)=\epsilon} a_i(s)Q_\epsilon(s_j^*), \tag{10}$$

where $a_i(s) = 0$ if length $(s) <$ length (s_i^*), and no set $\{s_i'\}$ satisfying (10) has maximum length $(s_i') <$ maximum length (s_i^*).

A similar procedure will obtain a set $\{t_j^*:1 \leqq j \leqq N\}$ such that for every t,

$$R_\epsilon(t) = \sum_{f(j)=\epsilon} b_j(t) R_\epsilon(t_j^*), \tag{11}$$

with corresponding properties for the b_j's and t_j^*'s. We shall assume from now on that the sequences have been chosen in the first place so that $s_i = s_i^*$, $t_j = t_j^*$, and shall drop the asterisk in the notation.

LEMMA 2. *If $\{Y_n\}$ has D states and is a regular function of a Markov chain with N states, and if $\{s_i\}$ and $\{t_j\}$ are the sets of sequences such that $\| p(s_i \epsilon t_j) \|$ is non-singular, chosen as in* (10) *and* (11), *then maximum length $(s_i) \leqq N - D$ and maximum length $(t_j) \leqq N - D$.*

PROOF: Let m be any integer. Suppose for all s such that length $(s) \leqq m$, and all ϵ, that $Q_\epsilon(s) = \sum a_i(s) Q_\epsilon(s_i)$, where length $(s_i) < m$. Then for all s with length $(s) \leqq m$ and all μ,

$$Q_\mu(s\epsilon) = Q_\epsilon(s) M_{\epsilon\mu} = \sum a_i(s) Q_\epsilon(s_i) M_{\epsilon\mu} = \sum a_i(s) Q_\mu(s_i \epsilon).$$

Now length $(s_i\epsilon) \leqq m$; so $Q_\mu(s_i\epsilon) = \sum_k a_k(s_i\epsilon) Q_\mu(s_k)$, and therefore

$$Q_\mu(s\epsilon) = \sum \sum a_i(s) a_k(s_i\epsilon) Q_\mu(s_k) = \sum a_k(s\epsilon) Q_\mu(s_k).$$

That is, for all s such that length $(s) \leqq m + 1$, and all ϵ., $Q_\epsilon(s) = \sum a_i(s) Q_\epsilon(s_i)$, where length $(s_i) < m$. Then by induction we obtain that the result holds for all s and all ϵ, with the maximum length of the sequences s_i being less than m.

Therefore in the set $\{s_i : 1 \leqq i \leqq N\}$ there must be at least one sequence of each length up to the maximum length. For if any length were skipped, then so would be all following. Since there are $N - D$ sequences not $\varnothing$ in the set, the maximum length for s_i is not greater than $N - D$. A similar argument holds for sequences t_i.

THEOREM 1. *If $\{Y_n\}$ has D states and is a regular function of a Markov chain having $N = \sum n(\epsilon)$ states, then the entire distribution of $\{Y_n\}$ is determined by the set of functions $\{p(s)$: length $(s) \leqq 2(N - D + 1)\}$.*

PROOF. Multiplying both sides of equation (10) by $R_\epsilon(t)$, we obtain for every s, ϵ, and t,

$$p(s\epsilon t) = \sum_{f(i)=\epsilon} a_i(s) p(s_i \epsilon t). \tag{12}$$

Setting t successively equal to t_j for each j such that $f(j) = \epsilon$, we get a set of N_ϵ independent linear equations in the N_ϵ functions $a_i(s)$. By Cramer's rule, we can solve the system of equations, obtaining $a_i(s)$ as a ratio of determinants which involve only $p(s_j \epsilon t_k)$ and $p(s\epsilon t_k)$ for $f(j) = f(k) = f(i) = \epsilon$.

Let J = maximum length $(s_i f(i) f(j) t_j)$. Then (12) expresses $p(s)$ for all s of length greater than J in terms of the probabilities of sequences of length $<$ length (s), and by repeated use, in terms of the probabilities of sequences of length $\leqq J$. But by lemma 2, $J \leqq 2(N - D + 1)$. This completes the proof.

We may obtain the same result, and another recurrence relation, by multiplying both sides of (11) by $Q_\epsilon(s)$, to get

$$p(s\epsilon t) = \sum_{f(j)=\epsilon} b_j(t) p(s\epsilon t_j). \tag{13}$$

The calculations in Lemma 2 indicate that a process which achieves the upper bound might be obtained by choosing one sequence s_i of each length from 1 to $N - D$, and similarly for t_j. By choosing a function which is one to one on $D - 1$ states and which groups the remaining $N - D + 1$ states together, we obtain for each N and D a pair (f, M) which attains the upper bound.

