PROBABILITY CONTENT OF REGIONS UNDER SPHERICAL NORMAL DISTRIBUTIONS, III: THE BIVARIATE NORMAL INTEGRAL[1]

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1. Introduction and summary. The bivariate normal distribution, with its numerous applications, is of considerable importance and has been studied fairly extensively. Among the first statisticians to investigate the distribution were Sheppard [12] and Karl Pearson [9], the latter from the point of view of his celebrated "tetrachoric functions", which were used as the basis for computing tables of the distribution. Pearson's tables have been extended by the University of California Statistical Laboratory [16] and, more fully, by the National Bureau of Standards [5].

In more recent years, the distribution has been studied among others by Nicholson [6], Pólya [10], Cadwell [1] and Owen [7], [8]. Owen has also provided useful tables from which the bivariate normal integral may be evaluated. These tables have been published in [7] and in extended form, together with auxiliary tables, in [8]. (The reader is referred to [8] and [5] for further references and for some interesting applications.) An essential part of the procedures used by Nicholson and Owen is to reduce the integral, which is a function of three parameters, the coordinates (x_0, y_0) of the vertex of the infinite rectangle over which integration is to be extended and the correlation coefficient ρ, to functions of only two parameters.

The series based on tetrachoric functions for the bivariate normal integral suffers from the disadvantage that it converges rather slowly except when $|\rho|$ is small. The need for an expression which shall be suitable for all values of ρ, but more especially for high $|\rho|$, has long been felt (see e.g., David [3]). Formula (3.16), taken in conjunction with (2.7), as well as formula (3.21), is designed to meet this need. These are two-parameter formulae and have the further advantage of being especially useful for high values of x_0 and/or y_0. Next, the formulae are used to provide equivalent rapidly convergent Stieltjes type continued fractions, known as S-fractions (equations (4.6) and (4.19)). These two sets of formulae constitute the basic results of this paper. They are, in fact, analogues of the corresponding known formulae for the univariate normal integral.

2. Reduction of the bivariate normal integral. We wish to evaluate the probability content, $L(x_0, y_0; \rho)$, of an infinite rectangle under a correlated bivariate

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normal distribution:

$$L(x_0, y_0; \rho) \tag{2.1}$$
$$= (2\pi)^{-1}(1-\rho^2)^{-\frac{1}{2}} \int_{x_0}^{\infty}\int_{y_0}^{\infty} \exp[-\tfrac{1}{2}(x^2+y^2-2\rho xy)/(1-\rho^2)]\, dx\, dy.$$

Diagonalisation of the 2 × 2 matrix involved in the quadratic form for the distribution of x and y, e.g., by an orthogonal transformation, or by a linear transformation corresponding to triangular resolution of the matrix, maps the infinite rectangle into an infinite sector with angle arc cos $-\rho$. A rotation of the transformed coordinate axes to orient one of the two final coordinate axes along the line joining the center of the distribution and the vertex of the sector maps the sector into a sector R of equal angle, but with vertex located along one of the latter coordinate axes. The following single, composite transformation compactly performs the required task:

$$x = \left(x_0\, u + \frac{y_0 - \rho x_0}{(1-\rho^2)^{\frac{1}{2}}}\, v\right)\Big/ c_0, \qquad y = \left(y_0\, u - \frac{x_0 - \rho y_0}{(1-\rho^2)^{\frac{1}{2}}}\, v\right)\Big/ c_0, \tag{2.2}$$

where

$$c_0^2 = (x_0^2 - 2\rho x_0 y_0 + y_0^2)/(1-\rho^2). \tag{2.3}$$

Under the transformation in (2.2), (2.1) becomes

$$L(x_0, y_0; \rho) = (2\pi)^{-1} \iint_R \exp[-\tfrac{1}{2}(u^2+v^2)]\, du\, dv, \tag{2.4}$$

where R is defined as follows:

$$R: x_0\, u + \frac{y_0 - \rho x_0}{(1-\rho^2)^{\frac{1}{2}}}\, v \geqq c_0\, x_0, \qquad y_0\, u - \frac{x_0 - \rho y_0}{(1-\rho^2)^{\frac{1}{2}}}\, v \geqq c_0\, y_0. \tag{2.5}$$

The vertex of R is located on the u-axis at a distance of $c_0 (c_0 > 0)$ from the origin. One possible orientation of R, corresponding to the case

$$x_0 < 0, \quad y_0 > 0, \quad y_0 - \rho x_0 > 0, \quad x_0 - \rho y_0 > 0, \quad \rho < 0,$$

is represented in Fig. 1. There are in all 32 possible cases (16 for $\rho > 0$ and 16 for $\rho < 0$), corresponding to the 4 possible quadrants in the original xy-plane for the location of (x_0, y_0) and all possible signs of the deviations, $y_0 - \rho x_0$ and $x_0 - \rho y_0$, of (x_0, y_0) from the lines of regressions. The angles of inclinations, θ_1, θ_2, of the bounding lines of R relative to the positive u-axis are given by

$$\tan\theta_1 = -\frac{x_0(1-\rho^2)^{\frac{1}{2}}}{y_0 - \rho x_0}, \qquad \tan\theta_2 = \frac{y_0(1-\rho^2)^{\frac{1}{2}}}{x_0 - \rho y_0} \qquad (0 \leqq \theta_1, \theta_2 < \pi). \tag{2.6}$$

