A CONVEXITY PROPERTY IN THE THEORY OF RANDOM VARIABLES DEFINED ON A FINITE MARKOV CHAIN

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1. Summary. Let $P = (p_{jk})$ be the transition matrix of an ergodic, finite Markov chain with no cyclically moving sub-classes. For each possible transition (j, k), let $H_{jk}(x)$ be a distribution function admitting a moment generating function $f_{jk}(t)$ in an interval surrounding $t = 0$. The matrix $P(t) = \{p_{jk}f_{jk}(t)\}$ is of interest in the study of the random variable $S_n = X_1 + \cdots + X_n$, where X_m has the distribution $H_{jk}(x)$ if the mth transition takes the chain from state j to state k. The matrix $P(t)$ is non-negative and therefore possesses a maximal positive eigenvalue $\alpha_1(t)$, which is shown to be a convex function of t. As an application of the convexity property, we obtain an asymptotic expression for the probability of tail values of the sum S_n, in the case where the X_m are integral random variables.

The results are related to those of Blackwell and Hodges [1], whose methods are followed closely in Section 5, and Volkov [4], [5], who treats in detail the case of integer-valued functions of the state of the chain, i.e., the case $f_{jk}(t) = \exp(\beta_k t)$ (β_k integral).

2. Introduction and notation. Let $k_m (m = 0, 1, 2, \ldots)$ be the state at time m of a finite N-state ergodic Markov chain with no cyclically moving subclasses and with transition matrix $P = (p_{jk})$, where $p_{jk} = \Pr(k_m = k \mid k_{m-1} = j)$, $j, k = 1, \cdots, N$. The distribution of k_0 is unspecified, since we shall mostly deal with probabilities conditional on k_0. It follows that P is a non-negative, primitive and irreducible matrix. Let $H_{jk}(x)$ be a distribution function associated with the transition (j, k) $(p_{jk} \neq 0)$ and let $f_{jk}(t)$ be the corresponding moment generating function, i.e.,

$$f_{jk}(t) = \int_{-\infty}^{\infty} e^{tx}\, dH_{jk}(x).$$

We shall suppose that each $f_{jk}(t)$ is analytic in a strip which strictly contains the imaginary axis of the complex t-plane. There will therefore be a maximal strip

$$u_0 < \mathrm{Re}(t) < u_0' \quad (-\infty \leqq u_0 < 0 < u_0' \leqq \infty), \tag{2.1}$$

in which all the $f_{jk}(t)$ are analytic.

Let X_m, $m = 1, 2, \cdots$, be a random variable having the distribution $H_{jk}(x)$ if $k_{m-1} = j$ and $k_m = k$, i.e., if the mth transition is (j,k), and let

Received October 18, 1960; revised February 20, 1961.

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$S_n = X_1 + \cdots + X_n$. Let $P(t)$ be the matrix $\{p_{jk}f_{jk}(t)\}$ and let

$$\{P(t)\}^n = \{p_{jk}^{(n)}f_{jk}^{(n)}(t)\}, \tag{2.2}$$

where $P^n = \{p_{jk}^{(n)}\}$. Then $f_{jk}^{(n)}(t)$ is the moment generating function of S_n conditional on the n-stage transition from state j at time 0 to state k at time n. Thus

$$f_{jk}^{(n)}(t) = E\{\exp(tS_n) \mid k_0 = j, k_n = k\}. \tag{2.3}$$

For real t, the matrix $P(t)$ is non-negative and therefore it has a maximal positive eigenvalue, the Perron root, which we denote by $\alpha_1(t)$. Thus $\alpha_1(0) = 1$, and, for real t, $\alpha_1(t)$ has the properties (i) $\alpha_1(t) > 0$, (ii) $\alpha_1(t) > |\alpha_j(t)|$, where $\alpha_j(t), j = 2, 3, \cdots, N$, are the remaining eigenvalues of $P(t)$.

We shall say that $f(t)$ is a degenerate moment generating function if it is of the form $e^{\beta t}$ (β real) and we shall say that $P(t)$ is degenerate if it is of the form

$$P(t) = e^{\beta t}D(t)P\{D(t)\}^{-1}, \tag{2.4}$$

where $D(t)$ is a diagonal matrix of degenerate moment generating functions. If $P(t)$ is degenerate, then the sum S_n is also degenerate in the sense that given $k_0 = j$, $k_n = k$, S_n is deterministic and of the form $S_n = n\beta + \beta_j - \beta_k$, where $D(t) = \text{diag}\,\{\exp(\beta_j t)\}$.