Let $f(i) = 0$ if $1 \leqq i \leqq N - D + 1$

$f(i) = i - (N - D + 1)$ if $N - D + 2 \leqq i \leqq N$.

Let $0 < a_1 < a_2 < \cdots < a_{N-D+1} < 1$.

Let $m_{ij} = 0$ if $1 \leqq i, j \leqq N - D + 1$ and $i \neq j$,

$m_{ij} = a_i$ if $1 \leqq i, j \leqq N - D + 1$ and $i = j$,

$m_{ij} = (1 - a_j)/(D - 1)$ if $N - D + 1 < i \leqq N$ and $1 \leqq j \leqq N - D + 1$,

$m_{ij} = (1 - a_i)(D - 1)$ if $1 \leqq i \leqq N - D + 1$ and $N - D + 1 < j \leqq N$,

$$m_{ij} = \frac{2(D - 1) - N + a_1 + a_2 + \cdots + a_{N-D+1}}{(D - 1)^2} \text{ if } N - D + 1 < i, j \leqq N.$$

It is easily verified that $M = \| m_{ij} \|$ is a doubly stochastic Markov matrix. If $\{X_n\}$ is a stationary Markov chain having transition matrix M, then $\{f(X_n)\}$ is a function of a Markov chain for which we may choose $s_1 = t_1 = \varnothing$, $s_2 = t_2 = 0$, $s_3 = t_3 = 00, \cdots, s_{n-D+1} = t_{n-D+1} =$ a sequence of $N - D$ 0's, $s_{N-D} = t_{N-D} = \cdots = s_N = t_N = \varnothing$.

Then if $P_0 = P_0(s_1, \cdots, s_{N-D+1}, t_1, \cdots, t_{N-D+1})$, then

$$| P_0 | = N^{D-N-1}\Big[\prod_{1 \leqq i < j \leqq N-D+2} (a_i - a_j)\Big]^2.$$

Since all a_i are distinct, $| P_0 | \neq 0$, and therefore the distribution of $\{f(X_n)\}$ is not determined unless the probability of a sequence of $2(N - D + 1)$ 0's is known.

We shall see in Section 3 that Theorem 1 is true in general for a function of a finite Markov chain, with N replaced by $\sum n(\epsilon)$. At this point, then, we have a partial answer to our question. The class of Markov matrices which could generate the observable process is determined by the set of functions $p(s)$ for s having length $\leqq 2(N - D + 1)$. More precisely, following Blackwell and Koopmans [1], let us say that a finite set S of functions p_i defined on the set of $N \times N$ irreducible aperiodic Markov matrices is a complete set of invariants relative to a function f if and only if $p_i(M_1) = p_i(M_2)$ for all p_i in S when and only when M_1 and M_2 give the same distribution for $\{f(X_n)\}$. A complete set of invariants is said to be minimal if no proper subset is complete. Then the result of Theorem 1 (as extended in Section 3) is that the set of functions

$\{p(s):\text{length } (s) \leqq 2(N - D + 1)\}$ is a complete set of invariants relative to any function taking N states into D states. It is not a minimal complete set relative to any particular function, as some of the probabilities listed are determined by others.

However, if for some ϵ, $n(\epsilon) < N_\epsilon$, there may be a pair (f', M') such that M' is a $\sum n(\epsilon) \times \sum n(\epsilon)$ Markov matrix, and $\{Y_n\}$ has the same distribution as $\{f'(X_n')\}$. Then a complete set of invariants relative to f' could be found which contained fewer functions than any complete set relative to f. In this case, (f', M') would seem to be a more natural representation than (f, M). This furnishes a second reason for looking at regular functions of a Markov chain.