Clearly, in all 32 cases the required probability content of R, under a centered circular normal distribution with unit variance in any direction, may be expressed in terms of the difference of probability contents of two sectors, each

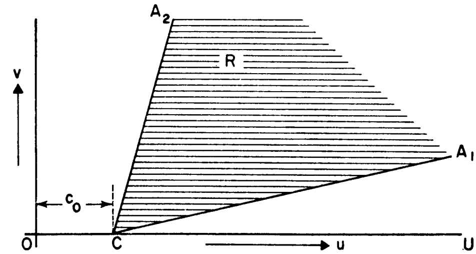

FIG. 1. Illustrating the orientation of R for (x_0, y_0) in 2nd quadrant, with the deviations of (x_0, y_0) from the regression lines both positive and negative correlation. $\angle A_1CU = \theta_1$, $\angle A_2CU = \theta_2$, $\angle A_1CA_2 = \text{arc} \cos -\rho$.

with vertex distant c_0 from the center of the distribution and having one arm oriented along the positive u-axis. Thus, in Fig. 1, the probability content of R = probability content of sector A_2CU-probability content of sector A_1CU. The probability content of a fundamental sector of the type A_1CU (or A_2CU) with parameters c_0, θ, i.e., having vertex C distant c_0 from the center of the distribution, angle θ, and one arm of the sector passing through the latter point, will be denoted by $W(c_0, \theta)$. The fundamental sector is depicted in Fig. 2.

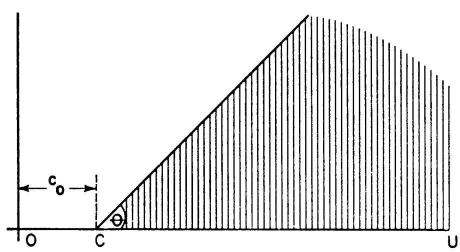

FIG. 2. The shaded portion represents the fundamental sector whose probability content is $W(c_0, \theta)$ under a standardized circular normal distribution centered at O.

Detailed examination of the 32 possible cases then gives

$$L(x_0, y_0; \rho) = W(c_0, \theta_2) - W(c_0, \theta_1) + C(x_0, y_0), \tag{2.7}$$

where

$$C(x_0, y_0) = \begin{cases} G(x_0), & (x_0, y_0) \text{ in 1st quadrant}, \\ 0, & (x_0, y_0) \text{ in 2nd quadrant}, \\ G(y_0), & (x_0, y_0) \text{ in 3rd quadrant}, \\ G(y_0) - G(-x_0), & (x_0, y_0) \text{ in 4th quadrant}, \end{cases} \tag{2.8}$$

and

$$G(x) = (2\pi)^{-\frac{1}{2}} \int_x^\infty e^{-\frac{1}{2}\xi^2} d\xi. \tag{2.9}$$

It will be noted that

$$W(0, \theta) = \theta/2\pi, \tag{2.10}$$

$$\begin{aligned} W(c_0, \pi/2) &= \tfrac{1}{2}G(c_0), \qquad W(c_0, \pi) = \tfrac{1}{2}, \\ W(c_0, 3\pi/2) &= 1 - \tfrac{1}{2}G(c_0), \qquad W(c_0, 2\pi) = 1, \end{aligned} \tag{2.11}$$

$$W(c_0, -\theta) = W(c_0, \theta) \qquad (0 \leqq \theta \leqq \pi), \tag{2.12}$$

$$W(c_0, \theta) + W(c_0, \pi - \theta) = G(c_0 \sin \theta) \qquad (0 \leqq \theta \leqq \pi). \tag{2.13}$$

(2.10) and (2.12) follow directly from the circular symmetry of the distribution, while (2.13) follows from (2.12) together with the fact that the left-hand member of (2.13) represents the probability content of the half-plane below a line distant $c_0 \sin \theta$ from the center of the distribution. In view of the preceding relationships, a knowledge of $W(c_0, \theta)$ for $0 \leqq \theta \leqq \pi/2$ is sufficient for a specification of all possible values of the function.

The W-function is closely related to the distribution of the non-central t with 1 degree of freedom. The latter statistic with non-centrality parameter c_0, $t_{1;c_0}$, is defined by $t_{1;c_0} = (u - c_0)/|v|$, where u and v are independent normal random variables with zero means and unit variances. On referring to Fig. 2, we find that

$$\text{Prob}\,(t_{1;c_0} \geqq t_0) = 2W(c_0, \text{arc cot}\, t_0) \qquad (t_0 > 0). \tag{2.14}$$

In particular, the asymptotic expansion for $W(c_0, \theta)$ obtained subsequently (Eq. (3.10)) may be used to evaluate the probability that $t_{1;c_0}$ is not less than t_0, with t replaced by $1/t_0$ in the coefficients of the expansion as given by (3.13).

The relationships between the present W-functions and functions introduced previously by Owen [7] and Nicholson [6] in their studies of the bivariate normal integral are of some interest. The functions in question are the T and V-functions, defined as the probability contents, under a centered circular normal distribution with unit variance in any direction, of an infinite quadrilateral and right triangle, respectively. These functions are represented in Fig. 3.