Let (p_k), $k = 1, \cdots, N$, be the unique ergodic distribution associated with P. Then, if k_0 has the distribution (p_k) and if we take expectation unconditional on k_1, it is easy to show that $E(X_1) = \alpha_1'(0)$. Thus $\alpha_1'(0)$ is a measure of the ultimate drift of S_n.

3. Some properties of non-negative square matrices. For the sake of clarity we quote the following properties of non-negative square matrices from the paper of Debreu and Herstein [3].

(a) Let $A \geqq 0$ be an irreducible (indecomposable) square matrix, and let α_1 be its maximal positive eigenvalue. Then α_1 is a simple root of the equation $|\alpha I - A| = 0$, and there exist strictly positive left and right eigenvectors corresponding to α_1. If σ is any other eigenvalue of A, then $|\sigma| \leqq \alpha_1$, and if $|\sigma| < \alpha_1$ then A is said to be primitive.

(b) A finite stochastic matrix is the transition matrix of a Markov chain which is ergodic and without cyclically moving sub-classes if and only if it is primitive and irreducible.

(c) Let $B = (b_{jk})$ be a square matrix with complex elements, and let $B^* = (|b_{jk}|)$. If β is any eigenvalue of B, if A $(\geqq 0)$ is irreducible, and if $B^* \leqq A$ then $|\beta| \leqq \alpha_1$. Moreover $|\beta| = \alpha_1$ and $B^* \leqq A$ together imply that $B^* = A$; if $\beta = \alpha_1 e^{i\phi}$, then $B = e^{i\phi}D^{-1}AD$ where $D^* = I$.

In addition we state the following lemma which we need in Sections 4 and 5. It is an immediate consequence of the left and right eigenvector relations.

LEMMA 3.1. *Let A be a non-negative, primitive and irreducible matrix of order $N \times N$. Let α_1 be its maximal positive eigenvalue with corresponding left and right positive eigenvectors $y = (y_j)$ and $x = (x_j)$ respectively, such that $yx = 1$. Let*

$X = \operatorname{diag}(x_j)$. *Then the matrix* $\alpha_1^{-1}X^{-1}AX$ *is a primitive, irreducible, stochastic matrix with limiting probability vector* (x_jy_j).

4. The properties of $\alpha_1(t)$. Let $t = u + iv$ (u, v real). Then for t lying in the strip (2.1), the $f_{jk}(t)$ and $P(t)$ satisfy the following conditions:

$$\text{(4.1)} \qquad \begin{aligned} &\text{(i) } f_{jk}(u) > 0, \qquad \text{(ii) } |f_{jk}(t)| \leqq f_{jk}(u), \qquad \text{(iii) } f_{jk}(0) = 1, \\ &\qquad\qquad\qquad\qquad \text{(iv) } P(u) \geqq 0, \qquad \text{(v) } \{P(t)\}^* \leqq P(u), \end{aligned}$$

where, in (v), we use the notation of Section 3(c).

THEOREM 1.

(a) *The function* $\alpha_1(t)$ *is regular at each point* $t = u$ *of the real axis in the strip* (2.1).

(b) *An eigenvalue of* $P(t)$ *is of the form* $e^{\beta t}$ (β real) *if and only if* $P(t)$ *is degenerate, i.e., of the form* (2.4).

(c) *In the strip* (2.1) *we have*

$$\alpha_1(u) \geqq |\alpha_j(t)| \quad (j = 2, 3, \cdots, N; t = u + iv)$$

PROOF.

(a) Since for each real t, $\alpha_1(t)$ is a simple root of the determinantal equation $|\alpha I - P(t)| = 0$, and since $|\alpha I - P(t)|$ is an analytic function of the two complex variables α and t, the result follows from the implicit function theorem for analytic functions (Bochner and Martin [2], p. 39).

(b) If $P(t)$ is of the form (2.4) then clearly $\alpha_1(t) = e^{\beta t}$. If $e^{\beta t}$ is an eigenvalue of $P(t)$, then we put $t = iv$ (v real), and it follows from (4.1) (v) and Section 3(c) that $P(iv) = e^{i\beta v}D(v)P\{D(v)\}^{-1}$, where $\{D(v)\}^* = I$. Thus $|f_{jk}^{(n)}(iv)| = 1$ for each j, k and n for which $p_{jk}^{(n)} > 0$, and since $f_{jk}^{(n)}(iv)$ is a characteristic function, we must have $D(v) = \operatorname{diag}\{\exp(i\beta_j v)\}$ (β_j real $j = 1, \cdots, N$). Hence $P(t)$ is degenerate.

(c) The inequalities follow from (4.1) (v) and Section 3(c).

THEOREM 2. *If* $P(t)$ *is not degenerate, then* $\alpha_1(t)$ (t *real*) *is a strictly convex function of* t.

