THE USE OF SAMPLE QUASI-RANGES IN ESTIMATING POPULATION STANDARD DEVIATION

H. LEON HARTER

Wright Air Development Center[1]

Summary. The use of sample quasi-ranges in estimating the standard deviation of normal, rectangular, and exponential populations is discussed. For the normal population, the expected value and the variance of the rth quasi-range for samples of size n are tabulated for $r = 0\ (1)\ 8$ and $n = (2r + 2)\ (1)\ 100$. The efficiency of the unbiased estimate of population standard deviation based on one sample quasi-range is tabulated for the same values of r, with $n = (2r + 2)\ (2)\ 50\ (5)\ 100$. Estimates based on a linear combination of two quasi-ranges are considered and a method is given for determining the weighting factor which maximizes the efficiency. The most efficient unbiased estimates based on one quasi-range for $n = 2\ (1)\ 100$ and on linear combinations of two adjacent quasi-ranges and of two quasi-ranges among those with $r < r' \leqq 8$ for $n = 4\ (1)\ 100$ are tabulated, along with their efficiencies. An example illustrates the use of these estimates. For the rectangular population, the efficient estimate of population standard deviation, which is based on the sample range, is tabulated for $n = 2\ (1)\ 100$. The bias, when estimates which assume normality are used, is tabulated for $n = 2\ (1)\ 100$ for rectangular and exponential populations.

0. Introduction. It is well known that, for small samples, the standard deviation of a normal population can be estimated quite efficiently from the sample range. However, the efficiency of the estimate based on the range decreases rather rapidly as the sample size increases, being less than 35% for samples of 100. There appears to be a need for substitute estimates which are reasonably efficient for moderate sample sizes, yet much simpler to compute than the efficient estimate based on the sample standard deviation. A number of authors, including Jones [9], Nair [12], [13], Godwin [6] and Sarhan and Greenberg [17] have proposed methods based on order statistics. This paper will be concerned with estimates, based on sample quasi-ranges, that satisfy these requirements quite well. Up to the present such estimates have been used relatively little, mainly because of the lack of suitable tables, though estimates based on quasi-ranges of samples of moderate size were proposed by Mosteller [11] in 1946, and estimates based on quantiles of large samples had been advocated much earlier by a number of authors, notably Edgeworth [5], Sheppard [18], and K. Pearson [15]. More recently, Benson [1] has explored further aspects of estimates of the latter type.

The rth quasi-range, w_r, of a sample of size n is defined as the range of $(n - 2r)$ sample values, omitting the r largest and the r smallest. Symbolically,

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[1] Applied Mathematics Research Branch, Aeronautical Research Laboratory.

$w_r = x_{n-r} - x_{r+1}$, where $x_1 \leqq x_2 \leqq \cdots \leqq x_n$ are the ordered sample values. Cadwell [2] has shown that the range, w_0, is the most efficient statistic of this type for estimating the standard deviation of a normal population from samples of sizes up through $n = 17$, beyond which point w_1 is optimum up through $n = 31$, where w_2 becomes better. Cadwell [2] has also proposed the use of linear combinations of quasi-ranges, while Dixon [4] has advocated the use of sums of from two to four quasi-ranges with equal weights.

Section 1 of this paper will deal with the most efficient unbiased estimates of the standard deviation of a normal population, based on one sample quasi-range and on linear combinations of two sample quasi-ranges. Section 2 will be concerned with the efficient estimate of the standard deviation of a rectangular distribution, based on the range, and with the bias of the estimates which assume normality when the population is actually rectangular or exponential.

1. Estimates of σ for a normal population.

1.1. *Expected values of quasi-ranges.* In order to determine the factor by which the rth quasi-range, w_r, must be multiplied in order to obtain an unbiased estimate of the population standard deviation σ, it is necessary to know the expected value $E(w_r)$ of the rth quasi-range for samples of n from a standard normal population, which is given by Cadwell ([2], p. 606) in terms of an integral which cannot be evaluated in closed form. Expected values of the range (to five decimal places) have been tabulated for $n = 2\ (1)\ 1000$ by Tippett [20]. Cadwell [2] has tabulated $E(w_1)$ to four decimal places for $n = 10\ (1)\ 30$. The author has computed tables of $E(w_r)$, accurate to within a unit in the sixth decimal place, for $r = 0\ (1)\ 8$ and $n = (2r + 2)\ (1)\ 100$, using the Burroughs E101 computer. The trapezoidal rule was employed for the numerical integration. The results, which are given in Table 1, agree with those obtained by Tippett and Cadwell to within a unit in the last place published by them. The values in Table 1 also agree with those found by doubling the expected values of order statistics, which have been tabulated to ten decimal places for $n = 2\ (1)\ 20$ by Teichroew [19], and rounding to six decimal places.

1.2. *Variances of quasi-ranges.* In order to determine the variances of unbiased estimates based on quasi-ranges (and hence their efficiencies), it is necessary to know the variance of the rth quasi-range for samples of n from a standard normal population. This is given by the equation $\operatorname{var} w_r = E(w_r^2) - [E(w_r)]^2$, where $E(w_r^2)$ can be obtained by multiplying the probability density function of w_r (see Cadwell [2], p. 604) by w_r^2 and integrating with respect to w_r between the limits 0 and ∞. Tippett [20] and E. S. Pearson [14] have computed approximate values of the variance of the range for a few values of n. Cadwell [2] has tabulated $\operatorname{var} w_1$ to four decimal places for $n = 10\ (1)\ 30$. The variance of all quasi-ranges for samples of $n = 2\ (1)\ 20$ can be obtained quite easily from ten-decimal-place values of the variances and covariances of order statistics, which have been tabulated by Sarhan and Greenberg [17]. These tables are based on ten-decimal-place expected values of order statistics and of products of order statistics tabulated by Teichroew [19]. The author, with the assistance of Eugene

TABLE 1

Expected Value of the r^{th} Quasi-Range for Samples of n from $N(\mu, 1)$

n	r = 0	r = 1	r = 2	r = 3	r = 4	r = 5	r = 6	r = 7	r = 8
2	1.128379								
3	1.692569								
4	2.058751	0.594023							
5	2.325929	0.990038							
6	2.534413	1.283510	0.403094						
7	2.704357	1.514749	0.705414						
8	2.847201	1.704450	0.945645	0.305029					
9	2.970026	1.864595	1.143942	0.549052					
10	3.077505	2.002714	1.312118	0.751529	0.245336				
11	3.172873	2.123833	1.457679	0.923957	0.449782				
12	3.258455	2.231464	1.585676	1.073686	0.624498	0.205179			
13	3.335980	2.328154	1.699669	1.205700	0.776654	0.381047			
14	3.406763	2.415805	1.802253	1.323527	0.911132	0.534594	0.176318		
15	3.471827	2.495870	1.895378	1.429755	1.031402	0.670592	0.330597		
16	3.531983	2.569488	1.980542	1.526333	1.140019	0.792446	0.467503	0.154575	
17	3.587884	2.637564	2.058922	1.614770	1.238915	0.902667	0.590373	0.291975	
18	3.640064	2.700827	2.131456	1.696250	1.329589	1.003163	0.701674	0.415471	0.137605
19	3.688963	2.759877	2.198906	1.771724	1.413223	1.095415	0.803285	0.527486	0.261450
20	3.734950	2.815208	2.261896	1.841963	1.490766	1.180594	0.896664	0.629866	0.373915
21	3.778336	2.867236	2.320945	1.907604	1.562992	1.259644	0.982970	0.724051	0.476816
22	3.819385	2.916311	2.376488	1.969174	1.630538	1.333334	1.063136	0.811183	0.571570
23	3.858323	2.962731	2.428893	2.027118	1.693938	1.402301	1.137928	0.892185	0.659305
24	3.895348	3.006755	2.478476	2.081814	1.753638	1.467076	1.207975	0.967812	0.740931
25	3.930629	3.048602	2.525506	2.133585	1.810021	1.528108	1.273807	1.038691	0.817195
26	3.964316	3.088468	2.570219	2.182707	1.863411	1.585779	1.335872	1.105347	0.888717
27	3.996539	3.126520	2.612819	2.229423	1.914092	1.640416	1.394549	1.168222	0.956017
28	4.027414	3.162907	2.653484	2.273942	1.962307	1.692302	1.450167	1.227697	1.019535
29	4.057044	3.197761	2.692372	2.316449	2.008271	1.741683	1.503007	1.284097	1.079647
30	4.085522	3.231200	2.729624	2.357108	2.052170	1.788775	1.553316	1.337704	1.136678
31	4.112928	3.263326	2.765363	2.396061	2.094171	1.833766	1.601311	1.388764	1.190907
32	4.139338	3.294235	2.799700	2.433439	2.134420	1.876825	1.647180	1.437492	1.242580
33	4.164817	3.324009	2.832734	2.469355	2.173049	1.918099	1.691092	1.484078	1.291910
34	4.189425	3.352725	2.864556	2.503912	2.210174	1.957721	1.733196	1.528690	1.339088
35	4.213219	3.380451	2.895245	2.537204	2.245901	1.995809	1.773626	1.571478	1.384281
36	4.236247	3.407249	2.924875	2.569314	2.280326	2.032471	1.812500	1.612575	1.427639
37	4.258554	3.433177	2.953513	2.600317	2.313532	2.067802	1.849927	1.652101	1.469294
38	4.280183	3.458286	2.981218	2.630284	2.345600	2.101890	1.886002	1.690164	1.509367
39	4.301171	3.482623	3.008047	2.659277	2.376598	2.134813	1.920814	1.726860	1.547965
40	4.321554	3.506233	3.034049	2.687353	2.406591	2.166643	1.954443	1.762279	1.585186
41	4.341364	3.529154	3.059272	2.714565	2.435639	2.197445	1.986961	1.796500	1.621118
42	4.360631	3.551424	3.083756	2.740962	2.463796	2.227280	2.018434	1.829596	1.655842
43	4.379382	3.573076	3.107544	2.766588	2.491111	2.256203	2.048923	1.861634	1.689430
44	4.397644	3.594143	3.130670	2.791484	2.517629	2.284264	2.078483	1.892675	1.721950
45	4.415439	3.614654	3.153169	2.815688	2.543394	2.311510	2.107166	1.922775	1.753463
46	4.432790	3.634635	3.175071	2.839235	2.568444	2.337984	2.135019	1.951985	1.784025
47	4.449718	3.654111	3.196406	2.862158	2.592816	2.363725	2.162086	1.980354	1.813688
48	4.466242	3.673108	3.217201	2.884486	2.616542	2.388771	2.188406	2.007924	1.842500
49	4.482379	3.691645	3.237481	2.906249	2.639654	2.413155	2.214017	2.034738	1.870505
50	4.498147	3.709744	3.257268	2.927472	2.662181	2.436910	2.238954	2.060832	1.897744

