A NECESSARY CONDITION FOR EXISTENCE OF REGULAR AND SYMMETRICAL EXPERIMENTAL DESIGNS OF TRIANGULAR TYPE, WITH PARTIALLY BALANCED INCOMPLETE BLOCKS[1]

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A necessary condition for the existence of a symmetrical balanced incomplete block (B.I.B.) design in terms of the Hasse-Minkowski p-invariant was obtained by S. S. Shrikhande [1]. Similar necessary conditions for regular symmetrical group divisible designs and for regular symmetrical L_2 type designs were obtained by R. C. Bose and W. S. Connor [2] and S. S. Shrikhande [3] respectively.

The purpose of this note is to give a necessary condition for the existence of a regular symmetrical partially balanced incomplete block (P.B.I.B.) design of triangular type in terms of the Hasse-Minkowski p-invariant.

1. A necessary theorem and lemmas. Two symmetric and non-singular matrices A and B of the same order n with rational elements are said to be *rationally congruent* or *congruent in the field of rational numbers*, if there exists a non-singular and rational matrix C of the same order such that

$$C'AC = B, \tag{1.1}$$

where C' stands for the transposed matrix of C [4]. This relation is denoted by the symbol

$$A \sim B. \tag{1.2}$$

By the very definition of the rational congruence, it will be clear that (i) $A \sim A$ (reflexive), (ii) if $A \sim B$, then $B \sim A$ (symmetric), (iii) if $A \sim B$ and $B \sim C$, then $A \sim C$ (transitive), (iv) $A \sim A^{-1}$, and (v) if $A \sim B$, then $A^{-1} \sim B^{-1}$.

Hasse's Theorem [4, 5]. The necessary and sufficient conditions for two positive-definite, rational and symmetric matrices A and B of the same order to be rationally congruent are that, in the first place, the square-free parts of the determinants of both matrices are the same, and in the second, the Hasse-Minkowski p-invariants of both matrices coincide with each other for all primes p including p_∞.

If we denote the n leading principal minor determinants of A by

$$D_1, D_2, \cdots, D_{n-1}, \qquad D_n = |A|$$

and let $D_0 = 1$, then [4] the Hasse-Minkowski p-invariant of A is given by

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$$C_p(A) = (-1, -1)_p \cdot \prod_{i=0}^{n-1} (D_{i+1}, -D_i)_p \tag{1.3}$$

for each prime p, where the symbol $(a, b)_p$ denotes the extended Hilbert symbol of norm residue [4, 6], which is defined by

$$(a, b)_p = \begin{cases} +1, & \text{if } ax^2 + by^2 = 1 \text{ has a } p\text{-adic solution} \\ -1, & \text{otherwise.} \end{cases} \tag{1.4}$$

Now we shall list some useful properties of $C_p(A)$ as lemmas.

LEMMA 1.1 [4]: *If A and B are rational and symmetric and if*

$$U = \begin{Vmatrix} A & 0 \\ 0 & B \end{Vmatrix},$$

then

$$C_p(U) = (-1, -1)_p(|A|, |B|)_p C_p(A)\, C_p(B). \tag{1.5}$$

LEMMA 1.2 [4]: *For an $n \times n$ diagonal matrix Δ_n, whose i, i element is d,*

$$C_p(\Delta_n) = (-1, -1)_p(-1, d)_p^{\frac{n(n+1)}{2}} \tag{1.6}$$

LEMMA 1.3. *For a $(v - 1) \times (v - 1)$ diagonal matrix U, whose i, i element is*

$$(v - i + 1)\,(v - i),$$

$$C_p(U) = (-1, -1)_p. \tag{1.7}$$

LEMMA 1.4 [4]:

$$C_p(\rho A) = (-1, \rho)_p^{\frac{n(n+1)}{2}} (\rho, |A|)_p^{n-1} C_p(A). \tag{1.8}$$

LEMMA 1.5: *If the $n - 1$ rational vectors*

$$\mathbf{a}_2, \cdots, \mathbf{a}_n$$

of dimensionality n are linearly independent and are orthogonal to

$$\mathbf{1}' = (1\ 1 \cdots 1),$$

then the Gramian of the set, i.e.,

$$U = \begin{Vmatrix} \mathbf{a}_2' \\ \vdots \\ \mathbf{a}_n' \end{Vmatrix} \cdot \| \mathbf{a}_2 \cdots \mathbf{a}_n \|$$

has the p-invariant $C_p(U) = (-1, -1)_p$.