An example used in another connection by Blackwell and Koopmans furnishes a good illustration. Let

$$M = \begin{Vmatrix} \tfrac{1}{2} & x & \tfrac{1}{2} - x \\ \tfrac{1}{2} & y & \tfrac{1}{2} - y \\ \tfrac{1}{2} & z & \tfrac{1}{2} - z \end{Vmatrix}$$

and $f(1) = 0, f(2) = f(3) = 1$. $\{p(1), p(11), p(111), p(1111)\}$ is a minimal complete set of invariants relative to f. However, for the particular matrix M, $\{Y_n\}$ is a Markov chain, and its distribution is determined by $\{p(1), p(11)\}$, which is a minimal complete set relative to $f':f'(1) = 0, f'(2) = 1$, with

$$M' = \begin{Vmatrix} \tfrac{1}{2} & \tfrac{1}{2} \\ \tfrac{1}{2} & \tfrac{1}{2} \end{Vmatrix}$$

being the only 2×2 matrix which gives the proper distribution for $\{Y_n\}$. The parameters x, y, and z are all unidentifiable by observation of the process $\{Y_n\}$.

Next we shall obtain a parametric representation of the equivalence class of all $N \times N$ Markov matrices which give the same distribution for a given regular function of a Markov chain. In the process, we shall need to look more closely at individual sequences s_i and t_i and functions a_i than our present notation conveniently allows. So if z is any of these symbols, let $z_{\epsilon i} = z_{K_\epsilon + i}$; that is, $z_{\epsilon i}$ is the ith z associated with the state ϵ. Also, if $W = \| w_{ij} \|$ is any $N \times N$ matrix, let $W_{\epsilon\mu}$ be the $n(\epsilon) \times n(\mu)$ submatrix for which $f(i) = \epsilon, f(j) = \mu$.

Let A be the $N \times N$ matrix whose (i, j)th element is $a_j(s_i f(i))$, where $a_j(s)$ is the function defined by (12). Then as a consequence of (12), for every ϵ, μ, and t,

$$\begin{aligned} P_\mu(s_{\mu 1}, \cdots, s_{\mu n(\mu)}; \epsilon t) &= P_\epsilon(s_{\mu 1}\mu, \cdots, s_{\mu n(\mu)}\mu; t) \\ &= A_{\mu\epsilon} P_\epsilon(s_{\epsilon 1}, \cdots, s_{\epsilon n(\epsilon)}; t). \end{aligned} \tag{14}$$

By induction, then, if $t = \epsilon_1 \epsilon_2 \cdots \epsilon_n \epsilon$,

$$P_\mu(s_{\mu 1}, \cdots, s_{\mu n(\mu)}; t) = A_{\mu\epsilon_1} A_{\epsilon_1\epsilon_2} \cdots A_{\epsilon_n\epsilon} P_\epsilon(s_{\epsilon 1}, \cdots, s_{\epsilon n(\epsilon)}; \varnothing). \tag{15}$$

Using these facts, we may now prove

THEOREM 2. *Let $\{Y_n\}$ be a regular function of a Markov chain $\{X_n\}$ with $N \times N$ transition matrix M. Let A be the $N \times N$ matrix whose (i, j)th element is $a_j(s_i f(i))$.*

Then there is a nonsingular $N \times N$ *matrix* X *such that* (i) $x_{ij} = 0$ *unless* $f(i) = f(j)$, (ii) *the first row of each* $X_{\epsilon\epsilon}$ *consists of strictly positive numbers,* (iii) $\sum_j x_{ij} = p(s_i f(i))$, *and* $M = X^{-1}AX$. Any matrix equivalent to M can be written in this form with the same A, and for any nonsingular X satisfying (i), (ii), and (iii) which makes all elements of $X^{-1}AX$ nonnegative, $X^{-1}AX$ is a Markov matrix equivalent to M.

PROOF. Let P, Q, and R be the $N \times N$ matrices whose (i, j)th elements are respectively $p(s_i \epsilon t_j)$, $q_j(s_i)$, and $r_i(t_j)$ if $f(i) = f(j) = \epsilon$, and zero if $f(i) \neq f(j)$. Let C be the $N \times N$ matrix whose (i, j)th element is $p(s_i f(i) f(j) t_j)$. Then by (14) $C = AP$, and by (8), $C = QMR$. So $M = Q^{-1}APR^{-1} = Q^{-1}AQ$. Thus Q satisfies the requirements on X.

Any matrix M' equivalent to M defines a matrix Q' which satisfies the requirements in the same fashion. Since A is completely determined by the distribution of $\{Y_n\}$, it is not changed by substituting for M a matrix equivalent to M.