TABLE 1 (continued)

Expected Value of the r^{th} Quasi-Range for Samples of n from $N(\mu, 1)$

n	r = 0	r = 1	r = 2	r = 3	r = 4	r = 5	r = 6	r = 7	r = 8
51	4.513562	3.727424	3.276586	2.948181	2.684150	2.460065	2.263249	2.086242	1.924256
52	4.528637	3.744702	3.295455	2.968397	2.705587	2.482647	2.286933	2.110999	1.950074
53	4.543388	3.761597	3.313894	2.988142	2.726514	2.504683	2.310032	2.135136	1.975233
54	4.557827	3.778123	3.331921	3.007438	2.746955	2.526197	2.332574	2.158679	1.999762
55	4.571967	3.794295	3.349553	3.026301	2.766929	2.547210	2.354583	2.181655	2.023691
56	4.585818	3.810128	3.366805	3.044750	2.786457	2.567746	2.376082	2.204090	2.047045
57	4.599393	3.825635	3.383695	3.062803	2.805556	2.587823	2.397093	2.226007	2.069851
58	4.612701	3.840828	3.400234	3.080474	2.824244	2.607460	2.417635	2.247427	2.092132
59	4.625752	3.855719	3.416437	3.097778	2.842538	2.626675	2.437728	2.268371	2.113909
60	4.638556	3.870319	3.432316	3.114730	2.860452	2.645484	2.457390	2.288858	2.135204
61	4.651122	3.884639	3.447884	3.131343	2.878001	2.663904	2.476638	2.3089[illegible]	2.15[illegible]36
62	4.663457	3.898688	3.463151	3.147629	2.895199	2.681948	2.495488	2.328534	2.176423
63	4.675569	3.912477	3.478128	3.163599	2.912058	2.699632	2.513954	2.347756	2.[illegible]
64	4.687467	3.926014	3.492827	3.179267	2.928591	2.716968	2.532052	2.366588	2.215931
65	4.699157	3.939308	3.507255	3.194641	2.944809	2.733968	2.549794	2.385044	2.235084
66	4.710646	3.952367	3.521423	3.209732	2.960724	2.750646	2.567194	2.403139	2.253856
67	4.721941	3.965199	3.535339	3.224550	2.976347	2.767011	2.584263	2.420886	2.272261
68	4.733047	3.977811	3.549011	3.239104	2.991685	2.783075	2.601013	2.438295	2.290312
69	4.743971	3.990210	3.562448	3.253403	3.006751	2.798849	2.617456	2.455380	2.308021
70	4.754718	4.002402	3.575656	3.267455	3.021552	2.814341	2.633601	2.472152	2.325401
71	4.765294	4.014395	3.588644	3.281267	3.036096	2.829561	2.649458	2.488620	2.342462
72	4.775704	4.026195	3.601418	3.294848	3.050393	2.844517	2.665037	2.504796	2.359216
73	4.785953	4.037806	3.613984	3.308204	3.064450	2.859219	2.680347	2.520688	2.375673
74	4.796045	4.049236	3.626349	3.321343	3.078275	2.873674	2.695396	2.536305	2.391841
75	4.805985	4.060488	3.638519	3.334271	3.091874	2.887890	2.710192	2.551658	2.407731

n	r = 0	r = 1	r = 2	r = 3	r = 4	r = 5	r = 6	r = 7	r = 8
76	4.815777	4.071569	3.650499	3.346994	3.105254	2.901874	2.724744	2.566752	2.423351
77	4.825426	4.082483	3.662296	3.359519	3.118422	2.915633	2.739059	2.581598	2.438709
78	4.834935	4.093235	3.673914	3.371850	3.131384	2.929174	2.753143	2.596201	2.453815
79	4.844308	4.103829	3.685358	3.383994	3.144146	2.942503	2.767004	2.610571	2.468674
80	4.853549	4.114270	3.696633	3.395957	3.156714	2.955626	2.780649	2.624712	2.483296
81	4.862661	4.124561	3.707745	3.407742	3.169094	2.968550	2.794083	2.638633	2.497686
82	4.871648	4.134708	3.718696	3.419355	3.181289	2.981279	2.807312	2.652339	2.511851
83	4.880513	4.144713	3.729492	3.430800	3.193307	2.993819	2.820343	2.665836	2.525799
84	4.889259	4.154581	3.740137	3.442082	3.205150	3.006176	2.833180	2.679131	2.539534
85	4.897890	4.164315	3.750635	3.453206	3.216825	3.018355	2.845830	2.692229	2.553063
86	4.906407	4.173918	3.760988	3.464176	3.228335	3.030359	2.858296	2.705135	2.566393
87	4.914814	4.183393	3.771203	3.474994	3.239685	3.042194	2.870585	2.717855	2.579527
88	4.923114	4.192745	3.781280	3.485666	3.250879	3.053864	2.882700	2.730393	2.592471
89	4.931308	4.201975	3.791225	3.496196	3.261921	3.065374	2.894647	2.742754	2.605232
90	4.939401	4.211087	3.801040	3.506585	3.272815	3.076727	2.906429	2.754944	2.617812
91	4.947393	4.220084	3.810729	3.516839	3.283565	3.087928	2.918051	2.766965	2.630218
92	4.955288	4.228968	3.820294	3.526960	3.294173	3.098980	2.929517	2.778824	2.642452
93	4.963087	4.237743	3.829739	3.536952	3.304644	3.109887	2.940830	2.790523	2.654521
94	4.970794	4.246410	3.839066	3.546818	3.314980	3.120652	2.951995	2.802066	2.666428
95	4.978409	4.254972	3.848278	3.556560	3.325186	3.131279	2.963015	2.813458	2.678176
96	4.985935	4.263431	3.857378	3.566181	3.335264	3.141772	2.973894	2.824702	2.689771
97	4.993374	4.271790	3.866368	3.575685	3.345216	3.152132	2.984634	2.835802	2.701215
98	5.000728	4.280051	3.875251	3.585074	3.355047	3.162364	2.995240	2.846761	2.712512
99	5.007998	4.288217	3.884029	3.594350	3.364759	3.172471	3.005713	2.857582	2.723666
100	5.015187	4.296289	3.892705	3.603517	3.374354	3.182455	3.016059	2.868269	2.734680

H. Guthrie, has computed tables of var w_r, accurate to within a unit in the fifth decimal place, for $r = 0$ (1) 8 and $n = (2r + 2)$ (1) 100, using the Univac Scientific (ERA 1103) computer. Since Cadwell's expression for the probability density function of w_r involves an integral with respect to another variable x, it was necessary to integrate numerically with respect to both w_r and x. A seven-point integration formula was employed for a few cases where the trapezoidal rule did not give sufficient accuracy for the integral with respect to w_r, while the trapezoidal rule was used for all cases in the integration with respect to x. The variances, which are shown in Table 2, agree with those obtained by Tippett, Pearson, and Cadwell to within a unit in the last place published by them. The values in Table 2 also agree, to within a unit in the fifth decimal place, with results computed from the Sarhan and Greenberg table of variances and covariances of order statistics for $n = 2$ (1) 20.

1.3. *Covariance of two quasi-ranges.* In order to determine the variances of unbiased estimates based on linear combinations of two quasi-ranges (and hence their efficiencies), it is necessary to know not only the variances of the two quasi-ranges but also their covariance. The covariance of the rth and r'th quasi-ranges for samples of size n is given by $\text{cov}(w_r, w_{r'}) = E(w_r w_{r'}) - E(w_r)E(w_{r'})$, in which $E(w_r w_{r'}) = 2[E(x_{r+1}x_{r'+1}) - E(x_{r+1}x_{n-r'})]$. Wilks ([21], p. 20) has given an expression for the joint probability density function of the kth and k'th order statistics. Godwin [7] has tabulated (to five decimal places) the covariances of all order statistics for samples of $n = 2$ (1) 10. The more extensive and more precise tables of Sarhan and Greenberg [17] were mentioned in the preceding paragraph. The author, again with the assistance of Eugene H. Guthrie, undertook the task of computing $\text{cov}(w_r, w_{r'})$ for $0 \leqq r < r' \leqq 8$ and $n = (2r' + 2)$ (1) 100, accurate to within a unit in the fifth decimal place, using the Univac Scientific (ERA 1103) computer. Finding that the complete tabulation required too much machine time, he decided to limit the computations to those values required to determine the most efficient estimates of the population standard deviation based on linear combinations of two adjacent quasi-ranges and of two quasi-ranges among those with $r < r' \leqq 8$ for $n = 4$ (1) 100, together with the numerical values of the efficiencies of these estimates. Wilks' expression for the joint probability density function of x_k and $x_{k'}$ was first integrated with respect to $x_{k'}$ between the limits x_k and ∞ by using a seven-point integration formula; the result was then integrated with respect to x_k between the limits $-\infty$ and ∞ by employing the trapezoidal rule. For $n = 4$ (1) 20, the expected values of products of order statistics computed in this manner agree with Teichroew's values to within a unit in the sixth decimal place and the covariances of quasi-ranges computed from these results agree to within a unit in the fifth decimal place with those computed from the covariances of order statistics tabulated by Sarhan and Greenberg.