LEMMA 1.6: *So long as we restrict ourselves to rational vectors, the p-invariant of a vector set, i.e., the p-invariant of the Gramian of the set is uniquely determined by the linear subspace generated by the vectors of the set.*

LEMMA 1.7: *For a matrix A of the form*

$$A = eI_n + fG_n,$$

where I_n is the unit matrix of order n and G_n is the $n \times n$ matrix whose elements are all unity,

$$(1.9) \quad C_p(A) = (-1, -1)_p(-1, e)_p^{\frac{n(n-1)}{2}}(-1, g)_p(n, g)_p(n, e)_p(g, e)_p^{n-1},$$

where we have put

$$(1.10) \qquad g = e + nf.$$

Next we shall summarize the necessary properties of Hilbert's symbol [4, 6] and some of the fundamental properties of the Legendre symbol (a/p) of the quadratic residue [6].

First of all, from the definition of $(a, b)_p$, it is clear that

$$(1.11) \qquad (a, b)_p = (b, a)_p,$$

and for any rational numbers t and u,

$$(1.12) \qquad (at^2, bu^2)_p = (a, b)_p.$$

Hence in any calculation handling the Hilbert symbol, the square part of any rational number can be replaced by 1.

$$(1.13) \qquad \begin{aligned} (a, -a)_p &= +1 \\ (a, a)_p &= (-1, a)_p \\ (a, b_1b_2)_p &= (a, b_1)_p(a, b_2)_p \end{aligned}$$

and [2, 4]

$$(1.14) \qquad (a, b)_p = (-ab, a + b)_p.$$

As a special case of (1.14), we have for every positive integer n:

$$(1.15) \qquad (n, n + 1)_p = (-1, n + 1)_p.$$

(1.16) If $(ab, p) = 1$, then $(a, b)_p = +1$.

(1.17) For an odd prime p, $(p, a)_p = (a/p)$.

For the even prime 2, we have

$$(1.18) \qquad \begin{aligned} (p, q)_2 = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}, \quad (2, p)_2 = (2/p), \quad (-1, p)_2 = (-1/p), \\ (-1, 2)_2 = +1, \quad (-1, -1)_2 = -1. \end{aligned}$$

And for $p = \infty$, we have

$$(1.19) \qquad \begin{aligned} (p, q)_\infty = (-1, 1)_\infty &= (2, p)_\infty \\ &= (-1, p)_\infty = +1, \quad (-1, -1)_\infty = -1. \end{aligned}$$

In the above and hereafter, p and q denote odd primes.

For the Legendre symbol, the following properties are fundamental:

$$(a/p) = (b/p) \quad \text{if} \quad a \equiv b(\text{mod}\, p), \tag{1.20}$$

$$(ab/p) = (a/p)(b/p) \tag{1.21}$$

and the reciprocity law [6]

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}} \tag{1.22}$$

supplemented by

$$\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}, \qquad \left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}. \tag{1.23}$$

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2. A P.B.I.B. design of triangular type. Triangular association is defined as follows: The number of elements is $v = n(n-1)/2$, where n is a positive integer. We take an $n \times n$ square, and fill the $n(n-1)/2$ positions above the main diagonal by the different elements, taken in order. The positions in the main diagonal are left blank, while the positions below the main diagonal are filled so that the scheme is symmetrical with respect to the main diagonal. Two elements in the same column are 1st associates, whereas two elements which do not occur in the same column are 2nd associates.