PROOF. We have the factorization

$$\text{(4.2)} \quad |\alpha I - P(t)| = \alpha^N\{1 - \alpha^{-1}\alpha_1(t)\}\{1 - \alpha^{-1}\alpha_2(t)\} \cdots \{1 - \alpha^{-1}\alpha_N(t)\}$$

and we consider the t-roots of the equation

$$\text{(4.3)} \qquad |\alpha I - P(t)| = 0.$$

If $|\alpha| > \alpha_1(u)$ ($t = u + iv$) it follows from (4.2) and Theorem 1(c) that $|\alpha I - P(t)| \neq 0$. Thus there can be no t-roots of (4.3) in any part of the t-plane for which $|\alpha| > \alpha_1(u)$.

Now suppose $\alpha_1(u)$ is a concave function of u in some interval (u', u''). (The argument will be simpler to follow with the aid of a diagram of the u, $\alpha_1(u)$ plane). We may choose real numbers a and b so that the linear function $a + bu$

satisfies

$$a + bu > \alpha_1(u) \qquad (u' < u < u''), \tag{4.4}$$

i.e., the line $a + bu$ lies above the curve $\alpha_1(u)$ in the interval (u', u''). In (4.2) let $\alpha = a + bt$. Since

$$|a + bt| = \{(a + bu)^2 + b^2v^2\}^{\frac{1}{2}} \geqq a + bu > \alpha_1(u) \qquad (u' < u < u''),$$

there are no roots of the equation

$$|(a + bt)I - P(t)| = 0 \tag{4.5}$$

in the strip of the t-plane $u' < u < u''$. But the t-roots of (4.5) are continuous functions of a, and we may choose values of a and b so that the line $a + bu$ cuts the curve $\alpha_1(u)$ in two points, thus producing two roots of (4.5) in the strip (u', u''). Thus for a suitable b, there is a value of a, say a', such that for $a > a'$ there are no roots of (4.5) in the strip (u', u''), while for $a < a'$ there are two roots. This contradicts the continuity of the t-roots of (4.5) and therefore $\alpha_1(u)$ cannot be concave in any interval.

Further, $\alpha_1(u)$ cannot be a linear function. For if $\alpha_1(u) = 1 + cu$ $(c \neq 0)$, say, we can choose a real number β so that the function $e^{\beta u}(1 + cu)$ is concave near the point $u = 0$. But $e^{\beta t}\alpha_1(t)$ is the maximal eigenvalue of the matrix $e^{\beta t}P(t)$, which is of the same type as $P(t)$, and which cannot therefore have a concave maximal eigenvalue.

It follows that $\alpha_1(u)$ is strictly convex.

We may specialize our results to integral random variables. To this end, let $\phi_{jk}(z)$ be a probability generating function associated with the transition (j, k) and suppose that there is an annulus $r_0 < |z| < r_0'$ $(0 \leqq r_0 < 1 < r_0' \leqq \infty)$ in which all the $\phi_{jk}(z)$ have convergent Laurent series. Let $Q(z)$ denote the matrix $\{p_{jk}\phi_{jk}(z)\}$ and we suppose that $Q(z)$ is not of the degenerate form

$$Q(z) = z^{\beta}ZPZ^{-1}, \tag{4.6}$$

where β is an integer and Z is a diagonal matrix of integral powers of z. For real and positive z let $a_1(z)$ be the maximal positive eigenvalue of $Q(z)$. If we set $z = e^t$, then, by Theorem 2, $a_1(e^t)$ is a strictly convex function of t (t real) and therefore $a_1(z)$, though not necessarily convex, has the property of not having a local maximum for real positive z. This generalizes the result of Volkov [4] who demonstrated this property in the special case where $\phi_{jk}(z) = z^{\alpha_k}$.

We return to the matrix $P(t)$ as defined in Section 2. The convexity property of $\alpha_1(u)$ $(t = u + iv)$ raises the question of whether $\alpha_1(u)$ attains its unique minimum at a finite value of u. The answer is clearly affirmative if $\alpha_1'(0) = 0$. If $\alpha_1'(0) < 0$ say, then either $\alpha_1(u)$ continues to decrease as u increases or it reaches a minimum and then starts increasing. We distinguish between the cases where the strip (2.1) includes the entire right half-plane $(u_0' = \infty)$ and where it is bounded to the right $(u_0' < \infty)$. Modifications for the left half plane will be obvious (i.e., for the case where $\alpha_1'(0) > 0$).