1.4. *Unbiased estimates of population standard deivation.* The minimum variance unbiased estimate (which will hereafter be called the efficient estimate) of population standard deviation σ is the one based on the sample standard deviation s, and given by the equation $\hat{\sigma} = s/c_2$, where $s = [\sum (x - \bar{x})^2/(n - 1)]^{\frac{1}{2}}$ and $c_2 = [2/(n - 1)]^{\frac{1}{2}}\Gamma(n/2)/\Gamma[(n - 1)/2]$. The unbiased estimate of σ based on one sample quasi-range is given by the equation $\tilde{\sigma}_r = w_r/E(w_r)$ while the unbiased estimate of σ based on a linear combination of two sample quasi-ranges is given by the equation

$$\tilde{\sigma}_{r,r'} = \frac{w_r + \lambda_{r,r'} w_{r'}}{E(w_r) + \lambda_{r,r'} E(w_{r'})}, \tag{1}$$

TABLE 2

Variance of the r^{th} Quasi-Range for Samples of n from N(μ, 1)

n	r = 0	r = 1	r = 2	r = 3	r = 4	r = 5	r = 6	r = 7	r = 8
2	.72676								
3	.78920								
4	.77406	.24902							
5	.74664	.32315							
6	.71917	.34734	.12588						
7	.69423	.35350	.18023						
8	.67212	.35231	.20590	.07602					
9	.65259	.34796	.21829	.11578					
10	.63529	.34231	.22394	.13799	.05091				
11	.61986	.33619	.22594	.15081	.08089				
12	.60603	.33003	.22589	.15830	.09943	.03649			
13	.59354	.32402	.22466	.16256	.11125	.05980			
14	.58220	.31827	.22275	.16481	.11892	.07525	.02744		
15	.57185	.31280	.22045	.16576	.12391	.08577	.04604		
16	.56235	.30763	.21796	.16588	.12713	.09305	.05902	.02138	
17	.55361	.30275	.21537	.16544	.12913	.09814	.06828	.03655	
18	.54551	.29816	.21277	.16462	.13028	.10171	.07499	.04757	.01714
19	.53799	.29382	.21019	.16356	.13084	.10419	.07991	.05572	.02973
20	.53098	.28973	.20765	.16234	.13097	.10588	.08353	.06183	.03918
21	.52442	.28586	.20518	.16101	.13079	.10700	.08620	.06646	.04637
22	.51827	.28220	.20279	.15963	.13040	.10768	.08817	.07000	.05191
23	.51249	.27874	.20048	.15822	.12984	.10805	.08959	.07271	.05622
24	.50703	.27545	.19825	.15679	.12917	.10817	.09061	.07478	.05960
25	.50188	.27233	.19611	.15537	.12841	.10810	.09131	.07637	.06226
26	.49699	.26936	.19404	.15396	.12760	.10789	.09175	.07757	.06436
27	.49236	.26653	.19205	.15258	.12675	.10757	.09201	.07848	.06602
28	.48796	.26383	.19014	.15122	.12587	.10717	.09211	.07914	.06733
29	.48377	.26126	.18830	.14989	.12498	.10671	.09209	.07961	.06836
30	.47977	.25879	.18652	.14859	.12409	.10619	.09198	.07992	.06916
31	.47595	.25644	.18481	.14732	.12319	.10565	.09178	.08011	.06977
32	.47229	.25417	.18316	.14609	.12230	.10507	.09153	.08020	.07023
33	.46879	.25201	.18158	.14489	.12141	.10448	.09123	.08020	.07056
34	.46544	.24992	.18004	.14372	.12054	.10388	.09089	.08014	.07079
35	.46221	.24792	.17857	.14259	.11969	.10327	.09052	.08002	.07094

TABLE 2 (continued)

Variance of the r^{th} Quasi-Range for Samples of n from $N(\mu, 1)$

n	r = 0	r = 1	r = 2	r = 3	r = 4	r = 5	r = 6	r = 7	r = 8
36	.45910	.24599	.17714	.14148	.11884	.10265	.09013	.07985	.07101
37	.45612	.24413	.17576	.14042	.11801	.10204	.08972	.07965	.07103
38	.45325	.24234	.17443	.13937	.11719	.10142	.08930	.07941	.07099
39	.45048	.24062	.17313	.13836	.11640	.10081	.08886	.07916	.07091
40	.44780	.23894	.17188	.13738	.11562	.10021	.08842	.07888	.07080
41	.44521	.23733	.17067	.13642	.11486	.09961	.08797	.07858	.07066
42	.44271	.23577	.16950	.13549	.11411	.09902	.08753	.07827	.07049
43	.44029	.23426	.16837	.13458	.11337	.09844	.08708	.07796	.07031
44	.43794	.23280	.16726	.13370	.11266	.09786	.08663	.07763	.07010
45	.43567	.23137	.16619	.13284	.11196	.09730	.08619	.07730	.06988
46	.43347	.23000	.16515	.13201	.11128	.09674	.08574	.07696	.06965
47	.43133	.22867	.16414	.13119	.11061	.09619	.08530	.07662	.06942
48	.42925	.22736	.16316	.13040	.10996	.09565	.08486	.07629	.06917
49	.42724	.22611	.16220	.12963	.10932	.09512	.08443	.07594	.06892
50	.42527	.22488	.16127	.12888	.10869	.09460	.08401	.07560	.06866
51	.42336	.22369	.16036	.12814	.10808	.09409	.08359	.07526	.06840
52	.42151	.22253	.15948	.12742	.10748	.09359	.08317	.07492	.06813
53	.41970	.22140	.15862	.12673	.10690	.09310	.08276	.07458	.06787
54	.41794	.22030	.15778	.12604	.10633	.09262	.08235	.07425	.06760
55	.41621	.21923	.15696	.12537	.10577	.09215	.08195	.07392	.06733
56	.41454	.21819	.15616	.12473	.10522	.09168	.08156	.07359	.06707
57	.41291	.21716	.15538	.12408	.10469	.09122	.08117	.07327	.06680
58	.41131	.21617	.15462	.12346	.10417	.09078	.08079	.07294	.06653
59	.40976	.21519	.15388	.12286	.10365	.09034	.08042	.07262	.06627
60	.40823	.21424	.15315	.12226	.10315	.08991	.08005	.07231	.06600
61	.40674	.21332	.15244	.12167	.10265	.08948	.07968	.07200	.06574
62	.40528	.21241	.15175	.12110	.10217	.08907	.07932	.07169	.06547
63	.40387	.21153	.15107	.12055	.10170	.08866	.07897	.07138	.06522
64	.40247	.21066	.15040	.12000	.10123	.08826	.07862	.07108	.06496
65	.40111	.20981	.14975	.11946	.10078	.08787	.07828	.07079	.06470
66	.39978	.20898	.14911	.11894	.10033	.08748	.07794	.07050	.06445
67	.39847	.20816	.14849	.11843	.09989	.08710	.07761	.07021	.06420
68	.39719	.20737	.14788	.11792	.09947	.08673	.07729	.06993	.06396
69	.39594	.20659	.14728	.11743	.09904	.08636	.07697	.06965	.06371
70	.39471	.20583	.14669	.11694	.09863	.08600	.07665	.06937	.06347

TABLE 2 (continued)

Variance of the r^{th} Quasi-Range for Samples of n from $N(\mu, 1)$

n	r = 0	r = 1	r = 2	r = 3	r = 4	r = 5	r = 6	r = 7	r = 8
71	.39351	.20508	.14612	.11647	.09822	.08565	.07634	.06910	.06324
72	.39232	.20434	.14555	.11600	.09783	.08531	.07604	.06882	.06300
73	.39116	.20362	.14500	.11554	.09743	.08496	.07574	.06856	.06276
74	.39002	.20291	.14446	.11509	.09704	.08463	.07544	.06830	.06253
75	.38890	.20222	.14392	.11465	.09667	.08430	.07515	.06804	.06230
76	.38780	.20154	.14340	.11422	.09630	.08397	.07486	.06779	.06208
77	.38672	.20087	.14289	.11379	.09593	.08365	.07458	.06753	.06186
78	.38566	.20021	.14238	.11337	.09557	.08334	.07430	.06729	.06164
79	.38463	.19957	.14188	.11296	.09522	.08303	.07403	.06704	.06142
80	.38360	.19894	.14140	.11255	.09487	.08273	.07376	.06680	.06120
81	.38260	.19832	.14092	.11216	.09453	.08242	.07349	.06656	.06099
82	.38161	.19770	.14045	.11177	.09420	.08213	.07323	.06633	.06079
83	.38064	.19711	.13999	.11138	.09386	.08184	.07297	.06610	.06058
84	.37969	.19651	.13953	.11101	.09354	.08156	.07272	.06588	.06037
85	.37874	.19593	.13908	.11064	.09322	.08127	.07247	.06565	.06018
86	.37782	.19536	.13865	.11027	.09290	.08100	.07223	.06543	.05997
87	.37691	.19480	.13820	.10991	.09259	.08072	.07198	.06521	.05978
88	.37601	.19424	.13778	.10956	.09229	.08046	.07174	.06499	.05959
89	.37513	.19370	.13736	.10920	.09199	.08019	.07150	.06479	.05939
90	.37426	.19316	.13695	.10886	.09169	.07993	.07127	.06457	.05920
91	.37340	.19264	.13654	.10852	.09139	.07967	.07104	.06437	.05901
92	.37256	.19212	.13614	.10819	.09111	.07941	.07081	.06416	.05883
93	.37173	.19160	.13575	.10786	.09082	.07916	.07059	.06396	.05865
94	.37091	.19109	.13536	.10753	.09055	.07892	.07037	.06376	.05847
95	.37010	.19060	.13498	.10721	.09027	.07868	.07015	.06357	.05829
96	.36931	.19011	.13460	.10691	.08999	.07843	.06994	.06337	.05812
97	.36852	.18963	.13423	.10659	.08973	.07820	.06973	.06318	.05794
98	.36774	.18916	.13386	.10629	.08946	.07797	.06952	.06299	.05777
99	.36699	.18868	.13350	.10599	.08920	.07773	.06931	.06281	.05760
100	.36624	.18822	.13314	.10569	.08894	.07750	.06911	.06262	.05744

where $\lambda_{r,r'}$ is a weighting factor. In the expressions for $\tilde{\sigma}_r$ and $\tilde{\sigma}_{r,r'}$, the expected values are understood to be those for samples drawn from $N(0, 1)$, the standard normal population.