In this association each element has n_i ith associates, where

$$n_1 = 2n - 4, \qquad n_2 = \frac{(n-2)(n-3)}{2}.$$

The parameters of association are as follows:

$$p_{11}^1 = n - 2, \qquad p_{12}^1 = n - 3 = p_{21}^1, \qquad p_{22}^1 = \frac{(n-3)(n-4)}{2},$$

$$p_{11}^2 = 4, \qquad p_{12}^2 = n - 8 = p_{21}^2, \qquad p_{22}^2 = \frac{(n-4)(n-5)}{2}.$$

Let the association matrices be $A_0 = I_v$, A_1, A_2, then it is known that these matrices generate a commutative linear associative algebra $\varkappa$ of rank 3, and the regular representation $(\varkappa)$ is given [7] by

$$(\varkappa): \begin{cases} A_0 \to I_3, \\[2ex] A_1 \to \mathcal{P}_1 = \begin{Vmatrix} 0 & 1 & 0 \\ 2n-4 & n-2 & 4 \\ 0 & n-3 & 2n-8 \end{Vmatrix}, \\[2ex] A_2 \to \mathcal{P}_2 = \begin{Vmatrix} 0 & 0 & 1 \\ 0 & n-3 & (n-8) \\ \dfrac{(n-2)(n-3)}{2} & \dfrac{(n-3)(n-4)}{2} & \dfrac{(n-4)(n-5)}{2} \end{Vmatrix}, \end{cases} \tag{2.1}$$

This regular representation $(\varkappa)$ decomposes into three non-equivalent linear representations

$$(2.2)\quad \begin{aligned} &\varkappa_0 : A_0 \to 1, && A_1 \to n_1 = 2n - 4, && A_2 \to n_2 = (n-2)(n-3)/2, \\ &\varkappa_1 : A_0 \to 1, && A_1 \to n - 4, && A_2 \to -(n-3), \\ &\varkappa_2 : A_0 \to 1, && A_1 \to -2, && A_2 \to 1, \end{aligned}$$

having respective multiplicities

$$(2.3)\qquad \alpha_0 = 1, \qquad \alpha_1 = n - 1, \qquad \alpha_2 = n(n-3)/2$$

in the algebra $\varkappa$.

Suppose that we are given v treatments having triangular association among them and b blocks each having k experimental units in such a way that

(1) each block contains k different treatments,

(2) each treatment occurs in r blocks, and

(3) any two treatments occur together in λ_i blocks, if they are ith associates.

This design is called a P.B.I.B. *design of triangular type.*

If the incidence matrix of this design is denoted by N, it is also well known ([7], [8]) that

$$(2.4)\qquad NN' = rA_0 + \lambda_1 A_1 + \lambda_2 A_2 .$$

Hence NN' has eigenvalues

$$(2.5)\quad \begin{aligned} \rho_0 &= r + (2n-4)\lambda_1 + \frac{(n-2)(n-3)}{2}\lambda_2 = rk, \\ \rho_1 &= r + (n-4)\lambda_1 - (n-3)\lambda_2, \\ \rho_2 &= r - 2\lambda_1 + \lambda_2, \end{aligned}$$

with multiplicities 1, $(n - 1)$ and $n(n - 3)/2$ respectively.

It can be shown from the elements of linear associative algebra [9] that there exist three mutually orthogonal and symmetric idempotents $A_0^* = (1/v)G_v$, A_1^*, and A_2^* with respective ranks 1, $n - 1$, and $n(n - 3)/2$, such that

$$(2.6)\qquad NN' = \rho_0 A_0^* + \rho_1 A_1^* + \rho_2 A_2^*.$$

The column vectors of A_i^* generate the eigenspace of NN' corresponding to the eigenvalue ρ_i. Let us assume, without any loss of generality, that

$$\mathbf{a}_1^{0*}, \mathbf{a}_2^{1*}, \cdots, \mathbf{a}_n^{1*}, \mathbf{a}_{n+1}^{2*}, \cdots, \mathbf{a}_v^{2*}$$

