ON THE LAWS OF CAUCHY AND GAUSS

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1. Introduction and summary. Let x and y be two independent normal variates each distributed with zero mean and a common variance; it is then well-known that the quotient x/y follows the Cauchy law distributed symmetrically about the origin. Now the question that naturally arises is whether we can obtain a characterization of the normal distribution by this property of the quotient. This converse problem can be more precisely formulated as follows:

Let x and y be two independently and identically distributed random variables having a common distribution function $F(x)$. Let the quotient $w = x/y$ follow the Cauchy law distributed symmetrically about the origin $w = 0$. Then the question is whether $F(x)$ is normal.

But this converse is not true in general. The author [1] has recently constructed a very simple example of a non-normal distribution where the quotient x/y follows the Cauchy law. Steck [7] has also given some examples of non-normal distributions with this property of the quotient.[2]

In the present paper we shall first derive some interesting general properties possessed by the class of distribution laws $F(x)$ [Section 2]. In Section 3 we deduce a characterization of the normal distribution under some conditions on the distribution function $F(x)$. Finally in Section 4 we construct an example of a non-normal distribution function $F(x)$ having finite moments of all orders where the quotient x/y follows the Cauchy law. The method of proof is essentially based on the applications of Fourier transforms of distribution functions. For the proof of Theorem 3.1 we require somewhat deeper results in the theory of analytic functions.

2. Some general properties of $F(x)$. We shall here discuss some general properties of the class of distribution laws $F(x)$. We first prove a lemma which is instrumental in the proofs of the subsequent results.

Lemma 2.1. *Let x and y be two independently and identically distributed proper random variables having a common distribution function $F(x)$ which is continuous at the origin $x = 0$. Let the quotient $w = x/y$ have a distribution function $G(w)$ symmetric about the origin. Then $F(x)$ is also symmetric about the origin.*

Proof. As usual we assume that each of the distribution functions $F(x)$ and

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[2] *Note added in proof*: While this paper was in press, the author learned that some examples of non-normal distributions have also been constructed by J. G. Mauldon (*Quart. J. Math., Oxford (2)*, Vol. 7 (1956), pp. 155–160).

$G(w)$ is everywhere continuous to the right. Then we have the following notations:

$$F(a) = \text{Prob } (x \leqq a) \qquad G(a) = \text{Prob } (w \leqq a)$$
$$F(a - 0) = \text{Prob } (x < a) \qquad G(a - 0) = \text{Prob } (w < a).$$

We also note that the origin $x = 0$ must be a continuity point of $F(x)$, as otherwise the quotient w assumes the indeterminate value 0/0 with a positive probability. Now for $w > 0$ we have

$$\begin{aligned} G(w) - G(0) &= \text{Prob}\left[0 < \frac{x}{y} \leqq w\right] \\ &= \text{Prob } [0 < x \leqq wy; y > 0] \\ &\qquad + \text{Prob } [wy \leqq x < 0; y < 0] \\ &= \int_0^\infty [F(wy) - F(0)]\, dF(y) \\ &\qquad + \int_{-\infty}^0 [F(0) - F(wy - 0)]\, dF(y) \\ &= \int_0^\infty [F(wy) - F(0)]\, dF(y) \\ &\qquad + \int_0^\infty [F(-wy - 0) - F(0)]\, dF(-y - 0). \end{aligned} \tag{2.1}$$

Similarly we can show that for any $w > 0$

$$\begin{aligned} G(0) - G(-w - 0) &= \text{Prob}\left[-w \leqq \frac{x}{y} < 0\right] \\ &= \int_0^\infty [F(0) - F(-wy - 0)]\, dF(y) \\ &\qquad + \int_0^\infty [F(0) - F(wy)]\, dF(-y - 0). \end{aligned} \tag{2.2}$$

Since $G(w)$ is symmetric about the origin $w = 0$, we have the relation

$$G(w) - G(0) = G(0) - G(-w - 0) \tag{2.3}$$

holding for all w.

Then using (2.1) and (2.2) together, we get from (2.3) the relation

$$\begin{aligned} &\int_0^\infty [F(wy) + F(-wy - 0) - 2F(0)]\, dF(y) \\ &\quad + \int_0^\infty [F(wy) + F(-wy - 0) - 2F(0)]\, dF(-y - 0) = 0 \end{aligned} \tag{2.4}$$

holding for all $w > 0$.