THEOREM 3. *Let $t = u + iv$ and suppose that $P(t)$ is not degenerate.*

(a) *Suppose that $\alpha_1(u)$ is defined for all $u > 0$ (i.e., $u_0' = \infty$). Then a necessary and sufficient condition for $\alpha_1(u)$ to be uniformly bounded (and so monotonic decreasing) for all $u > 0$ is that there exists a diagonal matrix $D(t)$ of degenerate moment generating functions such that each element of the matrix*

$$Q(t) = \{D(t)\}^{-1}P(t)D(t) \tag{4.7}$$

is of the form $p_{jk}q_{jk}(t)$, $q_{jk}(t)$ being the moment generating function of a non-positive random variable. In the case of integral random variables, each element of $D(t)$ and each $q_{jk}(t)$ will be the moment generating function of an integral random variable.

(b) *Let $\alpha_1'(0) < 0$ and suppose $u_0' < \infty$. Then $\alpha_1(u)$ attains its unique stationary minimum at a finite positive value of u if one of the following conditions is satisfied:*

(i) *There exists a number u_1 $(0 < u_1 < u_0')$ such that for each j, k for which $f_{jk}(t)$ is defined, $f_{jk}'(u_1) \geqq 0$.*

(ii) *For some j, k, $f_{jk}(u) \to \infty$ as $u \to u_0'-$.*

PROOF.

(a) If (4.7) is satisfied, then $P(t)$ and $Q(t)$ have the same eigenvalues. Since each element of $Q(t)$ is non-increasing for $t > 0$, it follows from Section 3(c) that $\alpha_1(t)$ is non-increasing for $t > 0$ and therefore bounded for all $t > 0$.

Conversely, if $\alpha_1(u)$ is bounded for $u > 0$, we note that for each j, k for which $p_{jk} > 0$, and for some finite, real β_{jk}, Pr $(X > \beta_{jk}) = 0$, where X is a random variable with moment generating function $f_{jk}(t)$. For if not, then we can find n and j such that Pr $(S_n > 0 \mid k_0 = j, k_n = j) > 0$, which implies that $f_{jj}^{(n)}(u) \to \infty$ as $u \to \infty$. But this contradicts the boundedness of $\alpha_1(u)$ since $\{\alpha_1(u)\}^n \geqq p_{jj}^{(n)}(u)$. Thus for each j,k, $f_{jk}(t)$ represents a random variable which is bounded above and we may write

$$f_{jk}(t) = \exp(\beta_{jk}t)g_{jk}(t), \tag{4.8}$$

where β_{jk} is real for each j, k and

$$g_{jk}(t) = o(e^{\epsilon t}) \qquad (t \to +\infty) \tag{4.9}$$

for every $\epsilon > 0$. Let $\{x_j(t)\}$ be a right eigenvector of $P(t)$ corresponding to the eigenvalue $\alpha_1(t)$. We can choose $x_j(t)$ to be the co-factor of, say, the element in position $(1, j)$ of the matrix $[\alpha_1(t)I - P(t)]$ $(j = 1, \cdots, N)$. Thus, for each j, $x_j(t)$ is expressible as a sum of products of the elements of $[\alpha_1(t)I - P(t)]$. Hence from (4.8), (4.9) and the boundedness of $\alpha_1(t)$ $(t > 0)$, it follows that there is a finite real number β_j such that

$$x_j(t) = y_j(t)\exp(\beta_j t) \qquad (j = 1, \cdots, N), \tag{4.10}$$

where for each j and every $\epsilon > 0$

$$y_j(t) = o(e^{\epsilon t}) \qquad (t \to +\infty). \tag{4.11}$$

Let $T(t) = \text{diag}\{x_j(t)\}$. Then the matrix

$$\{r_{jk}(t)\} = [\alpha_1(t)]^{-1}[T(t)]^{-1}P(t)T(t) \tag{4.12}$$

is a stochastic transition matrix for each real t by Lemma 3.1 and hence for all real t we have $0 \leqq r_{jk}(t) \leqq 1$. From (4.12) we have for each j, k

$$p_{jk}f_{jk}(t)x_k(t) = x_j(t)\alpha_1(t)r_{jk}(t),$$

and from the relations (4.8) to (4.11) it follows that $\beta_{jk} + \beta_k \leqq \beta_j$ for each j, k for which $p_{jk} > 0$. The result now follows by taking $D(t) = \text{diag}\{\exp(\beta_j t)\}$.

In the case where each $f_{jk}(t)$ is the moment generating function of an integral random variable, each β_{jk} and β_j will be an integer.