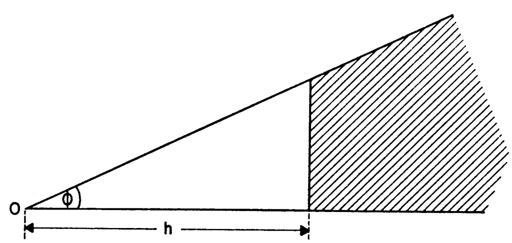

FIG. 3. The probability contents of the shaded region and unshaded triangle define Owen's $T(h, a)$ and Nicholson's $V(h, h \tan \phi)$, respectively, with $\tan \phi = a(h \geqq 0, 0 \leqq \theta \leqq \pi/2)$. The underlying distribution is a standardized circular normal distribution centered at O.

We have

$$T(c_0 \sin \theta, \cot \theta) + W(c_0, \theta) = \tfrac{1}{2}G(c_0 \sin \theta) \qquad (0 \leqq \theta \leqq \pi/2) \tag{2.15}$$

and

$$\begin{aligned} V(c_0 \sin \theta, c_0 \cos \theta) &- W(c_0, \theta) \\ &= \tfrac{1}{4} - \theta/(2\pi) - \tfrac{1}{2}G(c_0 \sin \theta) \quad (0 \leqq \theta \leqq \pi/2). \end{aligned} \tag{2.16}$$

The derivation of (2.15) is made sufficiently clear by the following diagram:

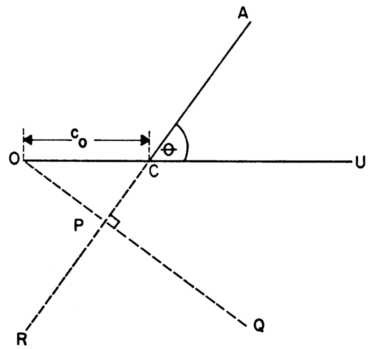

FIG. 4. Illustrating the relationship between the T and W-functions. O is the center of a standardized circular normal distribution.

The probability content of the sector ACU is $W(c_0, \theta)$. Produce AC to intersect the line through O which is orthogonal to AC in P. Then $\angle COP = \pi/2 - \theta$, $OP = c_0 \sin \theta$, and the probability content of the infinite quadrilateral $UCPQ$ is, in Owen's notation, $T(c_0 \sin \theta, \cot \theta)$. On the other hand, the sum of the last two probability contents is equal to the probability content of the quadrant $ACPQ$, which, by symmetry, is equal to one-half of the probability content of the half-plane below the line $ACPR$. Hence (2.15) is proved. Again, the probability content of the triangle OCP is, in Nicholson's notation,

$$V(c_0 \sin \theta, c_0 \cos \theta),$$

and the probability content of the sector UOQ is $(\pi/2 - \theta)/2\pi$, so that

$$V(c_0 \sin \theta, c_0 \cos \theta) + T(c_0 \sin \theta, \cot \theta) = (\pi/2 - \theta)/2\pi.$$

Equation (2.16) then follows directly from this last relationship and (2.15).[3]

It should be remarked that the V-functions, and therefore also the related T and W-functions, are of intrinsic interest quite apart from their relationship to the bivariate normal integral. The V-function has been tabulated in [5] and [6] and some of its applications discussed in [5].

[3] Formulae (2.15) and (2.16) can be extended to the range $\pi/2 < \theta \leqq \pi$ if, in accordance with the integral representations of the T and V-functions ([7], [5]), $T(h, -a) = -T(h, a)$ and $V(h, -k) = -V(h, k)$.

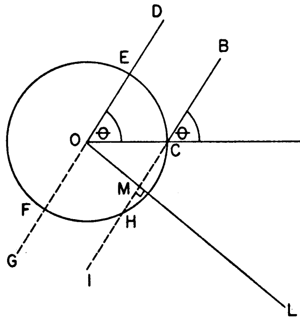

FIG. 5. The probability content of the infinite strip DECB is $K(c_0, \theta)$, where $OC = c_0$ O is the center of a standardized circular normal distribution.

Finally, it will be convenient to introduce a new function $K(c_0, \theta)$ defined by

$$K(c_0, \theta) + W(c_0, \theta) = \theta/(2\pi)e^{-\frac{1}{2}c_0^2}. \tag{2.17}$$

Geometrically, $K(c_0, \theta)$ represents the probability content, under a centered circular normal distribution with unit variance in any direction, of an infinite strip $DECB$ which is bounded by an arc of a circle, with center at the center of the distribution O and of radius c_0, subtending an angle θ at O (Fig. 5). To verify this, note that $\exp(-\frac{1}{2}c_0^2)$ represents the amount of probability outside the circle of radius c_0 and center at $O(u^2 + v^2$ is a chi-square with two degrees of freedom), whence, by symmetry, $\exp(-\frac{1}{2}c_0^2)\theta/2\pi$ represents the amount of probability in $DECU$. However, $DECU$ is the union of $DECB$ (probability content $K(c_0, \theta)$) and BCU (probability content $W(c_0, \theta)$), thereby proving the required result.