Substituting

$$H(wy) = F(wy) + F(-wy - 0) - 2F(0)$$

in (2.4), we obtain

$$\int_0^\infty H(wy)\, dH(y) = 0 \tag{2.5}$$

holding for all $w > 0$. Here $H(y)$ is a function of bounded variation. We now use the transformation $w = e^u$ and $y = e^v(-\infty \leqq u \leqq \infty; -\infty \leqq v \leqq \infty)$ and denote

$$H(y) = H(e^v) = H_1(v) \quad \text{and} \quad H(wy) = H(e^{u+v}) = H_1(u + v).$$

Here we note that $H_1(v)$ is also a function of bounded variation. Thus (2.5) reduces to

$$\int_{-\infty}^\infty H_1(u + v)\, dH_1(v) = 0 \tag{2.6}$$

holding for all $u(-\infty \leqq u \leqq +\infty)$. From (2.6) we see easily that the relation

$$\int_{-\infty}^\infty e^{iut}\, d\left[\int_{-\infty}^\infty H_1(u + v)\, dH_1(v)\right] = 0 \tag{2.7}$$

holds identically for all real t. Let

$$\psi(t) = \int_{-\infty}^\infty e^{ivt}\, dH_1(v) \tag{2.8}$$

denote the Fourier transform of $H_1(v)$ which is a function of bounded variation. Then using the theorem of Fourier transforms of convolutions of functions of bounded variation we get from (2.7)

$$\psi(t)\psi(-t) = |\psi(t)|^2 = 0;$$

that is,

$$|\psi(t)| = 0 \tag{2.9}$$

holding identically for all real t, where $\psi(t)$ is defined in (2.8). Finally from the uniqueness property of Fourier transforms of functions of bounded variation, it follows immediately from (2.9) that $H_1(v)$ is a constant almost everywhere. Hence

$$H(y) = F(y) + F(-y - 0) - 2F(0) = c, \quad \text{a.e.} \tag{2.10}$$

Next substituting $y = 0$ in (2.10) and noting that the origin $y = 0$ is a continuity point of $F(y)$, we get $c = 0$ and thus (2.10) reduces to

$$F(y) + F(-y - 0) = 2F(0). \tag{2.11}$$

Finally we note $F(-\infty) = 0$ and $F(+\infty) = 1$ and obtain from (2.11) that $F(0) = \frac{1}{2}$. Thus we have

$$F(y) = 1 - F(-y - 0), \tag{2.12}$$

which completes the proof.

Lemma 2.2. *Let x and y be two independently and identically distributed random variables having a common distribution function $F(x)$. Let the quotient $w = x/y$ follow the Cauchy law distributed symmetrically about the origin $w = 0$. Then $F(x)$ is absolutely continuous and has a continuous probability density function $f(x) = F'(x) > 0$.*

Proof. As a direct consequence of Lemma 2.1 it follows that $F(x)$ is also symmetric about the origin $x = 0$. Let $F_0(x)$ denote the distribution function of $|x|$. Then we can verify easily that

$$F_0(x) = \begin{cases} 0 & x < 0 \\ F(x) - F(-x - 0) = 2F(x) - 1 & \text{for } x \geqq 0. \end{cases} \tag{2.13}$$

Thus we note that in this case the distribution functions of x and w are uniquely determined by the distribution functions of $|x|$ and $|w|$ respectively. We can easily verify after elementary integration that the characteristic function of the distribution of $\ln |w|$ is given by

$$E(e^{it\ln|w|}) = \frac{1}{\cosh\left(\frac{\pi}{2}t\right)}.$$

Then noting that $\ln|w| = \ln|x| - \ln|y|$ we get finally the relation

$$\varphi(t)\varphi(-t) = \frac{1}{\cosh\left(\frac{\pi}{2}t\right)} \tag{2.14}$$

holding for all real t, where $\varphi(t)$ denotes the characteristic function of the distribution of $\ln|x|$. The relation (2.14) has also been derived independently by Steck [7]. From (2.14) we get at once

$$|\varphi(t)| = \frac{1}{\left[\cosh\left(\frac{\pi}{2}t\right)\right]^{1/2}} \tag{2.15}$$

and then verify easily that $\int_{-\infty}^{\infty} |\varphi(t)|\, dt < \infty$; that is, the characteristic function $\varphi(t)$ is absolutely integrable. Then using the well-known theorem ([2], p. 188), we deduce easily that the distribution function of $\ln|x|$ is absolutely continuous and has a continuous probability density function. Thus it follows as an immediate consequence that $|x|$ has an absolutely continuous distribution function. Finally from the relation (2.13) we see easily that $F(x)$ is also absolutely continuous and has a continuous probability density function.