(b) In (i), we have $\alpha_1(u_1) \geqq 0$ since $\alpha_1(u)$ is a nondecreasing function of each of the elements, and thus $\alpha_1(u)$ must attain its minimum in the interval $0 < u \leqq u_1$. In (ii) suppose that for some fixed j, k, $f_{jk}(u) \to \infty$ as $u \to u_0'-$. We choose n so that $P^n > 0$ and since $u_0' < \infty$, we can find $C > 0$ such that $f_{kj}^{(n)}(u) \geqq C$ as $u \to u_0'-$. Then we have

$$\begin{aligned}\{\alpha_1(u)\}^{n+1} \geqq f_{jj}^{(n+1)}(u) &\geqq p_{jk}p_{kj}^{(n)}f_{jk}(u)f_{kj}^{(n)}(u)\\ &\geqq Cp_{jk}p_{kj}^{(n)}f_{jk}(u) \to \infty \quad \text{as } u \to u_0'-.\end{aligned}$$

Thus $\alpha_1(u) \to \infty$ as $u \to u_0'-$ and the result follows.

We now explore further the properties of $\alpha_1(t)$ in the case of integral random variables. We first state a well known result concerning characteristic functions of integral random variables.

LEMMA 4.1. *Let $f(iv) = E(e^{ivX})$ (v real) where X is a non-degenerate integral random variable. Then $|f(iv_1)| = 1$ for some v_1 ($v_1 \neq 0$, $-\pi \leqq v_1 \leqq \pi$) if and only if $v_1/2\pi$ is a rational number, say $v_1 = 2\pi p/q$* (g.c.d. $[p, q] = 1$, $q > 1$), *and $f(iv)$ is of the form $e^{imv}g(iv)$, where m is an integer and $g(iv)$ is a characteristic function of period $2\pi/q$ in v, or equivalently if and only if X only takes values of the form $m + nq$ ($n = 0, \pm 1, \pm 2, \cdots$; m, q integral, $q > 1$)*

In the following theorem we prove a corresponding result for the functions $\alpha_j(iv)$ and it is sufficient to suppose that each $f_{jk}(t)$ exists only on the imaginary axis.

THEOREM 4. *Let $t = iv$ (v real) and suppose that each of the functions $f_{jk}(iv)$ is a characteristic function of an integral random variable. Let $\alpha_j(iv)$ ($j = 1, \cdots, N$) be the eigenvalues of $P(iv)$ where $\alpha_1(0) = 1$. Then there exists a number $v_1 \neq 0$ ($-\pi \leqq v_1 \leqq \pi$) satisfying $\alpha_j(iv_1) = 1$ for some j if and only if $v_1/2\pi$ is a rational number, say $v_1 = 2\pi p/q$* (g.c.d. $[p, q] = 1$, $q > 1$), *and $P(iv)$ is of the form*

$$P(iv) = e^{imv}D(iv)Q(iv)\{D(iv)\}^{-1}, \tag{4.13}$$

where

(i) *$Q(iv) = \{p_{jk}g_{jk}(iv)\}$, each g_{jk} being a characteristic function of period $2\pi/q$ in v, possibly $g_{jk}(iv) \equiv 1$;*

(ii) $D(iv) = \text{diag}\{\exp(im_j v)\}$ (*$m_1, \cdots, m_N$ integral*);

(iii) *m is integral.*

PROOF. If $P(iv)$ is of the form (4.13) then we may take $v_1 = 2\pi/q$ and then $\exp(imv_1)$ will be an eigenvalue of $P(iv_1)$.

Conversely, suppose that $e^{i\sigma}$ is an eigenvalue of $P(iv_1)$ ($v_1 \neq 0, -\pi \leqq v_1 \leqq \pi$). Since $\{P(iv_1)\}^* \leqq P$ it follows from Section 3(c) that

$$P(iv_1) = e^{i\sigma} DPD^{-1}, \tag{4.14}$$

where $D^* = I$, and hence that

$$|f_{jk}^{(n)}(iv_1)| = 1, \qquad \text{each } j, k \text{ and } n \text{ such that } p_{jk}^{(n)} > 0. \tag{4.15}$$

If $f_{jk}^{(n)}(iv)$ is degenerate for each j, k and n for which $p_{jk}^{(n)} > 0$, then $[\{P(iv)\}^n]^* = P^n$ for all v and n and thus $|\alpha_1(iv)| = 1$ (all v). Hence, again by Section 3(c) $P(iv) = \alpha_1(iv)D(v)P\{D(v)\}^{-1}$ where $D(v) = \operatorname{diag}\{d_j(v)\}$ ($|d_j(v)| = 1$ $j = 1, \cdots, N$). For each j, k and all sufficiently large n, therefore,

$$\{\alpha_1(iv)\}^n d_j(v)\{d_k(v)\}^{-1}$$

is a degenerate characteristic function, so that $\alpha_1(iv)$ must be of the form $e^{i\beta v}$ (β real and constant). It follows from Theorem 1(b) that $P(iv)$ is of the form (4.13) with $Q(iv) \equiv P$.