1.5. *Efficiency of unbiased estimates of σ.* The efficiency of the efficient estimate $\hat{\sigma}$ is by definition 1 (100%). The efficiency of a substitute estimate is defined as the ratio of the variance of the efficient estimate to the variance of the substitute estimate. Thus the efficiency of $\tilde{\sigma}_r$ is given by Eff $\tilde{\sigma}_r = \text{var}\,\hat{\sigma}/\text{var}\,\tilde{\sigma}_r$, while the efficiency of $\tilde{\sigma}_{r,r'}$ is given by Eff $\tilde{\sigma}_{r,r'} = \text{var}\,\hat{\sigma}/\text{var}\,\tilde{\sigma}_{r,r'}$, where $\text{var}\,\hat{\sigma} = [(1 - c_2^2)/c_2^2]\sigma^2$. By varying the weighting factor $\lambda_{r,r'}$, one may obtain a one-parameter family of unbiased estimates $\tilde{\sigma}_{r,r'}$. However, there is just one

TABLE 3

Efficiency (Percent) of Estimate of Population Standard Deviation Based on the r^{th} Quasi-Range for Samples of n from $N(\mu, \sigma^2)$

n	r = 0	r = 1	r = 2	r = 3	r = 4	r = 5	r = 6	r = 7	r = 8
2	100.00								
4	97.52	25.24							
6	93.30	49.55	13.48						
8	89.00	60.84	32.05	9.03					
10	84.99	66.80	43.83	23.33	6.74				
12	81.36	70.07	51.69	33.82	18.21	5.36			
14	78.09	71.83	57.12	41.64	27.34	14.88	4.44		
16	75.13	72.69	60.95	47.57	34.63	22.86	12.54	3.78	
18	72.46	72.98	63.70	52.14	40.48	29.52	19.58	10.82	3.30
20	70.02	72.91	65.67	55.70	45.23	35.09	25.65	17.10	9.51
22	67.80	72.59	67.08	58.51	49.11	39.76	30.88	22.64	15.16
24	65.75	72.11	68.07	60.73	52.31	43.72	35.38	27.52	20.24
26	63.86	71.52	68.76	62.49	54.96	47.07	39.28	31.81	24.78
28	62.12	70.86	69.20	63.90	57.17	49.94	42.66	35.59	28.85
30	60.49	70.15	69.46	65.01	59.01	52.39	45.61	38.93	32.49
32	58.98	69.41	69.57	65.90	60.56	54.50	48.19	41.89	35.74
34	57.56	68.66	69.57	66.59	61.86	56.32	50.45	44.51	38.66
36	56.23	67.89	69.48	67.12	62.95	57.89	52.44	46.85	41.29
38	54.98	67.13	69.32	67.53	63.86	59.26	54.19	48.93	43.66
40	53.81	66.38	69.10	67.82	64.63	60.44	55.74	50.80	45.79
42	52.70	65.63	68.83	68.03	65.27	61.46	57.11	52.47	47.72
44	51.64	64.89	68.53	68.16	65.80	62.35	58.32	53.97	49.47
46	50.64	64.17	68.20	68.22	66.23	63.13	59.39	55.31	51.05
48	49.70	63.46	67.84	68.23	66.59	63.80	60.35	56.52	52.49
50	48.79	62.76	67.47	68.20	66.87	64.38	61.20	57.61	53.79
55	46.71	61.08	66.49	67.95	67.33	65.50	62.92	59.89	56.58
60	44.85	59.50	65.46	67.53	67.51	66.25	64.20	61.66	58.79
65	43.18	58.01	64.42	67.00	67.49	66.71	65.14	63.02	60.55
70	41.65	56.60	63.38	66.39	67.32	66.98	65.80	64.07	61.96
75	40.26	55.27	62.36	65.74	67.04	67.07	66.26	64.88	63.08
80	38.99	54.02	61.36	65.05	66.69	67.04	66.56	65.47	63.97
85	37.81	52.84	60.38	64.35	66.27	66.92	66.72	65.91	64.67
90	36.73	51.72	59.43	63.63	65.81	66.72	66.77	66.22	65.21
95	35.72	50.66	58.51	62.92	65.33	66.46	66.74	66.41	65.62
100	34.77	49.65	57.62	62.21	64.82	66.16	66.65	66.52	65.93

value of $\lambda_{r,r'}$ which minimizes $V_{r,r'} = \text{var } \tilde{\sigma}_{r,r'}$, and hence maximizes Eff $\tilde{\sigma}_{r,r'}$. This value of $\lambda_{r,r'}$ which maximizes the efficiency of the estimate may be obtained by setting $dV_{r,r'}/d\lambda_{r,r'} = 0$ and solving for $\lambda_{r,r'}$, which yields

$$(2) \qquad \lambda_{r,r'} = \frac{E(w_r) \text{ var } w_{r'} - E(w_{r'}) \text{ cov } (w_r, w_{r'})}{E(w_{r'}) \text{ var } w_r - E(w_r) \text{ cov } (w_r, w_{r'})}.$$

Table 3 shows the efficiency of estimates based on w_r for $r = 0\,(1)\,8$ and $n = (2r + 2)\,(2)\,50\,(5)\,100$, accurate to within 0.01%. Table 4 gives the