An integral form for $K(c_0, \theta)$ may be obtained by the use of new coordinate axes OED, the η-axis, and OL, the ξ-axis, orthogonal to OED. Divide the region $DECB$ into indefinitely thin strips by lines parallel to the η-axis. The probability density at any point is

$$(2\pi)^{-1}e^{-\frac{1}{2}(\xi^2+\eta^2)} = (2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}\xi^2}\cdot(2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}\eta^2}.$$

Hence, by integration with respect to η (the lower limit of integration being $(c_0^2 - \xi^2)^{\frac{1}{2}}$, the probability content of the strip is

$$(2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}\xi^2}\, d\xi\cdot G((c_0^2 - \xi^2)^{\frac{1}{2}}),$$

and on integrating with respect to ξ between the limits 0 and $OM = c_0 \sin\theta$, where M is the intersection of $BCHI$ with OL,

$$K(c_0, \theta) = (2\pi)^{-\frac{1}{2}} \int_0^{c_0\sin\theta} e^{-\frac{1}{2}\xi^2}G((c_0^2 - \xi^2)^{\frac{1}{2}})\, d\xi \tag{2.18}$$

(cf., Ruben [11]).

A knowledge of $K(c_0, \theta)$ enables one to evaluate the probability content, say $M(c_0, \theta)$, of that portion of the circle $ECHFE$ with radius c_0 intercepted between a diameter and a parallel chord distant $c_0 \sin \theta$ from the diameter. For, the infinite parallel strip bounded by *DEOFG* and *BCMHI* may be expressed as the union of *BCED*, *IHFG* and *ECHFE*. On the other hand, the probability content of the strip is $\frac{1}{2} - G(c_0 \sin \theta)$. Evidently then,

$$2K(c_0, \theta) + M(c_0, \theta) = \tfrac{1}{2} - G(c_0 \sin \theta). \tag{2.19}$$

3. Asymptotic series developments for the *W* and *L*-functions. The following asymptotic series for the tail-end area under the standardized normal curve is well known (see e.g., [4]):

$$\begin{aligned} G(x_0) = (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}x_0^2} \cdot \frac{1}{x_0} \Big\{ 1 - \frac{1}{x_0^2} + \frac{1.3}{x_0^4} \\ + \cdots + (-1)^{m-1} \frac{1.3 \cdots (2m-3)}{x_0^{2m-2}} \Big\} + R_m(x_0)(x_0 > 0), \end{aligned} \tag{3.1}$$

where $R_m(x_0)$ is the "remainder after m terms",

$$R_m(x_0) = (-1)^m 1.3 \cdots (2m-1) \int_{x_0}^{\infty} (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}x^2} \frac{dx}{x^{2m}}. \tag{3.2}$$

Equation (3.1) is essentially a formula for Mill's ratio,

$$G(x_0)/(2\pi)^{-\frac{1}{2}} \exp(-\tfrac{1}{2}x_0^2).$$

It has the property, important for purposes of computation, that the error induced by stopping after m terms does not exceed numerically the value of the last term. For,

$$\begin{aligned} |R_m(x_0)| < 1.3 \cdots (2m-1)(2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}x_0^2} \int_{x_0}^{\infty} \frac{dx}{x^{2m}} \\ = \frac{1.3 \cdots (2m-3)}{x_0^{2m-1}} (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}x_0^2}. \end{aligned} \tag{3.3}$$

In this section, asymptotic expansions analogous to (3.1) will be derived for the bivariate normal integral (i.e., the *L*-function, expressible in terms of the difference of two *W*-functions, as in (2.7)), as well as for the *W*-function itself. Consider first $W(c_0, \theta)$, and assume that $0 \leqq \theta < \pi/2$. In actuality, both acute and obtuse angles may be needed in (2.7) (recall that θ_1 and θ_2 are defined by (2.6)), but in view of the remarks of the preceding section and, in particular, equation (2.13) there is no loss of generality in assuming that θ is acute.

Referring to Fig. 2 of the preceding section, let OP, the distance between O and any point P within the shaded sector be r. Let ξ and ϕ be the polar coordinates of P with respect to C as pole and CU as base line. The probability density at P is

$$(2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}r^2} = (2\pi)^{-1} \exp[-\tfrac{1}{2}(c_0^2 + \xi^2 + 2c_0\xi \cos\phi)] \tag{3.4}$$

and therefore

$$(3.5)\quad W(c_0\,,\theta) = (2\pi)^{-1}\int_0^{\infty}\int_0^{\theta} \exp\left[-\tfrac{1}{2}(c_0^2 + \xi^2 + 2c_0\,\xi\cos\phi)\right]\xi\,d\xi\,d\phi.$$

Now,

$$\begin{aligned}
&\int_0^{\infty} \xi \exp\left[-\tfrac{1}{2}\xi^2 - (c_0\cos\phi)\xi\right] d\xi \\
&\qquad = -\int_0^{\infty} \exp\left[-(c_0\cos\phi)\xi\right] \frac{d}{d\xi}\,(e^{-\frac{1}{2}\xi^2})\,d\xi \\
&\qquad = 1 - (c_0\cos\phi)\int_0^{\infty} \exp\left[-\tfrac{1}{2}\xi^2 - (c_0\cos\phi)\xi\right] d\xi \\
&\qquad = 1 - (c_0\cos\phi)\exp\left[\tfrac{1}{2}c_0^2\cos^2\phi\right]\cdot(2\pi)^{\frac{1}{2}}G(c_0\cos\phi) \\
&\qquad = 1 - \sum_{j=1}^{m}(-1)^{j-1}\frac{1.3\cdots(2j-3)}{(c_0\cos\phi)^{2j-2}} \\
&\qquad\qquad - (c_0\cos\phi)e^{\frac{1}{2}c_0^2\cos^2\phi}\cdot(2\pi)^{\frac{1}{2}}R_m(c_0\cos\phi)
\end{aligned}\tag{3.6}$$