are linearly independent, and let us put

$$(2.7)\qquad S = \| \mathbf{a}_1^{0*}\mathbf{a}_2^{1*} \cdots \mathbf{a}_n^{1*}\mathbf{a}_{n+1}^{2*} \cdots \mathbf{a}_v^{2*} \|,$$

then S is a non-singular $v \times v$ matrix with rational elements. Further let

$$(2.8)\quad Q_1 = \left\| \begin{matrix} \mathbf{a}_2^{1*\prime} \\ \vdots \\ \mathbf{a}_n^{1*\prime} \end{matrix} \right\| \| \mathbf{a}_2^{1*} \cdots \mathbf{a}_n^{1*} \|, \quad \text{and} \quad Q_2 = \left\| \begin{matrix} \mathbf{a}_{n+1}^{2*\prime} \\ \vdots \\ \mathbf{a}_v^{2*\prime} \end{matrix} \right\| \| \mathbf{a}_{n+1}^{2*} \cdots \mathbf{a}_v^{2*} \|,$$

then from (2.6) it follows that

$$S'NN'S = \left\| \begin{matrix} \rho_0 a_{11}^{0*} & 0 & 0 \\ 0 & \rho_1 Q_1 & 0 \\ 0 & 0 & \rho_2 Q_2 \end{matrix} \right\|,$$

or

$$NN' \sim \left\| \begin{matrix} \frac{rk}{v} & 0 & 0 \\ 0 & \rho_1 Q_1 & 0 \\ 0 & 0 & \rho_2 Q_2 \end{matrix} \right\|. \tag{2.9}$$

Since

$$S'S = \left\| \begin{matrix} \frac{1}{v} & 0 & 0 \\ 0 & Q_1 & 0 \\ 0 & 0 & Q_2 \end{matrix} \right\|,$$

we get

$$v \mid Q_1 \| Q_2 \mid \sim 1. \tag{2.10}$$

It has been shown by Corsten [10] that

$$\left\| \begin{matrix} \frac{1}{v} & 0 \\ 0 & Q_1 \end{matrix} \right\| \sim \left\| \begin{matrix} n-1 & 1 & \cdots & 1 \\ 1 & n-1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & n-1 \end{matrix} \right\|, \tag{2.11}$$

hence

$$\mid Q_1 \mid \sim n(n-2)^{n-1}. \tag{2.12}$$

3. Necessary conditions for the existence of a regular symmetrical P.B.I.B. **design of triangular type.** In this section, we shall show the non-existence of certain regular symmetrical P.B.I.B. designs of triangular type.

If the design is symmetrical, i.e., $v = b$ and $r = k$, then the incidence matrix N is a square matrix with elements 0 and 1, hence in the regular case $\mid NN' \mid$ must be a perfect square. Thus first of all

$$\rho_1^{n-1} \rho_2^{\frac{1}{2}n(n-3)} = [r + (n-4)\lambda_1 - (n-3)\lambda_2]^{n-1}[r - 2\lambda_1 + \lambda_2]^{\frac{1}{2}n(n-3)} \sim 1 \tag{3.1}$$

and then, since $NN' \sim I_v$, we have

$$C_p(NN') = (-1, -1)_p \tag{3.2}$$

for all primes p. (3.1) and (3.2) are necessary conditions for the existence.

Now, from (2.9) we get

$$C_p(NN') = (-1, -1)_p(-1, v)_p(v, \rho_1^{n-1} \rho_2^{\frac{1}{2}n(n-3)} \mid Q_1 \| Q_2 \mid)_p \tag{3.3}$$
$$(\rho_1^{n-1} \mid Q_1 \mid, \rho_2^{\frac{1}{2}n(n-3)} \mid Q_2 \mid)_p \, C_p(\rho_1 Q_1) \cdot C_p(\rho_2 Q_2).$$

By Lemma 1.4,

$$C_p(\rho_1 Q_1) = (-1, \rho_1)_p^{\frac{n(n-1)}{2}} (\rho_1, |Q_1|)_p^{n-2} C_p(Q_1) \tag{3.4}$$

$$C_p(\rho_2 Q_2) = (-1, \rho_2)_p^{\frac{n(n-1)(n-2)(n-3)}{8}} (\rho_2, |Q_2|)_p^{\frac{1}{2}n(n-3)-1} C_p(Q_2). \tag{3.5}$$