TABLE 4

Most Efficient Unbiased Estimates of Standard Deviation of Normal Population

Sample size, n	Based on one quasi-range: Estimate	Eff(%)	Based on a linear combination of two adjacent quasi-ranges: Estimate	Eff (%)	Based on a linear combination of two quasi-ranges among those with $r < r' \le 8$: Estimate	Eff(%)	Efficient Estimate
2	$.886227\, w_0$	100.00					$s/.797885$
3	$.590818\, w_0$	99.19					$s/.886227$
4	$.485731\, w_0$	97.52	$.45394\,(w_0 + 0.2427\, w_1)$	98.92	$.45394\,(w_0 + 0.2427\, w_1)$	98.92	$s/.921318$
5	$.429936\, w_0$	95.48	$.37238\,(w_0 + 0.3631\, w_1)$	98.84	$.37238\,(w_0 + 0.3631\, w_1)$	98.84	$s/.939986$
6	$.394569\, w_0$	93.30	$.31803\,(w_0 + 0.4752\, w_1)$	98.66	$.31803\,(w_0 + 0.4752\, w_1)$	98.66	$s/.951533$
7	$.369774\, w_0$	91.12	$.27922\,(w_0 + 0.5790\, w_1)$	98.32	$.27922\,(w_0 + 0.5790\, w_1)$	98.32	$s/.959369$
8	$.351222\, w_0$	89.00	$.25010\,(w_0 + 0.6754\, w_1)$	97.84	$.25010\,(w_0 + 0.6754\, w_1)$	97.84	$s/.965030$
9	$.336697\, w_0$	86.95	$.22745\,(w_0 + 0.7651\, w_1)$	97.23	$.22745\,(w_0 + 0.7651\, w_1)$	97.23	$s/.969311$
10	$.324939\, w_0$	84.99	$.20931\,(w_0 + 0.8489\, w_1)$	96.54	$.20931\,(w_0 + 0.8489\, w_1)$	96.54	$s/.972659$
11	$.315172\, w_0$	83.13	$.19444\,(w_0 + 0.9276\, w_1)$	95.78	$.19444\,(w_0 + 0.9276\, w_1)$	95.78	$s/.975350$
12	$.306894\, w_0$	81.36	$.18203\,(w_0 + 1.0017\, w_1)$	94.97	$.21177\,(w_0 + 0.9231\, w_2)$	95.17	$s/.977559$
13	$.299762\, w_0$	79.68	$.17150\,(w_0 + 1.0717\, w_1)$	94.12	$.19848\,(w_0 + 1.0015\, w_2)$	95.00	$s/.979406$
14	$.293534\, w_0$	78.09	$.16244\,(w_0 + 1.1381\, w_1)$	93.26	$.18704\,(w_0 + 1.0762\, w_2)$	94.77	$s/.980971$
15	$.288033\, w_0$	76.57	$.15457\,(w_0 + 1.2011\, w_1)$	92.39	$.17708\,(w_0 + 1.1477\, w_2)$	94.50	$s/.982316$
16	$.283127\, w_0$	75.13	$.14765\,(w_0 + 1.2612\, w_1)$	91.52	$.16834\,(w_0 + 1.2161\, w_2)$	94.18	$s/.983484$
17	$.278716\, w_0$	73.76	$.14153\,(w_0 + 1.3186\, w_1)$	90.65	$.16060\,(w_0 + 1.2817\, w_2)$	93.82	$s/.984506$
18	$.370257\, w_1$	72.98	$.13606\,(w_0 + 1.3736\, w_1)$	89.78	$.15369\,(w_0 + 1.3448\, w_2)$	93.43	$s/.985410$
19	$.362335\, w_1$	72.98	$.13114\,(w_0 + 1.4263\, w_1)$	88.92	$.14750\,(w_0 + 1.4055\, w_2)$	93.02	$s/.986214$
20	$.355214\, w_1$	72.91	$.12670\,(w_0 + 1.4769\, w_1)$	88.08	$.14192\,(w_0 + 1.4640\, w_2)$	92.59	$s/.986934$
21	$.348768\, w_1$	72.77	$.12266\,(w_0 + 1.5256\, w_1)$	87.24	$.13684\,(w_0 + 1.5206\, w_2)$	92.14	$s/.987583$
22	$.342899\, w_1$	72.59	$.11897\,(w_0 + 1.5726\, w_1)$	86.42	$.14637\,(w_0 + 1.5298\, w_3)$	91.78	$s/.988170$
23	$.337526\, w_1$	72.37	$.11558\,(w_0 + 1.6179\, w_1)$	85.62	$.14129\,(w_0 + 1.5881\, w_3)$	91.61	$s/.988705$
24	$.332584\, w_1$	72.11	$.11246\,(w_0 + 1.6617\, w_1)$	84.83	$.13663\,(w_0 + 1.6446\, w_3)$	91.42	$s/.989193$
25	$.328019\, w_1$	71.82	$.10958\,(w_0 + 1.7040\, w_1)$	84.05	$.13233\,(w_0 + 1.6996\, w_3)$	91.21	$s/.989640$
26	$.323785\, w_1$	71.52	$.10691\,(w_0 + 1.7451\, w_1)$	83.29	$.12836\,(w_0 + 1.7529\, w_3)$	90.98	$s/.990052$
27	$.319844\, w_1$	71.20	$.10442\,(w_0 + 1.7849\, w_1)$	82.54	$.12468\,(w_0 + 1.8050\, w_3)$	90.73	$s/.990433$
28	$.316165\, w_1$	70.86	$.10209\,(w_0 + 1.8235\, w_1)$	81.81	$.12126\,(w_0 + 1.8556\, w_3)$	90.48	$s/.990786$
29	$.312719\, w_1$	70.51	$.09992\,(w_0 + 1.8610\, w_1)$	81.10	$.11807\,(w_0 + 1.9050\, w_3)$	90.21	$s/.991113$
30	$.309483\, w_1$	70.15	$.09788\,(w_0 + 1.8975\, w_1)$	80.40	$.11508\,(w_0 + 1.9532\, w_3)$	89.93	$s/.991418$
31	$.306436\, w_1$	69.78	$.09596\,(w_0 + 1.9329\, w_1)$	79.71	$.11229\,(w_0 + 2.0002\, w_3)$	89.63	$s/.991703$
32	$.357181\, w_2$	69.57	$.09416\,(w_0 + 1.9675\, w_1)$	79.04	$.10967\,(w_0 + 2.0461\, w_3)$	89.35	$s/.991969$
33	$.353016\, w_2$	69.58	$.09245\,(w_0 + 2.0011\, w_1)$	78.38	$.11551\,(w_0 + 2.0673\, w_4)$	89.11	$s/.992219$
34	$.349094\, w_2$	69.57	$.14484\,(w_1 + 1.2398\, w_2)$	78.08	$.11282\,(w_0 + 2.1150\, w_4)$	88.97	$s/.992454$
35	$.345394\, w_2$	69.53	$.14201\,(w_1 + 1.2646\, w_2)$	77.82	$.11028\,(w_0 + 2.1616\, w_4)$	88.82	$s/.992675$
36	$.341895\, w_2$	69.48	$.13934\,(w_1 + 1.2888\, w_2)$	77.55	$.10788\,(w_0 + 2.2073\, w_4)$	88.66	$s/.992884$
37	$.338580\, w_2$	69.41	$.13680\,(w_1 + 1.3126\, w_2)$	77.27	$.10561\,(w_0 + 2.2520\, w_4)$	88.48	$s/.993080$
38	$.335433\, w_2$	69.32	$.13440\,(w_1 + 1.3358\, w_2)$	76.99	$.10345\,(w_0 + 2.2962\, w_4)$	88.31	$s/.993267$
39	$.332442\, w_2$	69.21	$.13211\,(w_1 + 1.3586\, w_2)$	76.71	$.10141\,(w_0 + 2.3393\, w_4)$	88.12	$s/.993443$
40	$.329593\, w_2$	69.10	$.12995\,(w_1 + 1.3806\, w_2)$	76.42	$.09947\,(w_0 + 2.3816\, w_4)$	87.92	$s/.993611$
41	$.326875\, w_2$	68.97	$.12788\,(w_1 + 1.4026\, w_2)$	76.12	$.09762\,(w_0 + 2.4232\, w_4)$	87.73	$s/.993770$
42	$.324280\, w_2$	68.83	$.12592\,(w_1 + 1.4236\, w_2)$	75.82	$.09586\,(w_0 + 2.4640\, w_4)$	87.52	$s/.993922$
43	$.321798\, w_2$	68.68	$.12403\,(w_1 + 1.4448\, w_2)$	75.53	$.09418\,(w_0 + 2.5042\, w_4)$	87.32	$s/.994066$
44	$.319420\, w_2$	68.53	$.12223\,(w_1 + 1.4652\, w_2)$	75.23	$.09258\,(w_0 + 2.5435\, w_4)$	87.10	$s/.994203$
45	$.317141\, w_2$	68.37	$.12051\,(w_1 + 1.4853\, w_2)$	74.92	$.09647\,(w_0 + 2.5741\, w_5)$	86.97	$s/.994335$
46	$.352208\, w_3$	68.22	$.11886\,(w_1 + 1.5051\, w_2)$	74.62	$.09482\,(w_0 + 2.6150\, w_5)$	86.85	$s/.994460$
47	$.349387\, w_3$	68.24	$.11727\,(w_1 + 1.5246\, w_2)$	74.32	$.09323\,(w_0 + 2.6553\, w_5)$	86.73	$s/.994580$
48	$.346682\, w_3$	68.23	$.15270\,(w_2 + 1.1550\, w_3)$	74.15	$.09171\,(w_0 + 2.6951\, w_5)$	86.60	$s/.994695$
49	$.344086\, w_3$	68.22	$.15056\,(w_2 + 1.1714\, w_3)$	74.03	$.09025\,(w_0 + 2.7341\, w_5)$	86.47	$s/.994806$
50	$.341592\, w_3$	68.20	$.14850\,(w_2 + 1.1877\, w_3)$	73.90	$.08885\,(w_0 + 2.7726\, w_5)$	86.33	$s/.994911$

Table 4 (continued)

Most Efficient Unbiased Estimates of Standard Deviation of Normal Population

Sample size, n	Based on one quasi-range		Based on a linear combination of two adjacent quasi-ranges		Based on a linear combination of two quasi-ranges among those with $r < r' \le 8$		Efficient Estimate
	Estimate	Eff(%)	Estimate	Eff(%)	Estimate	Eff(%)	
51	.339192 w_3	68.17	.14651 $(w_2 + 1.2037\, w_3)$	73.77	.08751 $(w_0 + 2.8105\, w_5)$	86.19	s/.995013
52	.336882 w_3	68.12	.14460 $(w_2 + 1.2195\, w_3)$	73.63	.08621 $(w_0 + 2.8479\, w_5)$	86.04	s/.995110
53	.334656 w_3	68.07	.14278 $(w_2 + 1.2348\, w_3)$	73.48	.08497 $(w_0 + 2.8847\, w_5)$	85.89	s/.995204
54	.332509 w_3	68.02	.14100 $(w_2 + 1.2504\, w_3)$	73.33	.08377 $(w_0 + 2.9212\, w_5)$	85.73	s/.995294
55	.330436 w_3	67.95	.13931 $(w_2 + 1.2652\, w_3)$	73.18	.08262 $(w_0 + 2.9567\, w_5)$	85.57	s/.995381
56	.328434 w_3	67.87	.13763 $(w_2 + 1.2805\, w_3)$	73.02	.13787 $(w_1 + 1.6820\, w_8)$	85.44	s/.995465
57	.326498 w_3	67.80	.13606 $(w_2 + 1.2949\, w_3)$	72.86	.13592 $(w_1 + 1.7062\, w_8)$	85.46	s/.995546
58	.324625 w_3	67.71	.13453 $(w_2 + 1.3093\, w_3)$	72.70	.13403 $(w_1 + 1.7303\, w_8)$	85.47	s/.995624
59	.322812 w_3	67.62	.13304 $(w_2 + 1.3236\, w_3)$	72.53	.13222 $(w_1 + 1.7539\, w_8)$	85.48	s/.995699
60	.321055 w_3	67.53	.13161 $(w_2 + 1.3374\, w_3)$	72.36	.13046 $(w_1 + 1.7774\, w_8)$	85.48	s/.995772
61	.347463 w_4	67.52	.13022 $(w_2 + 1.3514\, w_3)$	72.19	.12875 $(w_1 + 1.8006\, w_8)$	85.47	s/.995842
62	.345399 w_4	67.52	.15710 $(w_3 + 1.1114\, w_4)$	72.06	.12710 $(w_1 + 1.8236\, w_8)$	85.46	s/.995910
63	.343400 w_4	67.52	.15534 $(w_3 + 1.1243\, w_4)$	71.99	.12551 $(w_1 + 1.8461\, w_8)$	85.44	s/.995976
64	.341461 w_4	67.50	.15370 $(w_3 + 1.1360\, w_4)$	71.91	.12397 $(w_1 + 1.8684\, w_8)$	85.42	s/.996040
65	.339581 w_4	67.49	.15210 $(w_3 + 1.1478\, w_4)$	71.83	.12247 $(w_1 + 1.8907\, w_8)$	85.40	s/.996102
66	.337755 w_4	67.46	.15052 $(w_3 + 1.1598\, w_4)$	71.75	.12102 $(w_1 + 1.9125\, w_8)$	85.36	s/.996161
67	.335982 w_4	67.44	.14895 $(w_3 + 1.1723\, w_4)$	71.67	.11961 $(w_1 + 1.9342\, w_8)$	85.33	s/.996219
68	.334260 w_4	67.40	.14750 $(w_3 + 1.1835\, w_4)$	71.57	.11826 $(w_1 + 1.9554\, w_8)$	85.29	s/.996276
69	.332585 w_4	67.36	.14607 $(w_3 + 1.1949\, w_4)$	71.48	.11693 $(w_1 + 1.9765\, w_8)$	85.24	s/.996330
70	.330956 w_4	67.32	.14466 $(w_3 + 1.2064\, w_4)$	71.39	.11564 $(w_1 + 1.9975\, w_8)$	85.20	s/.996383
71	.329370 w_4	67.27	.14333 $(w_3 + 1.2172\, w_4)$	71.29	.11439 $(w_1 + 2.0181\, w_8)$	85.14	s/.996435
72	.327827 w_4	67.22	.14201 $(w_3 + 1.2284\, w_4)$	71.18	.11317 $(w_1 + 2.0387\, w_8)$	85.09	s/.996485
73	.326323 w_4	67.16	.14071 $(w_3 + 1.2395\, w_4)$	71.08	.11198 $(w_1 + 2.0593\, w_8)$	85.04	s/.996534
74	.324857 w_4	67.11	.13943 $(w_3 + 1.2510\, w_4)$	70.98	.11083 $(w_1 + 2.0793\, w_8)$	84.98	s/.996581
75	.346274 w_5	67.07	.13821 $(w_3 + 1.2617\, w_4)$	70.87	.10971 $(w_1 + 2.0992\, w_8)$	84.91	s/.996627