Now let X be a nonsingular $N \times N$ matrix satisfying (i), (ii) and (iii) such that all elements of $X^{-1}AX$ are nonnegative. Define $M = \| m_{ij} \| = X^{-1}AX$. We wish to show that M is a Markov matrix with a unique all-positive stationary distribution, and that (f, M) generates the process $\{Y_n\}$.

$$\sum_{j=1}^{N} m_{ij} = \sum_{\mu=0}^{D-1} \sum_{f(j)=\mu} m_{ij}\, r_j(\varnothing),$$

and by (iii),

$$\begin{aligned}\sum_{\mu} M_{\epsilon\mu} R_\mu(\varnothing) &= \sum_{\mu} X_{\epsilon\epsilon}^{-1} A_{\epsilon\mu} X_{\mu\mu} R_\mu(\varnothing) = X_{\epsilon\epsilon}^{-1} \sum_{\mu} A_{\epsilon\mu} P_\mu(s_{\mu 1}, \cdots, s_{\mu n(\mu)}; \varnothing) \\ &= X_{\epsilon\epsilon}^{-1} P_\epsilon(s_{\epsilon 1}, \cdots, s_{\epsilon n(\epsilon)}; \varnothing) = R_\epsilon(\varnothing).\end{aligned}$$

Therefore $\sum_j m_{ij} = 1$ for every i; so M is a Markov matrix.

We next show that the collection of positive numbers in the first row of each of the $X_{\epsilon\epsilon}$ form a stationary distribution for M. Let $Q'_\epsilon(\varnothing) = \{\text{first row of } X_{\epsilon\epsilon}\}$. Now

$$\begin{aligned} Q'_\epsilon(\varnothing) M_{\epsilon\mu} &= Q'_\epsilon(\varnothing) X_{\epsilon\epsilon}^{-1} A_{\epsilon\mu} X_{\mu\mu} \\ &= \{1, 0, \cdots, 0\} A_{\epsilon\mu} X_{\mu\mu} \\ &= \{a_{\mu 1}(\epsilon), \cdots, a_{\mu n(\mu)}(\epsilon)\} X_{\mu\mu}. \end{aligned}$$

And

$$\sum_{\epsilon} a_{\mu j}(\epsilon) = \begin{cases} 1 \text{ if } j = 1 \\ 0 \text{ if } j \neq 1. \end{cases}$$

Therefore

$$\begin{aligned} \sum_{\epsilon} Q'_\epsilon(\varnothing) M_{\epsilon\mu} &= \{1, 0, \cdots, 0\} X_{\mu\mu} \\ &= Q'_\mu(\varnothing). \end{aligned}$$

Now by (ii), every element of $Q'_\epsilon(\varnothing)$ is positive; so this is indeed the stationary distribution associated with M.

Finally, we show that if $\{X_n\}$ is a stationary Markov chain with transition matrix M, then $\{f(X_n)\}$ has the same distribution as $\{Y_n\}$. Let $s = \mu\epsilon_1\epsilon_2 \cdots \epsilon_n\epsilon$ be any finite sequence of integers from $\{0, 1, \cdots, D-1\}$. Then

$$\begin{aligned}
\text{Prob}\{(f(X_1), \cdots, f(X_{n+2})) = s\} &= Q'_\mu(\varnothing)M_{\mu\epsilon_1}M_{\epsilon_1\epsilon_2} \cdots M_{\epsilon_n\epsilon}R_\epsilon(\varnothing) \\
&= Q'_\mu(\varnothing)X^{-1}_{\mu\mu}A_{\mu\epsilon_1}A_{\epsilon_1\epsilon_2} \cdots A_{\epsilon_n\epsilon}X_{\epsilon\epsilon}R_\epsilon(\varnothing) \\
&= Q'_\mu(\varnothing)X^{-1}_{\mu\mu}A_{\mu\epsilon_1}A_{\epsilon_1\epsilon_2} \cdots A_{\epsilon_n\epsilon}P_\epsilon(s_{\epsilon 1}, \cdots, \\
&\qquad s_{\epsilon n(\epsilon)}; \varnothing) \\
&= \{1, 0, \cdots, 0\}P_\mu(s_{\mu 1}, \cdots, s_{\mu n(\mu)}; \\
&\qquad \epsilon_1\epsilon_2 \cdots \epsilon_n\epsilon) \\
&= p(\mu\epsilon_1\epsilon_2 \cdots \epsilon_n\epsilon).
\end{aligned}$$

This completes the proof.