We are now in a position to prove the following theorem.

THEOREM 2.1. *Let x and y be two independently and identically distributed random variables having a common distribution function $F(x)$. Let the quotient $w = x/y$ follow the Cauchy law distributed symmetrically about the origin $w = 0$. Then $F(x)$ has the following general properties:*

(1) *it is symmetric about the origin $x = 0$;*

(2) *it is absolutely continuous and has a continuous probability density function $f(x) = F'(x) > 0$;*

(3) *the random variable x has an unbounded range;*

(4) *the probability density function $f(x)$ satisfies the integral equation*

$$\int_0^\infty f(x)f(ux)x\,dx = \frac{c_0}{1+u^2} \tag{2.16}$$

holding for all u, where c_0 is a constant.

PROOF. The properties (1) and (2) follow as direct consequences of Lemmas 2.1 and 2.2. For the proof of property (3) we proceed as follows:

Let us suppose that the random variable x has a bounded range, that is, $F(x)$ is contained in a finite interval $(-a, +a)$ of the x-axis. We introduce the polar transformation $x = r\cos\theta$ and $y = r\sin\theta$ and deduce easily that the joint probability density function of r and θ has the form

$$rf(r\cos\theta)f(r\sin\theta). \tag{2.17}$$

We now integrate (2.17) with respect to r and obtain the probability density function of θ as:

$$\begin{aligned} f_1(\theta) &= \int_0^{a/\cos\theta} f(r\cos\theta)f(r\sin\theta)r\,dr \qquad \text{for} \quad 0 \leqq \theta \leqq \pi/4 \\ f_2(\theta) &= \int_0^{a/\sin\theta} f(r\cos\theta)f(r\sin\theta)r\,dr \qquad \text{for} \quad \pi/4 \leqq \theta \leqq \pi/2 \end{aligned} \tag{2.18}$$

Finally substituting $\cot\theta = x/y$ we get at once from (2.18) that if the random variable x has a bounded range $(-a, +a)$ the form of the probability density function of $w = x/y$ in the range $(0 \leqq w \leqq 1)$ is different from that in the range $(1 \leqq w \leqq \infty)$. The contradiction thus obtained leads to the proof of (3).

For the proof of (4) we introduce as usual the polar transformation $x = r\cos\theta$ and $y = r\sin\theta$ and integrate (2.17) with respect to r over the range $(0, \infty)$. We further note that $\theta = \text{arc cot } x/y$ has a uniform distribution. Thus the equation for the probability density function of θ is given by

$$\int_0^\infty f(r\cos\theta)f(r\sin\theta)r\,dr = c_0 \tag{2.19}$$

where c_0 is a constant. Then substituting $x = r\cos\theta$ and $u = \tan\theta$ in (2.19) we get (2.16). Thus the problem of determining the entire class of distribution laws $F(x)$ is equivalent to that of complete enumeration of the solutions of the integral equation (2.16). This problem is very difficult and still remains to be solved.

3. A characterization of the normal law. We shall now derive a characterization of the normal distribution under some additional conditions on the distribution function $F(x)$. For this purpose, we give first some analytical lemmas which are also of independent interest.

LEMMA 3.1. *Let $\Phi(z)$ be a decomposable characteristic function which is regular (analytic) in a strip $-\alpha < \text{Im } z < +\alpha$ ($\alpha > 0$) of the complex z-plane. Let $\Phi_1(z)$ be a factor of $\Phi(z)$. Then the characteristic function $\Phi_1(z)$ is also regular at least in the strip $-\alpha < \text{Im } z < +\alpha$.*

This lemma on the factorization of analytic characteristic functions is due to Raikov [5]. A proof of this lemma is presented by Loève ([2], p. 213).

LEMMA 3.2. *Let $\Phi(z)$ be a decomposable regular (analytic) characteristic function and $\Phi_1(z)$ a factor of $\Phi(z)$. Let $\Phi(-iv)$ exist for some v ($v \neq 0$ real). Then for this v, $\Phi_1(-iv)$ must also exist. Further, there always exist two finite real numbers $K > 0$ and $a \geqq 0$ not depending on v such that the inequality*

$$\Phi_1(-iv) \leqq Ke^{a|v|}\Phi(-iv) \tag{3.1}$$

is satisfied.

This lemma is also due to Raikov [5]. A proof of this lemma is presented by Loève ([2], p. 214).