Sample size, n	Based on one quasi-range		Based on a linear combination of two adjacent quasi-ranges		Based on a linear combination of two quasi-ranges among those with $r < r' \le 8$		Efficient Estimate
	Estimate	Eff(%)	Estimate	Eff(%)	Estimate	Eff(%)	
76	.344605 w_5	67.08	.13703 $(w_3 + 1.2723\, w_4)$	70.75	.10862 $(w_1 + 2.1188\, w_8)$	84.85	s/.996672
77	.342979 w_5	67.07	.15847 $(w_4 + 1.0948\, w_5)$	70.71	.10756 $(w_1 + 2.1382\, w_8)$	84.78	s/.996716
78	.341393 w_5	67.07	.15704 $(w_4 + 1.1049\, w_5)$	70.66	.10653 $(w_1 + 2.1575\, w_8)$	84.71	s/.996759
79	.339847 w_5	67.06	.15568 $(w_4 + 1.1145\, w_5)$	70.61	.10552 $(w_1 + 2.1766\, w_8)$	84.63	s/.996800
80	.338338 w_5	67.04	.15431 $(w_4 + 1.1245\, w_5)$	70.56	.10453 $(w_1 + 2.1956\, w_8)$	84.56	s/.996841
81	.336865 w_5	67.03	.15305 $(w_4 + 1.1334\, w_5)$	70.50	.10357 $(w_1 + 2.2145\, w_8)$	84.48	s/.996880
82	.335427 w_5	67.01	.15176 $(w_4 + 1.1432\, w_5)$	70.44	.10264 $(w_1 + 2.2328\, w_8)$	84.40	s/.996918
83	.334022 w_5	66.98	.15056 $(w_4 + 1.1519\, w_5)$	70.38	.10171 $(w_1 + 2.2515\, w_8)$	84.32	s/.996956
84	.332649 w_5	66.95	.14934 $(w_4 + 1.1612\, w_5)$	70.31	.10082 $(w_1 + 2.2696\, w_8)$	84.23	s/.996993
85	.331306 w_5	66.92	.14809 $(w_4 + 1.1714\, w_5)$	70.25	.09996 $(w_1 + 2.2875\, w_8)$	84.15	s/.997028
86	.329994 w_5	66.89	.14697 $(w_4 + 1.1800\, w_5)$	70.18	.09910 $(w_1 + 2.3057\, w_8)$	84.07	s/.997063
87	.328710 w_5	66.85	.14584 $(w_4 + 1.1890\, w_5)$	70.11	.09826 $(w_1 + 2.3234\, w_8)$	83.97	s/.997097
88	.327454 w_5	66.81	.14475 $(w_4 + 1.1977\, w_5)$	70.03	.09746 $(w_1 + 2.3406\, w_8)$	83.88	s/.997131
89	.345465 w_6	66.77	.14363 $(w_4 + 1.2071\, w_5)$	69.96	.09665 $(w_1 + 2.3585\, w_8)$	83.79	s/.997163
90	.344065 w_6	66.77	.14259 $(w_4 + 1.2156\, w_5)$	69.88	.09588 $(w_1 + 2.3755\, w_8)$	83.70	s/.997195
91	.342694 w_6	66.77	.16064 $(w_5 + 1.0751\, w_6)$	69.82	.09511 $(w_1 + 2.3930\, w_8)$	83.61	s/.997226
92	.341353 w_6	66.77	.15934 $(w_5 + 1.0844\, w_6)$	69.79	.09437 $(w_1 + 2.4096\, w_8)$	83.51	s/.997257
93	.340040 w_6	66.76	.15823 $(w_5 + 1.0916\, w_6)$	69.75	.09364 $(w_1 + 2.4264\, w_8)$	83.42	s/.997286
94	.338754 w_6	66.75	.15705 $(w_5 + 1.0999\, w_6)$	69.71	.09293 $(w_1 + 2.4431\, w_8)$	83.32	s/.997315
95	.337494 w_6	66.74	.15592 $(w_5 + 1.1078\, w_6)$	69.67	.09223 $(w_1 + 2.4595\, w_8)$	83.22	s/.997344
96	.336259 w_6	66.73	.15480 $(w_5 + 1.1158\, w_6)$	69.63	.09155 $(w_1 + 2.4761\, w_8)$	83.12	s/.997371
97	.335049 w_6	66.71	.15366 $(w_5 + 1.1244\, w_6)$	69.59	.09087 $(w_1 + 2.4924\, w_8)$	83.02	s/.997399
98	.333863 w_6	66.69	.15261 $(w_5 + 1.1319\, w_6)$	69.54	.09022 $(w_1 + 2.5085\, w_8)$	82.92	s/.997426
99	.332700 w_6	66.67	.15161 $(w_5 + 1.1389\, w_6)$	69.49	.08957 $(w_1 + 2.5245\, w_8)$	82.82	s/.997452
100	.331559 w_6	66.65	.15059 $(w_5 + 1.1466\, w_6)$	69.44	.08895 $(w_1 + 2.5401\, w_8)$	82.71	s/.997478

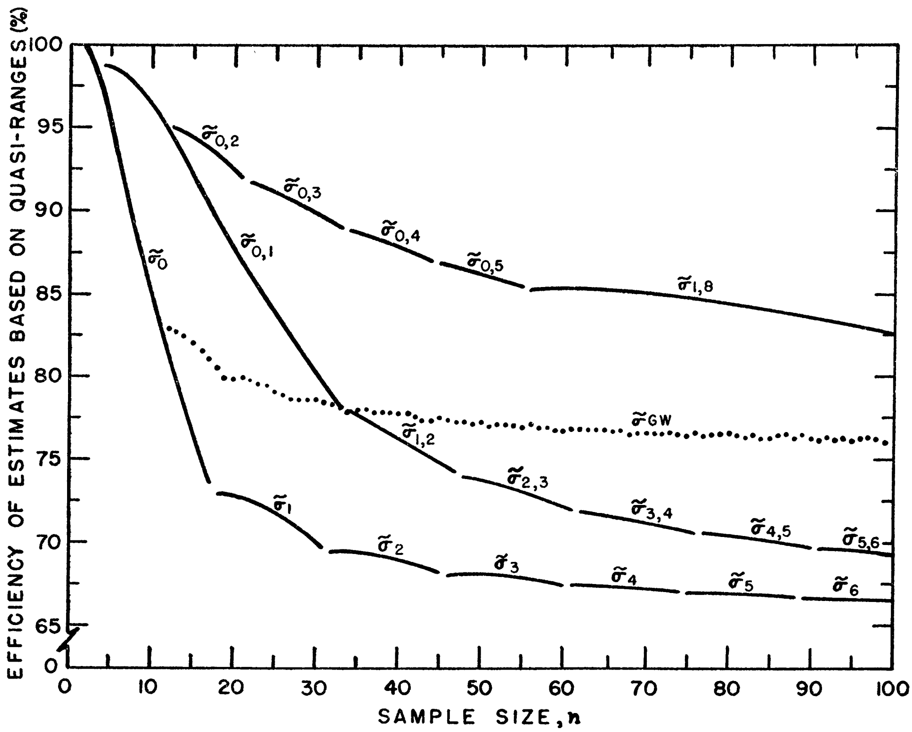

FIG. 1. Efficiency of Estimates of Standard Deviation for Normal Population

most efficient estimate based on one sample quasi-range, together with its efficiency, for $n = 2\ (1)\ 100$, also the most efficient estimates based on linear combinations of two adjacent quasi-ranges and of two quasi-ranges among those with $r < r' \leqq 8$, together with their efficiencies, for $n = 4\ (1)\ 100$, and the efficient estimate based on the sample standard deviation. For the estimates based on one sample quasi-range, the numerical coefficients $1/E(w_r)$ are accurate to within a unit in the sixth decimal place, and the efficiencies are accurate to within 0.01%. For the estimates based on a linear combination of two sample quasi-ranges, the numerical coefficients $1/E(w_r + \lambda_{r,r'}w_{r'})$ are accurate to within a unit in the fourth decimal place, the values of $\lambda_{r,r'}$ are accurate to within a unit in the third decimal place, and the efficiencies are accurate to within 0.01%. The efficiency of the estimates based on quasi-ranges is shown graphically by Figure 1. It will be noted that for $n < 56$ the estimate based on the best linear combination of two quasi-ranges among those with $r < r' \leqq 8$ always involves the range ($r = 0$), with $1 \leqq r' \leqq 5$ while the best such estimate for $56 \leqq n \leqq 100$ is $\tilde{\sigma}_{1,8}$, with the efficiency dropping to 82.71% for $n = 100$. It seems likely that slightly better estimates for n near 100 could be obtained by dropping the restriction $r' \leqq 8$, but it is doubtful whether the increase in efficiency would exceed 1%, which would hardly justify the additional computation of expected values, variances, and covariances required to obtain such estimates. One may wonder why the estimate based on the best linear combination of two quasi-ranges is not the one based on the two quasi-ranges which do best individually. The reason (see also K. Pearson [15]) is that these two quasi-ranges, which are certainly adjacent ones, are too highly correlated to do best together.