The "identifiable" parameters of M are contained in the matrix A, and the free entries in X are "unidentifiable," since they may be changed without changing the distribution of $\{Y_n\}$. Since the only real restrictions on elements of $X_{\epsilon\epsilon}$ are that each row have the proper sum, there are $\sum n(\epsilon)\{n(\epsilon) - 1\} = \sum\{n(\epsilon)\}^2 - N$ "unidentifiable" parameters associated with M. In general, there are $N(N-1)$ free parameters in a Markov matrix; so there are in general $N^2 - \sum\{n(\epsilon)\}^2$ "identifiable" parameters associated with M.

Since the distribution of $\{Y_n\}$ determines and is determined by the matrix A, any representation of $\{Y_n\}$ by (f', M'), where M' is larger than M, would have the same number of "identifiable" parameters and would simply include more "unidentifiable" parameters. Also no representation (f'', M'') with M'' smaller than M is possible. So the representation $(f, X^{-1}AX)$ would seem to be a complete solution of the identifiability problem for regular functions of a Markov chain.

3. General Case and Unsolved Problems. If $\{Y_n\}$ is a function of a finite Markov chain and $\sum n(\epsilon) < N$, it is still possible that a representation can be found for $\{Y_n\}$ as a function of a Markov chain having $\sum n(\epsilon)$ states. In this case, all the results of Section 2 apply. The special case still remaining, that $\{Y_n\}$ is a function of a finite Markov chain, but no representation can be given as a function of a chain having $\sum n(\epsilon)$ states, may be empty. At this time it is still an open question whether or not every function of a finite Markov chain is a regular function of a Markov chain. However, even if the case is not empty, a modified version of Theorem 1 still holds. The computations in proof of Theorem 2 prove the following:

LEMMA 3. *If* $\{Y_n\}$ *is a stationary sequence of random variables with values* $0, 1, \cdots, D-1$, *with* $p(\epsilon) > 0$ *for each* ϵ, *and with* $\sum n(\epsilon) = N' < \infty$, *and if f' is the function carrying the first $n(0)$ integers into 0, next $n(1)$ integers into 1, etc., then there is an $N' \times N'$ matrix M' of real numbers, and a set $\{m_i' : 1 \leqq i \leqq N'\}$ of positive real numbers such that*

(i) $\sum_j m_{ij}' = \sum_j m_j' = 1$,

(ii) $\sum_i m_i' m_{ij}' = m_j'$, *and*

(iii) *for any sequence* $s = \epsilon_1 \epsilon_2 \cdots \epsilon_n$,

$$p(s) = \sum \{m_{i_1}' m_{i_1 i_2}' \cdots m_{i_{n-1} i_n}' : f'(i_k) = \epsilon_k, 1 \leqq k \leqq n\}.$$

The fact that $p(s)$ satisfies a recurrence relation of the type (1) follows from the fact that for every ϵ, some determinant $| P_\epsilon(s_{\epsilon 1}, \cdots, s_{\epsilon n(\epsilon)}; t_{\epsilon 1}, \cdots, t_{\epsilon n(\epsilon)})|$ is nonzero, while for every s and t, $|P_\epsilon(s, s_{\epsilon 1}, \cdots, s_{\epsilon n(\epsilon)}; t, t_{\epsilon 1}, \cdots, t_{\epsilon n(\epsilon)})| = 0$. Then the matrix A defined by this recurrence relation, together with any X satisfying (i), (ii), and (iii) of Theorem 2 generates a matrix M'. The question of whether or not there is in this class of "pseudo-Markov" matrices one matrix with all elements nonnegative is the question of whether or not $\sum n(\epsilon) < \infty$ characterizes a regular function of a Markov chain. If $n(0) = 2, n(\epsilon) = 1$ for $0 < \epsilon < D$, then it can be shown that there is indeed such a nonnegative matrix. However, the writer has not yet been able to extend this result to the general case.

But for any M' having the properties of Lemma 3, we may define recursively, with $r_i'(\emptyset) = 1$, $q_i'(\emptyset) = m_i'$,

$$r_i'(\epsilon t) = \sum \{m_{ij}' r_j'(t) : f'(j) = \epsilon\},$$

$$q_i'(s\epsilon) = \sum \{q_j'(s) m_{ji}' : f'(j) = \epsilon\},$$

and note that for every s, ϵ, and t,

$$p(s\epsilon t) = \sum_{f'(i) = \epsilon} q_i'(s) r_i'(t).$$

All of the computations carried out in Lemma 2 and Theorem 1 go through for the functions q' and r', and Theorem 1 remains true with f replaced by f' and N by $\sum n(\epsilon)$.

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