Since

$$v\,|Q_1|\,|Q_2| \sim 1, \qquad \rho_1^{n-1} \rho_2^{\frac{1}{2}n(n-3)} \sim 1,$$

it follows that

$$(v, \rho_1^{n-1} \rho_2^{\frac{1}{2}n(n-3)} |Q_1|\,|Q_2|)_p = (v, v)_p = (-1, v)_p, \tag{3.6}$$

$$\begin{aligned} (\rho_1^{n-1} |Q_1|, \rho_2^{\frac{1}{2}n(n-3)} |Q_2|)_p &= (\rho_1^{n-1} |Q_1|, \rho_1^{n-1} |Q_2|)_p \\ &= (\rho_1, v)_p^{n-1} (-1, \rho_1)_p^{n-1} (|Q_1|, |Q_2|)_p \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} (\rho_2, |Q_2|)_p^{\frac{1}{2}n(n-3)-1} &= (\rho_2, |Q_2|)_p (\rho_1, v\,|Q_1|)_p^{n-1} \\ &= (\rho_2, |Q_2|)_p (\rho_1, v)_p^{n-1} (\rho_1, |Q_1|)_p^{n-1}. \end{aligned} \tag{3.8}$$

Substituting (3.4) to (3.8) into (3.3), we get

$$\begin{aligned} C_p(NN') &= (-1, -1)_p (\rho_1, v)_p^{n-1} (-1, \rho_1)_p^{n-1} (|Q_1|, |Q_2|)_p \\ &\quad \cdot (-1, \rho_1)_p^{\frac{n(n-1)}{2}} (\rho_1, |Q_1|)_p^{n-2} (-1, \rho_2)_p^{\frac{n(n-1)(n-2)(n-3)}{8}} \\ &\quad \cdot (\rho_2, |Q_2|)_p (\rho_1, v)_p^{n-1} (\rho_1, |Q_1|)_p^{n-1} C_p(Q_1) C_p(Q_2) \\ &= (-1, -1)_p (-1, \rho_1)_p^{\frac{(n-1)(n-2)}{2}} (-1, \rho_2)^{\frac{n(n-1)(n-2)(n-3)}{8}} (\rho_1, |Q_1|)_p \\ &\quad \cdot (\rho_2, |Q_2|)_p (|Q_1|, |Q_2|)_p C_p(Q_1)\, C_p(Q_2), \end{aligned}$$

whereas by Lemma 1.5

$$(|Q_1|, |Q_2|)_p C_p(Q_1) C_p(Q_2) = +1$$

and

$$|Q_1| \sim n(n-2)^{n-1}, \qquad |Q_2| \sim 2(n-1)(n-2)^{n-1},$$

therefore

$$\begin{aligned} C_p(NN') &= (-1, -1)_p (-1, \rho_1)_p^{\frac{(n-1)(n-2)}{2}} (\rho_1, n)_p (\rho_1, n-2)_p^{n-1} \\ &\quad \cdot (-1, \rho_2)_p^{\frac{n(n-1)(n-2)(n-3)}{8}} (\rho_2, 2)_p (\rho_2, n-1)_p (\rho_2, n-2)_p^{n-1} \end{aligned} \tag{3.9}$$

Consequently (3.2) becomes

$$\begin{aligned} O_p \equiv (-1, \rho_1)_p^{\frac{(n-1)(n-2)}{2}} (\rho_1, n)_p (\rho_1, n-2)_p^{n-1} (-1, \rho_2)_p^{\frac{n(n-1)(n-2)(n-3)}{8}} \\ \cdot (\rho_2, 2)_p (\rho_2, n-1)_p (\rho_2, n-2)_p^{n-1} = +1 \end{aligned} \tag{3.10}$$

for all primes p.