Otherwise, for some j, k and n, $f_{jk}^{(n)}(iv)$ is not degenerate and hence $v_1 = 2\pi p/q$ (g.c.d. $[p, q] = 1, q > 1$) by Lemma 4.1. In virtue of (4.15) we may write

$$f_{jk}^{(n)}(iv) = \exp(im_{jk}^{(n)}v)g_{jk}(iv), \qquad \text{each } j, k, n \text{ such that } p_{jk}^{(n)} > 0 \tag{4.16}$$

where $g_{jk}^{(n)}(iv)$ is a characteristic function of period $2\pi/q$ and $m_{jk}^{(n)}$ an integer. In (4.14) let $D = \operatorname{diag}\{\exp(i\beta_j)\}$ (β_j real, $j = 1, \cdots, N$). Then (4.16) (with $v = v_1 = 2\pi p/q$) implies that

$$2\pi p m_{jk}^{(n)}/q = n\sigma + \beta_j - \beta_k + 2N_{jk}^{(n)}\pi, \qquad N_{jk}^{(n)} \text{ integral}, \tag{4.17}$$

for each j, k and n such that $p_{jk}^{(n)} > 0$. By evaluating (4.17) at n and $n + 1$ (where n is such that $P^n > 0$) we obtain

$$\sigma = 2\pi pm/q + 2M\pi, \qquad m, M \text{ integral},$$

and (4.17) for j, h and k, h gives the result

$$\beta_j - \beta_k = (m_{jh}^{(n)} - m_{kh}^{(n)})2\pi p/q - 2(N_{jh}^{(n)} - N_{kh}^{(n)})\pi \tag{4.18}$$

Since the left hand side of (4.18) is independent of h we may take

$$\exp\{i(\beta_j - \beta_k)\} = \exp\{i(m_j - m_k)2\pi p/q\}, \qquad m_1, \cdots, m_N \text{ integral}.$$

Now from (4.17) we see that (writing $m_{jk} = m_{jk}^{(1)}$)

$$m_{jk}2\pi p/q = (m + m_j - m_k)2\pi p/q + 2N'_{jk}\pi, \qquad N'_{jk} \text{ integral}.$$

Hence $m_{jk} = m + m_j - m_k + N'_{jk}q/p$. Thus $N'_{jk}q/p$ must be an integer and since g.c.d. $(p, q) = 1$ we must have $m_{jk} = m + m_j - m_k + qM_{jk}$, M_{jk} integral. From (4.16) with $n = 1$ it follows that $P(iv)$ is of the form (4.13).

If $f(iv)$ is the characteristic function of an integral random variable, we may say that $f(iv)$ is expressed in its lowest terms if $f(iv) = e^{imv}g(iv)$, where $g(iv)$ has minimal period $2\pi/q$ (q integral, $q \geqq 1$) and m is an integer satisfying $0 \leqq m < q$. Analogously, in the matrix case we may say that $P(iv)$ is expressed in its lowest terms if it is written in the form (4.13) where $Q(iv)$ has minimal period $2\pi/q$ ($q \geqq 1$) and $0 \leqq m < q$. Hence an alternative statement of Theorem 4 is

THEOREM 4′. *If $P(iv)$ is expressed in its lowest terms in the form* (4.13), *then* $|\alpha_j(iv)| < 1$ $(0 < |v| \leqq \pi; j = 1, \cdots, N)$ *if and only if* $q = 1$.

5. The probability of tail values of the sums S_n. We use the notation and definitions of Section 2 and we suppose that each $f_{jk}(t)$ is an analytic moment generating function of an integral random variable. We suppose also that $P(iv)$, when expressed in its lowest terms, satisfies the conditions of Theorem 4′, i.e., if $Q(iv)$ has minimal period $2\pi/q$ (q integral, $q \geqq 1$) where

$$(5.1) \qquad P(t) = e^{mt}[\operatorname{diag}\{\exp(m_j t)\}]Q(t)[\operatorname{diag}\{\exp(m_j t)\}]^{-1},$$

$m, m_1, \cdots, m_N$ integral, then $q = 1$. If $P(iv)$ does not satisfy these conditions, i.e., if $q > 1$, then we write $Q_1(iv) = Q(iv/q)$. Now $Q_1(iv)$ has minimal period 2π, and it would be sufficient to study Q_1 instead of Q. Hence it is clearly no loss of generality to suppose that $q = 1$. Accordingly, we summarize our assumptions concerning $P(t)$ as follows:

(5.2)
(i) $P(iv)$ has minimal period 2π in v,
(ii) $P(t)$ is not reducible to the form (5.1) with $q > 1$,
(iii) $P(t)$ is not degenerate.