after substitution for $G(c_0\cos\phi)$ with the aid of (3.1).[4] Hence, using (3.6) in (3.5) and integrating with respect to ϕ, we obtain

$$\begin{aligned}
W(c_0\,,\theta) = (2\pi)^{-1}e^{-\frac{1}{2}c_0^2}\Big\{\theta &- \sum_{j=1}^{m}(-1)^{j-1}\frac{1.3\cdots(2j-3)}{c_0^{2j-2}}\int_0^{\theta}\sec^{2j-2}\phi\,d\phi \\
&- (2\pi)^{\frac{1}{2}}\int_0^{\theta} c_0\cos\phi\, e^{\frac{1}{2}c_0^2\cos^2\phi}R_m(c_0\cos\phi)\,d\phi\Big\}\,.
\end{aligned}\tag{3.7}$$

Equation (3.7) gives the desired formula for $W(c_0\,,\theta)$. We now show that the upper limit of summation may be replaced by ∞, i.e., that equation (3.7) provides an asymptotic expansion in the familiar sense that the error induced by using the first m terms of the expansion as an approximant for $W(c_0\,,\theta)$ does not exceed numerically the absolute value of the m-th term (cf. the remark about $R_m(x_0)$ in formula (3.1)). In fact, on using (3.3), this error (apart from the factor $(2\pi)^{-1}\exp(-\frac{1}{2}c_0^2)$) is numerically less than

$$\begin{aligned}
(2\pi)^{\frac{1}{2}}\int_0^{\theta} c_0\cos\phi\, e^{\frac{1}{2}c_0^2\cos^2\phi}\,.\,\frac{1.3\cdots(2m-3)}{(c_0\cos\phi)^{2m-1}}\,(2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}c_0^2\cos^2\phi}\,d\phi \\
= \frac{1.3\cdots(2m-3)}{c_0^{2m-2}}\int_0^{\theta}\sec^{2m-2}\phi\,d\phi,
\end{aligned}\tag{3.8}$$

which is the numerical value of the last term. Equation (3.7) may now be restated in the form

$$(3.9)\quad W(c_0\,,\theta) = (2\pi)^{-1}e^{-\frac{1}{2}c_0^2}\left\{\theta - \sum_{j=1}^{\infty}(-1)^{j-1}\frac{1.3\cdots(2j-3)}{c_0^{2j-2}}\int_0^{\theta}\sec^{2j-2}\phi\,d\phi\right\},$$

[4] $1.3\cdots(2j-3)$ is to be interpreted as 1 for $j = 1$.

or,

$$(3.10)\quad W(c_0, \theta) = (2\pi^{-1}e^{-\frac{1}{2}c_0^2} \sum_{j=1}^{\infty} (-1)^{j-1} \frac{1.3 \cdots (2j-1)}{c_0^{2j}} \int_0^{\theta} \sec^{2j} \phi \, d\phi.$$

Observe[5] from (3.9) that $K(c_0, \theta)$, defined in (2.17), has the asymptotic expansion

$$(3.11)\quad K(c_0, \theta) = (2\pi)^{-1}e^{-\frac{1}{2}c_0^2} \sum_{j=1}^{\infty} (-1)^{j-1} \frac{1.3 \cdots (2j-3)}{c_0^{2j-2}} \int_0^{\theta} \sec^{2j-2} \phi \, d\phi.$$

The coefficients

$$(3.12)\quad A_j(\theta) = \int_0^{\theta} \sec^{2j} \phi \, d\phi \qquad (j = 1, 2, \cdots)$$

in (3.10) may be evaluated in several ways. One way consists in expressing $\sec^{2j} \phi$ in terms of powers of $\tau \equiv \tan \phi$ and then integrating. Thus,

$$(3.13)\quad \begin{aligned} A_j(\theta) &= \int_0^{t} (1 + \tau^2)^{j-1} \, d\tau \\ &= \sum_{r=0}^{j-1} \binom{j-1}{r} \frac{t^{2r+1}}{2r+1}, \end{aligned}$$

where $t \equiv \tan \theta$, e.g.

$$A_1(\theta) = t, \qquad A_2(\theta) = t(1 + \tfrac{1}{3}t^2), \qquad A_2(\theta) = t(1 + \tfrac{2}{3}t^2 + \tfrac{1}{5}t^4).$$

Alternatively, integration by parts readily yields the recursion relationship

$$(3.14)\quad (2j-1)A_j(\theta) = 2(j-1)A_{j-1}(\theta) + \sec^{2j-2} \theta \tan \theta \quad (j = 1, 2, \cdots)$$

However, the most convenient method to use in computing (3.10) appears to consist in the employment of a recursion relationship between the total numerical coefficients. Let

$$(3.15)\quad B_j \equiv B_j(\theta) = 1.3 \cdots (2j-1)A_j(\theta).$$

Then

$$(3.16)\quad W(c_0, \theta) = (2\pi)^{-1}e^{-\frac{1}{2}c_0^2} \sum_{j=1}^{\infty} (-1)^{j-1} \frac{B_j}{c_0^{2j}},$$

and a recursion relationship for B_j, obtained from (3.14) and (3.15), is

$$(3.17)\quad \begin{aligned} B_j &= 2(j-1)B_{j-1} + 1.3 \cdots (2j-3)tu^{j-1} \qquad (j = 2, 3, \cdots), \\ B_1 &= t, \end{aligned}$$