4. Examples of non-existent P.B.I.B. designs of triangular type.

(1) $n = 7; \quad v = b = 21, \quad r = k = 6. \quad \lambda_1 = 0, \quad \lambda_2 = 3$

$$\rho_1 = -6, \quad \rho_2 = 9$$

$$O_p = (-1, 6)_p(-6, 7)_p = (-1, -1)_p(-1, 2)_p(-1, 3)_p(-1, 7)_p(2, 7)_p(3, 7)_p$$

$$O_3 = \left(\frac{-1}{3}\right)\left(\frac{7}{3}\right) = -1.$$

Hence this design is impossible.

(2) $n = 7; \quad v = b = 21, \quad r = k = 10, \quad \lambda_1 = 0, \quad \lambda_2 = 9$

$$\rho_1 = -26, \quad \rho_2 = 19$$

$$\begin{aligned} O_p &= (-1, -26)_p(-26, 7)_p(19, 2)_p(19, 6)_p(-1, 19)_p \\ &= (-1, -1)_p(-1, 2)_p(-1, 13)_p(-1, 7)_p(2, 7)_p(13, 7)_p(19, 3)_p(-1, 19)_p \end{aligned}$$

$$O_{13} = \left(\frac{-1}{13}\right)\left(\frac{7}{13}\right) = \left(\frac{13}{7}\right) = \left(\frac{-1}{7}\right) = -1.$$

Hence this design is impossible.

(3) $n = 7; \quad v = b = 21, \quad r = k = 10, \quad \lambda_1 = 1, \quad \lambda_2 = 8$

$$\rho_1 = -19, \quad \rho_2 = 16$$

$$O_p = (-1, -19)_p(-19, 7)_p = (-1, -1)_p(-1, 19)_p(-1, 7)_p(19, 7)_p$$

$$O_{19} = \left(\frac{-1}{19}\right)\left(\frac{7}{19}\right) = \left(\frac{19}{7}\right) = \left(\frac{5}{7}\right) = \left(\frac{2}{5}\right) = -1.$$

Hence this design is impossible.

(4) $n = 7; \quad v = b = 21, \quad r = k = 10, \quad \lambda_1 = 2, \quad \lambda_2 = 7$

$$\rho_1 = -12, \quad \rho_2 = 13$$

$$\begin{aligned} O_p &= (-1, -12)_p(-12, 7)_p(13, 2)_p(13, 6)_p(-1, 13)_p \\ &= (-1, -1)_p(-1, 3)_p(-1, 7)_p(3, 7)_p(13, 3)_p(-1, 13)_p \end{aligned}$$

$$O_3 = \left(\frac{-1}{3}\right)\left(\frac{7}{3}\right)\left(\frac{13}{3}\right) = \left(\frac{-1}{3}\right) = -1.$$

Hence this design is impossible.

(5) $n = 7; \quad v = b = 21, \quad r = k = 10, \quad \lambda_1 = 3, \quad \lambda_2 = 6$

$$\rho_1 = -5, \quad \rho_2 = 10$$

$$\begin{aligned} O_p &= (-1, -5)_p(-5, 7)_p(10, 2)_p(10, 6)_p(-1, 10)_p \\ &= (-1, -1)_p(5, 7)_p(-1, 7)_p(2, 7)_p(2, 3)_p(5, 3)_p \end{aligned}$$

$$O_2 = (-1, -1)_2(5, 7)_2(-1, 7)_2(2, 7)_2(2, 3)_2(5, 3)_2 = -1.$$

Hence this design is impossible.

(6) $n = 7; \quad v = b = 21, \quad r = k = 10, \quad \lambda_1 = 8, \quad \lambda_2 = 1$

$$\rho_1 = 30, \qquad \rho_2 = -5$$

$$O_p = (-1, 30)_p(7, 30)_p(-1, -5)_p(2, -5)_p(-5, 6)_p$$

$$O_3 = \left(\frac{-1}{3}\right)\left(\frac{7}{3}\right)\left(\frac{-5}{3}\right) = \left(\frac{-1}{3}\right) = -1.$$

Hence this design is impossible.

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