If a is any real number, then $e^{-at}\alpha_1(t)$ is the maximal positive eigenvalue of the matrix $e^{-at}P(t)$ and is therefore a strictly convex function for real t. We choose a so that the matrix $e^{-at}P(t)$ satisfies one of the conditions of Theorem 3 and also so that $a > \alpha_1'(0)$, thus ensuring that $e^{-at}\alpha_1(t)$ attains its unique minimum at a real, positive, finite value of t. Let

$$m(a) = \inf_{t>0} e^{-at}\alpha_1(t)$$

and let $t^*(a)$ satisfy $m(a) = \exp\{at^*(a)\}\alpha_1(t^*(a))$. Since $\alpha_1'(0) < a$ we have $t^*(a) > 0$ and $0 < m(a) < 1$. For brevity we write $t^* = t^*(a)$. We now define the matrices

$$\Phi_n(a) = \{p_{jk}^{(n)} \Pr(S_n = na \mid k_0 = j, k_n = k)\}$$

and

$$\Pi_n(a) = \{p_{jk}^{(n)} \Pr(S_n \geqq na \mid k_0 = j, k_n = k)\},$$

and our task will be to obtain asymptotic expressions for these as $n \to \infty$. We shall follow closely the methods used by Blackwell and Hodges [1].

The matrix $e^{-at^*}P(t^*)$ is non-negative, irreducible and primitive, so that it has positive right and left eigenvectors $x^* = (x_j^*)$, $y^* = (y_j^*)$ respectively such that $y^*x^* = 1$. Let

$$r_{jk} = e^{-at^*}\{m(a)\}^{-1}x_k^*(x_j^*)^{-1}p_{jk}f_{jk}(t^*).$$

Then it follows from Lemma 3.1 that $R = (r_{jk})$ is the transition matrix of an ergodic Markov chain with no cyclically moving sub-classes. Let K_n denote the state at time n in a realization of this chain ($n = 0, 1, 2, \cdots$). Let

$$(5.3)\qquad \begin{aligned} R(t) &= \{r_{jk}f_{jk}(t + t^*)/f_{jk}(t^*)\} \qquad \text{i.e.,} \\ R(t) &= \{m(a)\}^{-1}e^{-at^*}D^{-1}P(t + t^*)D, \end{aligned}$$

where $D = \operatorname{diag}(x_j^*)$. For each j, k for which $p_{jk} > 0$, $f_{jk}(t + t^*)/f_{jk}(t^*)$ is the moment generating function of an integral random variable. We define a sequence of random variables $Y_1, Y_2, \cdots$ associated with the Markov chain $K_0, K_1, K_2, \cdots$ in such a way that Y_n has the moment generating function $f_{jk}(t + t^*)/f_{jk}(t^*)$ if $K_{n-1} = j$ and $K_n = k$. Thus $Y_1, Y_2, \cdots$ are associated with $R(t)$ in the same way as $X_1, X_2, \cdots$ are associated with $P(t)$.

Let $R^n = (r_{jk}^{(n)})$ and $T_n = Y_1 + \cdots + Y_n$. If we raise each side of (5.3) to the power n and equate coefficients of e^{nat} (assuming na to be an integer) we obtain the relation

$$(5.4)\qquad \begin{aligned} &p_{jk}^{(n)} \Pr(S_n = na \mid k_0 = j, k_n = k) \\ &\qquad = \{m(a)\}^n x_j^*(x_k^*)^{-1}r_{jk}^{(n)} \Pr(T_n = na \mid K_0 = j, K_n = k) \end{aligned}$$

which corresponds to Theorem 1 of Blackwell and Hodges. Further, for any integer s, we have

$$(5.5)\qquad \begin{aligned} &p_{jk}^{(n)} \Pr(S_n = na + s \mid k_0 = j, k_n = k) \\ &\qquad = \{m(a)\}^n x_j^*(x_k^*)^{-1}e^{-st^*}r_{jk}^{(n)} \Pr(T_n = na + s \mid K_0 = j, K_n = k). \end{aligned}$$

Let $\beta_1(t) = \alpha_1(t + t^*)/\alpha_1(t^*)$, $\beta_2(t), \cdots \beta_N(t)$ be the eigenvalues of $R(t)$. Since $\beta_1'(0) = a$, the asymptotic expectation of the increment $T_n - T_{n-1}$ is a, whereas that of $S_n - S_{n-1}$ is $\alpha_1'(0)$. Thus we have achieved a shift of expectation similar to that of Blackwell and Hodges and others mentioned in [1].