[5] It should be noted that an asymptotic expansion for a function closely related to Owen's T-function (and the present W-function) for the case $0 < h < k$ in $T(h, k/h)$ was obtained by Pólya [10]. (This reproduced in essence an expansion first given by Sheppard [12].) Unfortunately, however, the expansion in question does not appear to be very manageable, and its physical or geometrical interpretation is obscure.

where $u \equiv 1 + t^2 = \sec^2 \theta$. On using (3.17), the first eight values of B_j are obtained as follows:

$$
\begin{aligned}
B_1/t &= 1, \\
B_2/t &= 2 + u, \\
B_3/t &= 8 + 4u + 3u^2, \\
B_4/t &= 3(16 + 8u + 6u^2 + 5u^3), \\
B_5/t &= 3(128 + 64u + 48u^2 + 40u^3 + 35u^4), \\
B_6/t &= 15(256 + 128u + 96u^2 + 80u^3 + 70u^4 + 63u^5), \\
B_7/t &= 45(1024 + 512u + 384u^2 + 320u^3 + 280u^4 + 252u^5 \\
&\quad + 231u^6), \\
B_8/t &= 315(2048 + 1024u + 768u^2 + 640u^3 + 560u^4 + 504u^5 \\
&\quad + 462u^6 + 429u^7).
\end{aligned} \tag{3.18}
$$

More generally, B_j/t is a polynomial of degree $j - 1$ in u. In fact, if

$$B_j/t = k_{j0} + k_{j1}u + \cdots + k_{j,j-1}u^{j-1}, \tag{3.19}$$

then, by (3.17),

$$k_{jp} = 2^{j-1}(j - 1)!\,\frac{(2p)!}{2^{2p}(p!)^2} \qquad (p = 0, 1, \cdots, j - 1). \tag{3.20}$$

In view of (2.7), the bivariate normal integral may be expressed in terms of the difference of two asymptotic expansions of the type (3.16). We now show that in certain situations the integral may be expressed in terms of a *single* asymptotic expansion. To achieve this, recall first that the asymptotic expansions (3.16) and (3.10) for $W(c_0, \theta)$ are valid only for $0 \leqq \theta < \pi/2$. Therefore, in order to exploit either of the two latter expansions for the derivation of the bivariate normal integral by means of (2.7), the angle arguments in each of the two W-functions must either be acute or rendered acute, the "rendering acute" being effected by (2.13). We then find that the bivariate integral may be expressed in terms of the difference between two W-functions, each of whose angle arguments is acute, either when θ_1 and θ_2 in (2.6) are both acute, or when θ_1 and θ_2 are both obtuse. When one of the two angles is acute and the other obtuse, the bivariate integral is expressed as the *sum* (not the difference) of two W-functions with acute angle arguments.

Assume then that $0 \leqq \theta_1, \theta_2 < \pi/2$ and, for convenience, assume further that $\theta_1 < \theta_2$ (if $\theta_1 > \theta_2$ interchange θ_1 and θ_2). Then, by (3.16),

$$W(c_0, \theta_2) - W(c_0, \theta_1) = (2\pi)^{-1}e^{-\frac{1}{2}c_0^2} \sum_{j=1}^{\infty} (-1)^{j-1} \frac{B_j'}{c_0^{2j}} \qquad (0 \leqq \theta_1 < \theta_2 < \pi/2), \tag{3.21}$$

where

$$B_j' \equiv B_j'(\theta_1, \theta_2) = B_j(\theta_2) - B_j(\theta_1). \tag{3.22}$$

Equation (3.21) gives the desired asymptotic expansion for the difference of two W-functions. By virtue of the fact that (3.21) can be derived directly from (3.4), just as was $W(c_0, \theta)$, with the limits 0 and θ replaced by θ_1 and θ_2, respectively, it follows once again (cf., (3.16)) that an upper bound to the error after m terms in (3.21) is given by the mth term of the series.

It is important to note that c_0 is large when either (i) $|x_0|$ and/or $|y_0|$ is large, or (ii) $|\rho|$ is high.[6] Further, from (2.6), a high value of $|\rho|$ implies generally a small value for both θ_1 and θ_2. This tends to increase the rate of "convergence" of the series (in the special sense (refer to (3.8)) in which one may speak of the totally divergent series in (3.16) and (3.21) being "convergent"). These series should therefore be particularly useful in extending the currently available range of tabulation of the bivariate normal integral, as well as of the V-function.

4. Continued fraction developments for the W and L-functions. A continued fraction for Mill's ratio has long been known. Kendall [4] attributes it to Laplace and rightly points out that Sheppard [13] was enabled to obtain "superb" tables for the tail-end area under the normal curve by the use of the fraction. The relevant formula is

$$(2\pi)^{-\frac{1}{2}} \int_x^\infty e^{-\frac{1}{2}\xi^2}\, d\xi = (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}x^2} \cfrac{1}{x + \cfrac{1}{x + \cfrac{2}{x + \cfrac{3}{x + \cdots}}}} \qquad (x > 0). \tag{4.1}$$

It will be observed that the coefficients 1, 1.3, 1.3.5, $\cdots$, in the series (3.1) are the even moments of a standardised normal distribution. This is not accidental; on the contrary, it provides the essential clue to the development of an analogous continued fraction for $W(c_0, \theta)$, as well as, in special cases, for the L-function.