LEMMA 3.3. *Under the same conditions as in Lemma 3.2, let $z = t + iv$ (t and v both real). Then we have the inequality*

$$|\Phi_1(-z)| \leqq Ke^{a|v|}\Phi(-iv). \tag{3.2}$$

The proof follows at once from (3.1) and the well-known property of the positive definite functions

$$\max_{-\infty \leqq t \leqq +\infty} |\Phi_1(t + iv)| \leqq \Phi_1(iv), \qquad (t \text{ and } v \text{ both real}).$$

LEMMA 3.4. *Let $f(x)$ be a continuous non-negative function of the real variable x. Let the integral $\int_0^\infty x^v f(x)\,dx$ exist for all real $v > 0$. Then the integral*

$$I(z) = \int_0^\infty x^{-iz} f(x)\,dx$$

as a function of the complex variable z is regular (analytic) in the upper half plane $\text{Im } z > 0$. Conversely if the function $I(z)$ is regular in the upper half plane $\text{Im } z > 0$, then the integral $\int_0^\infty x^v f(x)\,dx$ exists for all real $v > 0$.

PROOF. We first note that $I(z)$ is uniformly convergent in every closed domain of the half plane $\text{Im } z > 0$. Then using the well known theorems on regular functions ([6], pp. 107, 116) we derive that $I(z)$ is regular in the half plane $\text{Im } z > 0$. The proof of the converse statement is obvious.

From Lemma 3.4, it is also easy to see that if the integral $\int_0^\infty x^v f(x)\,dx$ exists for all $v > 0$, then the integral $\int_0^\infty x^{iz} f(x)\,dx$, ($z$ complex) is regular in the lower half plane $\text{Im } z < 0$.

LEMMA 3.5. *Under the same conditions as in Theorem 2.1, let the distribution law*

$F(x)$ have finite moments of all orders. Let $\varphi(t) = E(e^{it\ln|x|})$ denote the characteristic function of the distribution of $\ln |x|$. Then $\varphi(z) = E(e^{iz\ln|x|})$ as a function of the complex variable z is regular in the region $\operatorname{Im} z < 1$.

PROOF. Since $F(x)$ has finite moments of all orders, the integral $\int_0^\infty x^v f(x)\,dx$ is convergent for all $v > 0$, where $f(x)$ is the probability density function. We further note that $f(x)$ is symmetric about the origin $x = 0$. Then applying Lemma 3.4, we get easily that

$$\varphi(z) = E(e^{iz\ln|x|}) = 2\int_0^\infty x^{iz} f(x)\,dx \tag{3.3}$$

is regular at least in the lower half plane $\operatorname{Im} z < 0$.

Next we note that the characteristic function $1/\cosh[(\pi/2)t]$ can be continued in the complex z-plane since $1/\cosh[(\pi/2)z]$ is regular in the strip $|\operatorname{Im} z| < 1$. Then applying Lemma 3.1 to the relation (2.14), we deduce at once that $\varphi(t)$ can also be continued in the complex z-plane and further $\varphi(z)$ is also regular at least in the strip $|\operatorname{Im} z| < 1$. Thus combining the two results we conclude that $\varphi(z)$ is regular in the region $\operatorname{Im} z < 1$. Similarly we see that $\varphi(-z)$ is regular in the region $\operatorname{Im} z > -1$.

We are now in a position to prove the following theorem.

THEOREM 3.1. *In addition to the conditions of Theorem 2.1, if the following two conditions are satisfied:*

(1) $F(x)$ has finite moments of all orders,

(2) $\varphi(z) = E(e^{iz\ln|x|})$ has no zeros in its region of regularity (z complex), then $F(x)$ is normal.

We must note in this connection that the condition (2) is essential for the theorem. In the next section we shall give an example to show that the theorem is not true if the condition (2) is not satisfied.

PROOF. We examine more closely the equation

$$\varphi(z)\varphi(-z) = \frac{1}{\cosh\left(\frac{\pi}{2}z\right)} \tag{3.4}$$

for complex values of z.

For further investigation, we have to study the analytical behaviour of the function $\cosh[(\pi/2)z]$ in the complex z-plane. We note that $\cosh[(\pi/2)z]$ is an entire function of order unity having simple zeros at the points $z = \pm i(2k+1)$, $k = 0, 1, 2, \cdots$ on the imaginary axis. Then applying the decomposition theorem ([6], p. 299), we have the canonical representation of $\cosh[(\pi/2)z]$ as:

$$\cosh\left(\frac{\pi}{2}z\right) = \prod_{k=0}^{\infty}\left(1 + \frac{z^2}{\alpha_k^2}\right) \tag{3.5}$$

where $\alpha_k = 2k+1$; $k = 0, 1, 2, \cdots$. It is also easy to verify that the condition $\sum_{k=0}^\infty 1/\alpha_k^2 < \infty$ is satisfied.