1.6. *Comparison with Grubbs-Weaver estimates.* Grubbs and Weaver [8] have proposed estimates of the population standard deviation based on a weighted average of the ranges of random subgroups of the complete sample. Since the optimum size of such subgroups is 8, the sample is divided into subgroups which are as nearly as possible of size 8. If the sample size is an integral multiple of 8, all subgroups are of size 8; otherwise, some subgroups are not of size 8, since no observations are discarded. The Grubbs-Weaver estimate is always more efficient than the estimate $\tilde{\sigma}_r$ based on one sample quasi-range, except for $n < 12$, when it is identical with $\tilde{\sigma}_r$, both using the range of the complete sample, and always less efficient than the estimate based on the best linear combination of two quasi-ranges. The efficiency of the Grubbs-Weaver estimate $\tilde{\sigma}_{GW}$ for sample sizes up through 100 is shown in Figure 1 along with the efficiencies of the estimates based on quasi-ranges. The irregularities in $\tilde{\sigma}_{GW}$ are due partly to the inherent nature of the estimates and partly to the fact that the number of decimal places carried by Grubbs and Weaver is sufficient to yield values of the efficiency accurate only to within about 0.1%. The asymptotic efficiency of the Grubbs-Weaver estimate is 75.38%, since for samples of size 8, var w_0 = .67212 and $E(w_0) = 2.8472$, so that var $\tilde{\sigma}_0 = .08291$, and hence the variance of $\tilde{\sigma}_{GW}$, which is the mean of $n/8$ such estimates, is $.08291/(n/8) = .6633/n$, as compared with an asymptotic variance $1/2n = .5/n$ for $\hat{\sigma}$. By using results given by K. Pearson [15] and by Benson [1], one can easily show that the corresponding asymptotic efficiencies are 65.23% for estimates based on one quasi-range and also for those based on the best linear combination of two adjacent quasi-ranges, and approximately 80.08% for estimates based on the best linear combination of two quasi-ranges.

1.7. *Example.* As an example of the use of estimates based on sample quasi-ranges, consider the following data, given by Morse and Kimball ([10], p. 134) and assumed to come from a normal population, which represent the deviation (in one dimension) from the aiming point of the mean point of impact of salvos of two projectiles:

−237	−23	Quasi-ranges:
−133	−13	$w_0 = 270 - (-237) = 507$
−93	−10	$w_1 = 209 - (-133) = 342$
−77	57	$w_2 = 173 - (-93) = 266$
−75	65	Sample standard deviation: $s = 127.2$
−70	142	Estimates of population standard deviation:
−66	154	$\tilde{\sigma}_1 = .355214\, w_1 = 121.5$
−65	173	$\tilde{\sigma}_{0,1} = .12670\,(w_0 + 1.4769\, w_1) = 128.2$
−34	209	$\tilde{\sigma}_{0,2} = .14192\,(w_0 + 1.4640\, w_1) = 127.2$
−28	270	$\hat{\sigma} = s/.986934 = 128.9$

Morse and Kimball plotted the data on normal probability paper, fitted a straight line "by eye", and estimated the standard deviation as the difference between the 84% point and the 50% point, the result being 161, a value nearly 25% greater than the efficient estimate. A much better result could have been obtained by using an estimate based on a single quasi-range, and a still better one by using an estimate based on a linear combination of two quasi-ranges. It is also easier to arrange the data in order and make the simple quasi-range calculations shown above than to plot the data on normal probability paper (though one may want to do the latter for other reasons). Moreover, it is really not necessary to arrange all the data in order; it would suffice in this example to pick out the three largest and the three smallest values.

2. Estimates of σ for non-normal populations.

2.1. *Rectangular population.* For the standard rectangular population (mean zero and variance one), the probability density function is $f(x) = 1/2\sqrt{3}$, $-\sqrt{3} \leqq x \leqq \sqrt{3}$. It can easily be shown (see Cramér [3], p. 372) that the expected value and the variance of w_r are $E(w_r) = 2\sqrt{3}(n - 2r - 1)/(n + 1)$ and $\operatorname{var} w_r = 12(2r + 2)(n - 2r - 1)/(n + 1)^2(n + 2)$. An unbiased estimate of the standard deviation of a rectangular population is given by $\tilde{\sigma}_r = w_r/E(w_r)$, where $E(w_r)$ is understood to be taken for the standard rectangular population. The variance of $\tilde{\sigma}_r$ is $\operatorname{var} \tilde{\sigma}_r = (2r + 2)/[(n + 2)(n - 2r - 1)]$. It is evident that the range is more efficient than any of the quasi-ranges for estimating σ, since increasing r both increases the numerator and decreases the denominator of the expression for $\operatorname{var} \tilde{\sigma}_r$. As a matter of fact, it can be shown that the range is an efficient statistic for estimating the standard deviation of a rectangular population. Table 5 gives the unbiased estimates $\tilde{\sigma}_0$ for $n = 2\ (1)\ 100$. The numerical coefficients $1/E(w_0)$ are accurate to within a unit in the sixth decimal place. Since the efficiency is always 100%, it is not given in the table.

2.2. *Exponential population.* For the exponential population with mean and variance each equal to one, the probability density function is $f(x) = e^{-x}$, $0 \leqq x < \infty$. Rider [16] has shown that the expected value and the variance of w_r for samples of n from this population are

$$E(w_r) = \sum_{j=r+1}^{n-r-1} \frac{1}{j} \tag{3}$$

and

$$\operatorname{var} w_r = \sum_{j=r+1}^{n-r-1} \frac{1}{j^2}. \tag{4}$$

An unbiased estimate of σ for an exponential population is $\tilde{\sigma}_r = w_r/E(w_r)$ where $E(w_r)$ is understood to be taken for an exponential population with variance one. The variance of $\tilde{\sigma}_r$ is

$$\operatorname{var} \tilde{\sigma}_r = \sum_{j=r+1}^{n-r-1} \frac{1}{j^2} \bigg/ \left(\sum_{j=r+1}^{n-r-1} \frac{1}{j} \right)^2. \tag{5}$$

TABLE 5

ESTIMATES OF STANDARD DEVIATION OF RECTANGULAR POPULATION

Sample size, n	Estimate based on the range	Sample size, n	Estimate based on the range	Sample size, n	Estimate based on the range
		36	.305171 w_o	71	.296923 w_o
2	.866025 w_o	37	.304713 w_o	72	.296807 w_o
3	.577350 w_o	38	.304279 w_o	73	.296694 w_o
4	.481125 w_o	39	.303869 w_o	74	.296584 w_o
5	.433013 w_o	40	.303479 w_o	75	.296477 w_o
6	.404145 w_o	41	.303109 w_o	76	.296373 w_o
7	.384900 w_o	42	.302757 w_o	77	.296272 w_o
8	.371154 w_o	43	.302422 w_o	78	.296173 w_o
9	.360844 w_o	44	.302102 w_o	79	.296077 w_o
10	.352825 w_o	45	.301797 w_o	80	.295983 w_o
11	.346410 w_o	46	.301505 w_o	81	.295892 w_o
12	.341162 w_o	47	.301226 w_o	82	.295803 w_o
13	.336788 w_o	48	.300959 w_o	83	.295716 w_o
14	.333087 w_o	49	.300703 w_o	84	.295631 w_o
15	.329914 w_o	50	.300458 w_o	85	.295548 w_o
16	.327165 w_o	51	.300222 w_o	86	.295467 w_o
17	.324760 w_o	52	.299996 w_o	87	.295389 w_o
18	.322637 w_o	53	.299778 w_o	88	.295311 w_o
19	.320750 w_o	54	.299569 w_o	89	.295236 w_o
20	.319062 w_o	55	.299367 w_o	90	.295162 w_o
21	.317543 w_o	56	.299172 w_o	91	.295090 w_o
22	.316168 w_o	57	.298985 w_o	92	.295020 w_o
23	.314918 w_o	58	.298804 w_o	93	.294951 w_o
24	.313777 w_o	59	.298629 w_o	94	.294883 w_o
25	.312731 w_o	60	.298461 w_o	95	.294817 w_o
26	.311769 w_o	61	.298298 w_o	96	.294753 w_o
27	.310881 w_o	62	.298140 w_o	97	.294689 w_o
28	.310058 w_o	63	.297987 w_o	98	.294627 w_o
29	.309295 w_o	64	.297839 w_o	99	.294566 w_o
30	.308584 w_o	65	.297696 w_o	100	.294507 w_o
31	.307920 w_o	66	.297557 w_o		
32	.307299 w_o	67	.297423 w_o		
33	.306717 w_o	68	.297292 w_o		
34	.306171 w_o	69	.297166 w_o		
35	.305656 w_o	70	.297043 w_o		

For the exponential population with mean and standard deviation each equal to c, the probability density function is $f(x) = (1/c)e^{-x/c}$, $0 \leqq x < \infty$. The sample mean $\bar{x}$ is the efficient estimate of the parameter c, and has variance c^2/n. When $c = 1$, var $\bar{x} = 1/n$. Thus the efficiency, for an exponential population whose lower limit is zero (or some other known value x_0), of the estimate $\tilde{\sigma}_r$ based on the rth quasi-range is given by the ratio of the variance of the efficient estimate $\bar{x}$(or $\bar{x} - x_0$) to the variance of $\tilde{\sigma}_r$, that is by

$$\text{Eff } \tilde{\sigma}_r = \left(\sum_{j=r+1}^{n-r-1} \frac{1}{j}\right)^2 \Big/ n \sum_{j=r+1}^{n-r-1} \frac{1}{j^2}. \tag{6}$$

The most efficient estimates $\tilde{\sigma}_r$, together with their efficiencies, have been computed for $n = 2\ (1)\ 100$, but they will not be tabulated here, since the efficiencies are somewhat disappointing, varying from 50.00% to 61.73%. It should not be surprising that quasi-ranges, which are differences of symmetric order statistics, are not very efficient in estimating the standard deviation of an unsymmetric population. It is interesting, however, to note that the standard deviation of an exponential population whose lower limit is known can be estimated more efficiently from a single order statistic. The author is currently investigating the efficiency of estimates based on a linear combination of two order statistics, and preliminary results look promising,

2.3. *Bias when estimates which assume normality are used.* Paragraphs 2.1 and 2.2 cover the cases in which the population being sampled is known to be rectangular or exponential. Suppose, however, that the population is of one or the other of these two types, but the investigator who is interested in estimating the standard deviation is not aware of this fact, and proceeds to use one of the estimates which assume normality. In this case, the estimate is no longer unbiased. The bias of an estimate, based on one sample quasi-range, which assumes normality, when the population being sampled is actually of some other type, is given by

$$B_0 = [E_0(w_r) - E_n(w_r)]/E_n(w_r). \tag{7}$$

The bias of an estimate, based on a linear combination of two sample quasi-ranges, which assumes normality, when the population being sampled is actually of some other type, is given by

$$B_0 = \frac{[E_0(w_r) + \lambda_{r,r'} E_0(w_{r'})] - [E_n(w_r) + \lambda_{r,r'} E_n(w_{r'})]}{E_n(w_r) + \lambda_{r,r'} E_n(w_{r'})}. \tag{8}$$

In equations (7) and (8), E_n represents an expected value taken for the normal population, while E_0 represents an expected value taken for the other population, both populations having variance one. Table 6 gives the bias B_r for a rectangular population and the bias B_e for an exponential population when the estimates of Table 4, which assume normality, are used. In both cases, the values of the bias are accurate to within 0.01%.