It is known from Stieltjes' classic work on continued fractions (see Wall [17], Chapter XIX, for details, as well as for references to Stieltjes' work) that a sufficient condition for totally divergent series of the type (3.1) to be representable as continued fractions of the type (4.1), known as S-fractions, is that the coefficients of the series represent the moments of a distribution. Furthermore, the fraction is convergent if the distribution is uniquely determined by

[6] c_0 may usefully be regarded as a normed distance, in terms of oblique Cartesian coordinates, between the cut-off point (x_0, y_0) and the center of the distribution, where the angle between the coordinate axes is arc cos $-\rho$ and the standardization factor is $(1 - \rho^2)^{\frac{1}{2}}$.

its moments. Now, from (3.11),

$$K(c_0, \theta) = \frac{\theta}{2\pi} e^{-\frac{1}{2}c_0^2} \left\{ \frac{1}{\theta} \int_0^\theta d\phi - \left(\frac{1}{\theta} \int_0^\theta \sec^2\phi \, d\phi \right) \frac{1}{c_0^2} + \left(\frac{1.3}{\theta} \int_0^\theta \sec^4\phi \, d\phi \right) \frac{1}{c_0^4} - \cdots \right\}. \tag{4.2}$$

The $(2j)$th moment of a normal random variable with zero mean and standard deviation σ is $1.3 \cdots (2j - 1)\sigma^j$, the odd moments being, of course, zero. The coefficient of $1/c_0^{2j}$, $\mu_{2j}(X)$, in (4.2),

$$\mu_{2j}(X) \equiv \frac{1.3 \cdots (2j - 1)}{\theta} \int_0^\theta \sec^{2j} \phi \, d\phi, \tag{4.3}$$

may accordingly be identified as the $(2j)$th moment of a weighted normal random variable X with zero mean and a random standard deviation σ, $\sigma \equiv \sec \phi$, where ϕ is uniformly distributed over $(0, \theta)$. This defines a legitimate distribution

$$F(x; \theta) = \frac{1}{\theta} \int_0^\theta \Phi(x \cos \phi) \, d\phi, \tag{4.4}$$

where $\Phi(\cdot)$ is the standardized normal distribution function. Thus $K(c_0, \theta)$ may be represented as an S-fraction. The fraction is, moreover, convergent, since the moment sequence $\{\mu_{2j}(X)\}$ uniquely determines the distribution (4.4), the uniqueness property being a direct consequence of Carleman's criterion ([2], pp. 78–96). In fact,

$$\sum (\mu_{2j}(X))^{-1/(2j)} = \sum (1.3 \cdots (2j - 1))^{-1/(2j)} \left(\frac{1}{\theta} \int_0^\theta \sec^{2j} \phi \, d\phi \right)^{-1/(2j)}$$
$$> \sum (1.3 \cdots (2j - 1))^{-1/(2j)} \cos \theta = \infty,$$

and the uniqueness is established. Consequently, $K(c_0, \theta)$ may be represented as a convergent continued fraction, as follows:

$$K(c_0, \theta) = \frac{\theta}{2\pi} e^{-\frac{1}{2}c_0^2} c_0 \cdot \cfrac{a_0}{c_0 + \cfrac{a_1}{c_0 + \cfrac{a_2}{c_0 + \ddots}}} \qquad (c_0 > 0;\ 0 \leqq \theta < \pi/2). \tag{4.5}$$

It now follows from (2.17) that

$$W(c_0, \theta) = e^{-\frac{1}{2}c_0^2} \frac{\theta}{2\pi} \left[1 - c_0 \cfrac{a_0}{c_0 + \cfrac{a_1}{c_0 + \cfrac{a_2}{c_0 + \ddots}}} \right] \qquad (c_0 > 0; 0 \leqq \theta < \pi/2). \tag{4.6}$$

We remark that $K(c_0, \theta)$ is greater than every even approximant of the continued fraction in (4.5) and less than every odd approximant. This follows from a general property of S-fractions proved by Stieltjes [14]. Consequently, $W(c_0, \theta)$ is trapped between known bounds at each stage of computation.

It now remains to construct the coefficients a_i of the fractions in (4.5) and (4.6) which correspond to the divergent series (3.11) and (3.10), respectively. Two methods appear to be practically useful, so far as the present series are concerned.

The first method, which is a recursive one, consists in the employment of an algorithm to determine the orthogonal polynomials corresponding to the distribution function in (4.4). A knowledge of the coefficients of the first m polynomials (up to and including the polynomial of degree $m - 1$) allows the next cycle, the evaluation of a_{m-1} and then of the coefficients in the $(m + 1)$th polynomial to be completed. Formally, if the $(p + 1)$th polynomial, of degree p, is defined by

$$M_p(x) = \beta_{p0}x^p + \beta_{p1}x^{p-1} + \cdots + \beta_{pp} \qquad (\beta_{p0} = 1), \tag{4.7}$$

then the algorithm[7] may be stated in the form

$$\mu_{2n}\beta_{n0} + \mu_{2n-1}\beta_{n1} + \cdots + \mu_n\beta_{nn} = a_0a_1 \cdots a_n \qquad (n = 0, 1, \cdots), \tag{4.8}$$

$$\begin{aligned} \beta_{n+1,1} &= \beta_{n1}, \\ \beta_{n+1,j} &= \beta_{nj} - a_n\beta_{n-1,j-2} \qquad (j = 2, 3, \cdots, n), \\ \beta_{n+1,n+1} &= a_n\beta_{n-1,n-1}, \end{aligned} \tag{4.9}$$

where $\mu_k \equiv \mu_k(X)$ and $\mu_{2p+1} = 0$ $(p = 0, 1, \cdots)$. ($M_p(x)$ is here an odd or even polynomial according as to whether p is odd or even.)