From the conditions of the theorem 3.1 and lemma 3.5 it follows that the characteristic function $\varphi(z)$ is regular in the region $\operatorname{Im} z < 1$ and has no zeros in this region. We now factorize $\varphi(z)$ in the following manner:

$$\varphi(z) = \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1+iz}{2}\right) \theta(z). \tag{3.6}$$

From the elementary properties of the Gamma function ([6], p. 313) it can be verified easily that $\Gamma((1 + iz)/2)$ is a meromorphic function which is regular everywhere in the region $\operatorname{Im} z < 1$, real on the imaginary axis and has no zeros in its region of regularity. We also note that its reciprocal $1/\Gamma((1 + iz)/2)$ is an entire function of order unity having simple zeros at the points $z = i(2k + 1)$, $k = 0, 1, 2, \cdots$ all located on the imaginary axis ([6], p. 415). Hence using the factorization theorem of Hadamard ([6], p. 332) we get

$$\frac{\sqrt{\pi}}{\Gamma\left(\frac{1+iz}{2}\right)} = e^{i\rho z} \prod_{k=0}^{\infty} \left(1 - \frac{z}{i\alpha_k}\right) e^{z/i\alpha_k} \tag{3.7}$$

where $\rho \neq 0$ real; $\alpha_k = 2k + 1$, $k = 0, 1, 2, \cdots$. Thus the function $\theta(z)$ introduced in (3.6) must also be regular at least in the region $\operatorname{Im} z < 1$, real on the imaginary axis and without any zeros. From (3.6) we get

$$\varphi(z)\varphi(-z) = \frac{1}{\pi} \cdot \Gamma\left(\frac{1+iz}{2}\right) \cdot \Gamma\left(\frac{1-iz}{2}\right) \cdot \theta(z) \cdot \theta(-z). \tag{3.8}$$

Again it is easy to verify from (3.5) and (3.7)

$$\Gamma\left(\frac{1+iz}{2}\right) \cdot \Gamma\left(\frac{1-iz}{2}\right) = \frac{\pi}{\cosh\left(\frac{\pi}{2} z\right)}. \tag{3.9}$$

Hence using (3.8) and (3.9) we get easily from (3.4) that

$$\theta(z)\theta(-z) = 1 \tag{3.10}$$

holding for complex values of z. But we note that $\theta(-z)$ is regular at least in the region $\operatorname{Im} z > -1$ and has no zeros in its region of regularity. Hence $1/\theta(-z)$ is also regular at least in the region $\operatorname{Im} z > -1$ and without any zeros in this region. Then using the relation (3.10) it follows easily that $\theta(z)$ is regular everywhere throughout the complex plane, that is, it is an entire function. We note further that $\theta(z)$ has no zeros in the complex plane.

We next prove that the order of the entire function $\theta(z)$ cannot exceed unity. We apply the inequality (3.2) to the relation (3.4) and using the expression for $\varphi(z)$ in (3.6), we get after a little rearrangement

$$|\theta(z)| \cos\left(\frac{\pi}{2} v\right) \leqq K\sqrt{\pi} \cdot \frac{e^{a|v|}}{\left|\Gamma\left(\frac{1+iz}{2}\right)\right|} \leqq K\sqrt{\pi} \cdot \frac{e^{a|z|}}{\left|\Gamma\left(\frac{1+iz}{2}\right)\right|} \tag{3.11}$$

where $z = t + iv$ (t and v both real).

But the right hand side of (3.11) is an entire function of order unity. Hence from (3.11) it follows that the order of the entire function $\theta(z)$ cannot exceed unity. Again we have already proved that $\theta(z)$ has no zeros throughout the complex plane. Hence using the factorization theorem of Hadamard, we get $\theta(z) = e^{cz}$. Since $\theta(z)$ is real on the imaginary axis we get $\theta(z) = e^{i\alpha z}$ where α is real. Thus we obtain from (3.6)

$$\varphi(-z) = \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1 - iz}{2}\right) e^{-i\alpha z}. \tag{3.12}$$

Next we substitute $z = iv$ ($v > 0$ real) in (3.12) and get

$$\varphi(-iv) = 2\int_0^\infty x^v f(x)\, dx = \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1 + v}{2}\right) e^{\alpha v}. \tag{3.13}$$