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TABLE 6

Bias (%) of Estimates which Assume Normality

	When Population is Rectangular			When Population is Exponential		
Sample size, n	One quasi-range	Two adjacent quasi-ranges	Two quasi-ranges with r < r' ≤ 8	One quasi-range	Two adjacent quasi-ranges	Two quasi-ranges with r<r' ≤ 8
2	2.33			-11.38		
3	2.33			-11.38		
4	0.96	1.98	1.98	-10.95	-11.27	-11.27
5	-0.71	1.61	1.61	-10.43	-11.15	-11.16
6	-2.37	1.13	1.13	-9.91	-11.01	-11.01
7	-3.93	0.54	0.54	-9.41	-10.84	-10.84
8	-5.37	-0.11	-0.11	-8.93	-10.66	-10.66
9	-6.69	-0.80	-0.80	-8.49	-10.46	-10.46
10	-7.90	-1.51	-1.51	-8.08	-10.26	-10.26
11	-9.02	-2.22	-2.22	-7.69	-10.06	-10.06
12	-10.04	-2.92	-1.47	-7.32	-9.85	-10.07
13	-10.99	-3.60	-1.72	-6.98	-9.66	-10.00
14	-11.87	-4.27	-2.01	-6.65	-9.46	-9.92
15	-12.69	-4.92	-2.33	-6.34	-9.27	-9.84
16	-13.46	-5.54	-2.66	-6.05	-9.08	-9.75
17	-14.18	-6.14	-3.01	-5.77	-8.89	-9.65
18	1.26	-6.72	-3.37	-11.85	-8.71	-9.56
19	0.41	-7.28	-3.74	-11.61	-8.53	-9.46
20	-0.39	-7.82	-4.11	-11.37	-8.36	-9.36
21	-1.15	-8.33	-4.48	-11.14	-8.20	-9.25
22	-1.87	-8.83	-3.11	-10.92	-8.03	-9.43
23	-2.56	-9.31	-3.32	-10.71	-7.87	-9.38
24	-3.22	-9.78	-3.53	-10.51	-7.72	-9.33
25	-3.85	-10.23	-3.75	-10.31	-7.56	-9.28
26	-4.45	-10.66	-3.98	-10.12	-7.42	-9.22
27	-5.03	-11.08	-4.21	-9.93	-7.27	-9.16
28	-5.58	-11.48	-4.45	-9.75	-7.13	-9.10
29	-6.11	-11.87	-4.69	-9.58	-6.99	-9.04
30	-6.63	-12.25	-4.94	-9.41	-6.86	-8.98
31	-7.12	-12.61	-5.18	-9.24	-6.73	-8.91
32	1.23	-12.97	-5.43	-12.07	-6.60	-8.85
33	0.71	-13.31	-3.95	-11.92	-6.47	-9.09
34	0.20	-4.02	-4.11	-11.78	-10.32	-9.05
35	-0.29	-4.43	-4.28	-11.63	-10.19	-9.02

TABLE 6
(Continued)
Bias (%) of Estimates which Assume Normality

	When Population is Rectangular			When Population is Exponential		
Sample size, n	One quasi-range	Two adjacent quasi-ranges	Two quasi-ranges with r < r'≤ 8	One quasi-ranges	Two adjacent quasi-ranges	Two quasi-ranges with r<r'≤ 8
36	-0.77	-4.83	-4.45	-11.49	-10.06	-8.98
37	-1.23	-5.22	-4.63	-11.35	-9.94	-8.93
38	-1.68	-5.60	-4.81	-11.22	-9.82	-8.89
39	-2.11	-5.96	-4.99	-11.09	-9.70	-8.85
40	-2.53	-6.32	-5.17	-10.96	-9.58	-8.81
41	-2.94	-6.67	-5.36	-10.83	-9.47	-8.76
42	-3.34	-7.00	-5.54	-10.71	-9.36	-8.72
43	-3.73	-7.33	-5.72	-10.59	-9.25	-8.67
44	-4.10	-7.65	-5.91	-10.47	-9.14	-8.62
45	-4.47	-7.97	-4.45	-10.35	-9.03	-8.89
46	1.24	-8.27	-4.59	-12.18	-8.93	-8.86
47	0.86	-8.57	-4.73	-12.07	-8.83	-8.83
48	0.49	-2.46	-4.88	-11.96	-11.01	-8.80
49	0.12	-2.79	-5.02	-11.86	-10.91	-8.76
50	-0.23	-3.10	-5.17	-11.75	-10.81	-8.73
51	-0.58	-3.41	-5.32	-11.65	-10.72	-8.70
52	-0.92	-3.71	-5.46	-11.55	-10.62	-8.66
53	-1.25	-4.01	-5.61	-11.45	-10.53	-8.63
54	-1.57	-4.30	-5.76	-11.36	-10.44	-8.59
55	-1.89	-4.58	-5.91	-11.26	-10.35	-8.55
56	-2.20	-4.86	-0.63	-11.17	-10.26	-10.80
57	-2.50	-5.13	-0.76	-11.08	-10.17	-10.78
58	-2.79	-5.40	-0.89	-10.98	-10.09	-10.75
59	-3.08	-5.66	-1.02	-10.89	-10.00	-10.71
60	-3.37	-5.91	-1.15	-10.81	-9.92	-10.67
61	0.95	-6.16	-1.28	-12.16	-9.83	-10.64
62	0.66	-1.61	-1.41	-12.07	-11.36	-10.60
63	0.37	-1.87	-1.55	-11.99	-11.28	-10.56
64	0.09	-2.13	-1.67	-11.91	-11.20	-10.53
65	-0.19	-2.39	-1.81	-11.83	-11.13	-10.49
66	-0.46	-2.64	-1.94	-11.75	-11.05	-10.46
67	-0.73	-2.88	-2.07	-11.67	-10.97	-10.42
68	-0.99	-3.12	-2.20	-11.59	-10.90	-10.38
69	-1.25	-3.36	-2.34	-11.51	-10.83	-10.35
70	-1.50	-3.59	-2.47	-11.44	-10.75	-10.31

TABLE 6
(Continued)
Bias (%) of Estimates which Assume Normality

Sample size, n	When Population is Rectangular: One quasi-range	When Population is Rectangular: Two adjacent quasi-ranges	When Population is Rectangular: Two quasi-ranges with r < r'≤8	When Population is Exponential: One quasi-ranges	When Population is Exponential: Two adjacent quasi-ranges	When Population is Exponential: Two quasi-ranges with r<r'≤6
71	-1.75	-3.82	-2.59	-11.36	-10.68	-10.27
72	-1.99	-4.05	-2.73	-11.29	-10.61	-10.24
73	-2.23	-4.27	-2.85	-11.22	-10.54	-10.20
74	-2.47	-4.49	-2.98	-11.14	-10.47	-10.17
75	1.01	-4.70	-3.11	-12.21	-10.40	-10.13
76	0.77	-4.91	-3.24	-12.14	-10.33	-10.09
77	0.53	-1.29	-3.36	-12.07	-11.51	-10.05
78	0.30	-1.51	-3.49	-12.01	-11.45	-10.02
79	0.07	-1.73	-3.62	-11.94	-11.38	-9.98
80	-0.16	-1.94	-3.74	-11.87	-11.32	-9.95
81	-0.38	-2.15	-3.87	-11.81	-11.25	-9.91
82	-0.60	-2.36	-3.99	-11.74	-11.19	-9.88
83	-0.82	-2.56	-4.11	-11.68	-11.13	-9.84
84	-1.04	-2.76	-4.24	-11.62	-11.07	-9.81
85	-1.25	-2.96	-4.36	-11.55	-11.01	-9.77
86	-1.45	-3.15	-4.48	-11.49	-10.95	-9.74
07	-1.66	-3.34	-4.60	-11.43	-10.89	-9.70
88	-1.86	-3.53	-4.72	-11.37	-10.83	-9.67
89	1.06	-3.72	-4.83	-12.25	-10.77	-9.63
90	0.85	-3.91	-4.95	-12.19	-10.71	-9.60
91	0.65	-0.89	-5.07	-12.13	-11.67	-9.56
92	0.45	-1.08	-5.19	-12.08	-11.61	-9.53
93	0.25	-1.27	-5.30	-12.02	-11.55	-9.49
94	0.05	-1.45	-5.42	-11.96	-11.50	-9.46
95	-0.14	-1.63	-5.53	-11.91	-11.45	-9.43
96	-0.33	-1.81	-5.64	-11.85	-11.39	-9.39
97	-0.52	-1.99	-5.76	-11.80	-11.34	-9.36
98	-0.70	-2.17	-5.87	-11.74	-11.28	-9.32
99	-0.88	-2.34	-5.98	-11.69	-11.23	-9.29
100	-1.07	-2.51	-6.09	-11.63	-11.18	-9.26

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