The second method consists in the direct evaluation of the moment determinants of various orders, since according to the algorithm the a_i may be expressed in terms of these determinants. To prove this, note that (Szego [15])

$$M_n(x) = \Delta_{n-1}^{-1} \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix} \qquad (n = 1, 2, \cdots), \tag{4.10}$$

where

$$\Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \mu_n \\ \mu_1 & \mu_2 & \mu_{n+1} \\ \cdots & \cdots & \cdots \\ \mu_n & \mu_{n+1} & \mu_{2n} \end{vmatrix} \qquad (n = 1, 2, \cdots) \tag{4.11}$$

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[7] (4.8) and (4.9) have been extracted from Wall [17], Chapter VI, after considerable simplification.

and $\Delta_0 = 1$. Hence, (4.8) is equivalent to

$$\Delta_n/\Delta_{n-1} = a_0 a_1 \cdots a_n \qquad (n = 1, 2, \cdots), \tag{4.12}$$

so that

$$a_n = \Delta_n \Delta_{n-2}/\Delta_{n-1}^2 \qquad (n = 2, 3, \cdots). \tag{4.13}$$

Systematic application of (4.8) and (4.9) gives

$$\begin{cases} a_0 = 1, \\ a_1 = \mu_2, \\ a_2 = (\mu_4 - \mu_2^2)/\mu_2, \\ a_3 = \dfrac{\mu_6 - \mu_4^2/\mu_2}{\mu_4 - \mu_2^2}, \\ a_4 = \dfrac{\mu_8 \mu_2 - \mu_6 \mu_4}{\mu_6 \mu_2 - \mu_4^2} + \dfrac{\mu_4 \mu_2 - \mu_6}{\mu_4 - \mu_2^2}, \end{cases} \tag{4.14}$$

where, by (4.3)

$$\mu_{2j} = B_j/\theta \qquad (j = 1, 2, \cdots). \tag{4.15}$$

The formulae for high order a_i become progressively and rapidly more complicated, and for a specific computational need it is therefore more appropriate to use the algorithm ((4.8) and (4.9)) directly when the μ_{2j} have been numerically evaluated.

Similarly, a continued fraction development may be obtained for

$$W(c_0, \theta_2) - W(c_0, \theta_1)$$

(though for computational purposes it seems more convenient to determine the continued fractions for $W(c_0, \theta_1)$ and $W(c_0, \theta_2)$ separately), when

$$0 \leqq \theta_1 < \theta_2 < \pi/2.$$

Thus, from (3.11),

$$K(c_0, \theta_2) - K(c_0, \theta_1) = \frac{\theta_2 - \theta_1}{2\pi} e^{-\frac{1}{2}c_0^2} \left\{1 - \frac{\mu_2(X')}{c_0^2} + \frac{\mu_4(X')}{c_0^4} - \cdots\right\}. \tag{4.16}$$

where

$$\mu_{2j}(X') = (1 \cdot 3 \cdots (2j - 1)) \cdot \frac{1}{\theta_2 - \theta_1} \int_{\theta_1}^{\theta_2} \sec^{2j} \phi \, d\phi \quad (j = 1, 2, \cdots), \tag{4.17}$$

$\mu_m(X')$ being, then, the mth moment of a weighted normal random variable X' with zero mean and a random standard deviation σ, $\sigma \equiv \sec \phi$, where ϕ is uniformly distributed over (θ_1, θ_2). We then have (cf., (4.5))

$$K(c_0, \theta_2) - K(c_0, \theta_1) = \frac{\theta_2 - \theta_1}{2\pi} e^{-\frac{1}{2}c_0^2} \cfrac{c_0}{c_0 + \cfrac{a_1'}{c_0 + \cfrac{a_2'}{c_0 + \cdots}}} \tag{4.18}$$

$$(c_0 > 0; 0 \leqq \theta_1 < \theta_2 < \pi/2).$$

and so (cf., (4.6))

$$W(c_0, \theta_2) - W(c_0, \theta_1) = \frac{\theta_2 - \theta_1}{2\pi} e^{-\frac{1}{2}c_0^2} \left[1 - \cfrac{c_0}{c_0 + \cfrac{a_1'}{c_0 + \cfrac{a_2'}{c_0 + \cdots}}} \right] \tag{4.19}$$

$$(c_0 > 0; 0 \leqq \theta_1 < \theta_2 < \pi/2).$$

Equations (4.8) and (4.9) for computing the coefficients of the fractions still retain their validity, so that, correspondingly, the first few a_i' are given by (4.14), with μ_{2j} interpreted as $\mu_{2j}(X')$.[8] The convergence of the fractions in (4.18) and (4.19) follows from the uniqueness property of $\{\mu_{2j}(X')\}$.

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