Since the distribution of x is symmetric about the origin, all the moments of odd order are equal to zero and a moment of the even order $2k$ is given by

$$\mu_{2k} = \int_{-\infty}^\infty x^{2k} f(x)\, dx = 2\int_0^\infty x^{2k} f(x)\, dx. \tag{3.14}$$

Finally substituting $v = 2k$ (k a positive integer) in (3.13) we have

$$\mu_{2k} = \frac{1}{\sqrt{\pi}} \Gamma(k + \tfrac{1}{2}) e^{2\alpha k} = \frac{(2k)!}{k!2^k} \sigma^{2k} \tag{3.15}$$

where $\sigma = e^\alpha/\sqrt{2}$.

The proof of theorem (3.1) follows at once from the fact that the moments in (3.15) determine uniquely the normal distribution with mean zero and variance σ^2.

4. An example. The non-normal distribution functions constructed in [1], [7] have moments only up to a certain finite order. Here we give an example of a non-normal distribution having finite moments of all orders. We shall now construct a characteristic function $\varphi(z)$ which satisfies the basic equation (3.4), is regular in the region $\operatorname{Im} z < 1$, but having zeros in its region of regularity so that the condition (2) of Theorem 3.1 is violated. We give first two lemmas.

LEMMA 4.1. *Let*

$$\Phi(t) = \frac{\left(1 + \frac{it}{\gamma}\right)\left(1 + \frac{it}{\bar{\gamma}}\right)}{\left(1 - \frac{it}{\alpha}\right)\left(1 - \frac{it}{\gamma}\right)\left(1 - \frac{it}{\bar{\gamma}}\right)} \tag{4.1}$$

where $\gamma = \alpha + i\beta$; $\bar{\gamma} = \alpha - i\beta$ *and* $\alpha > 0$, $\beta > 0$ *both real. Then* $\Phi(t)$ *is always a characteristic function whenever the relation* $\beta \geqq 2\sqrt{2}\,\alpha$ *is satisfied.* The proof follows from a more general result on rational characteristic functions ([4], p. 721).

LEMMA 4.2. *Let* $Q(z)$ *be an entire function of order unity having only purely imaginary zeros. Then its reciprocal* $1/Q(z)$ *is always a characteristic function.*

The proof follows from the result ([3], p. 140).

Next we define the quantities

$$(4.2)\quad \begin{aligned} \alpha_k &= 2k + 1 & k &= 0, 1, 2, \cdots N, N + 1 \cdots \infty \\ \beta_k &\geqq 2\sqrt{2}\, \alpha_k & k &= 0, 1, 2, \cdots N \quad (N > 0) \\ \gamma_k &= \alpha_k + i\beta_k & k &= 0, 1, 2, \cdots N \\ \bar{\gamma}_k &= \alpha_k - i\beta_k & k &= 0, 1, 2, \cdots N \end{aligned}$$

and construct the function $\varphi(z)$ as:

$$(4.3)\quad \begin{aligned} \varphi(z) &= \prod_{k=0}^{N} \frac{\left(1 + \frac{z}{i\gamma_k}\right)\left(1 + \frac{z}{i\bar{\gamma}_k}\right)}{\left(1 - \frac{z}{i\alpha_k}\right)\left(1 - \frac{z}{i\gamma_k}\right)\left(1 - \frac{z}{i\bar{\gamma}_k}\right) e^{z/i\alpha_k}} \prod_{k=N+1}^{\infty} \frac{1}{\left(1 - \frac{z}{i\alpha_k}\right) e^{z/i\alpha_k}} \\ &= P_1(z) \cdot P_2(z). \end{aligned}$$

From Lemma 4.1 it follows that $P_1(z)$ is a characteristic function, while we get as an immediate consequence of Lemma 4.2 that $P_2(z)$ is also a characteristic function. Hence $\varphi(z)$ in (4.3) is a characteristic function. It is also easy to verify that $\varphi(z)$ is regular in the region $\operatorname{Im} z < 1$ and has simple zeros at the points $z = -i\alpha_k \pm \beta_k$ ($k = 0, 1, 2, \cdots N$) inside the region where α_k and β_k are defined in (4.2). We also see easily that $\varphi(z)$ satisfies the basic equation (3.4). Then we take $\varphi(z)$ in (4.3) as the characteristic function of the distribution of $\ln |x|$ and verify at once that the corresponding distribution function $F(x)$ has moments of all orders, but is not normal and the quotient x/y follows the Cauchy law.

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