For each j, k the possible values of Y_1 are identical to those of X_1 and so $|\beta_1(iv)| < 1$ $(0 < |v| \leqq \pi)$ by Theorem 4′. Since $\beta_1(0) = 1$ and $[R(iv)]^* \leqq R$, it follows from Section 3(c) that $|\beta_j(iv)| < 1$ $(j = 2, 3, \cdots, N;\ -\pi \leqq v \leqq \pi)$ and hence by continuity that there exists a number η $(0 < \eta < 1)$ such that $|\beta_j(iv)| \leqq \eta$ $(j = 2, 3, \cdots, N;\ -\pi \leqq v \leqq \pi)$. Let $x(t) = \{x_j(t)\}$ and $y(t) = \{y_j(t)\}$ be respectively right and left eigenvectors of $R(t)$ corresponding to the root $\beta_1(t)$, chosen so that $x_j(0) = 1, j = 1, \cdots, N$ and

$$\sum_{j=1}^{N} x_j(t)y_j(t) = 1.$$

It follows from the Jordan canonical form for $R(t)$ that

$$(5.6)\qquad \{R(iv)\}^n = x(iv)y(iv)\{\beta_1(iv)\}^n + O(\eta^n).$$

Let $\sigma^2 = \beta_1''(0) - a^2$ $(= \dot{\alpha}_1''(t^*)/\alpha_1(t^*) - a^2)$ and we have

THEOREM 5. *Provided that na is a possible value of S_n the following asymptotic matrix relations hold as $n \to \infty$*

(i) $\Phi_n(a) = \{m(a)\}^n\{\sigma(2\pi n)^{\frac{1}{2}}\}^{-1}x^*y^*\{I + O(n^{-1})\}$;

(ii) $\Pi_n(a) = \{m(a)\}^n[\sigma(2\pi n)^{\frac{1}{2}}(1 - e^{-t^*})]^{-1}x^*y^*\{I + O(n^{-1})\}$.

PROOF. It follows from (5.6) and the theory of Fourier series that

$$r_{jk}^{(n)} \Pr(T_n = na \mid K_0 = j, K_n = k)$$

$$= (2\pi)^{-1}\int_{-\pi}^{\pi} e^{-ianv}\{\beta_1(iv)\}^n x_j(iv)y_k(iv)\,dv + O(\eta^n) \tag{5.7}$$

$$= (2\pi)^{-1}\int_{-\pi}^{\pi} B_n(v)\,dv + O(\eta^n), \text{ say.}$$

Since $\beta_1'(0) = a$ and $\beta_1''(0) - a^2 = \sigma^2$, we may choose v_0 (>0) so that

$$|e^{-iav}\beta_1(iv)| \leqq 1 - \sigma^2v^2/3 \qquad (|v| \leqq v_0). \tag{5.8}$$

We break up the range of integration in (5.7) into the ranges $|v| \leqq n^{-\frac{1}{2}}\log n$, $n^{-\frac{1}{2}}\log n < |v| \leqq v_0$, and $v_0 < |v| \leqq \pi$. In the first of these ranges we have the expansions

$$\log[\{e^{-iav}\beta_1(iv)\}^n] = -n\sigma^2v^2/2 + n\sum_{r=3}^{\infty} b_rv^r$$

and

$$x_j(iv)y_k(iv) = x_k^*y_k^* + \sum_{r=1}^{\infty} c_rv^r \qquad (c_r = c_r(j,k),\ r = 1, 2 \cdots)$$

since, in the latter expansion, $y(0)x(0) = 1$ and therefore, by Lemma 3.1, $y_k(0) = x_k^*y_k^*$. Thus on setting $w = n^{\frac{1}{2}}\sigma v$, the integrand in (5.7) may be written, for $|w| \leqq \sigma\log n$,

$$e^{-\frac{1}{2}w^2}[x_k^*y_k^* + wC_1(w^2)n^{-\frac{1}{2}} + C_2(w^2)n^{-1} + o(n^{-1})]$$

where C_1 and C_2 are polynomials in w^2, depending on j and k but not on n. Using the result that

$$\int_{-\sigma\log n}^{\sigma\log n} t^pe^{-\frac{1}{2}t^2}dt = 2^{\frac{1}{2}(p+1)}\Gamma\{\tfrac{1}{2}(p+1)\} + o(n^{-2})$$

when p is even and vanishes when p is odd, we have

$$(2\pi)^{-1}\int_{-n^{-\frac{1}{2}}\log n}^{n^{-\frac{1}{2}}\log n} B_n(v)\,dv = (2\pi n\sigma^2)^{-\frac{1}{2}}x_k^*y_k^*\{1 + O(n^{-1})\}.$$

In virtue of (5.8) we have

$$\int_{n^{-\frac{1}{2}}\log n<|v|\leqq v_0} B_n(v)\,dv = 0\left(\int_{n^{-\frac{1}{2}}\log n}^{\infty}\exp(-\tfrac{1}{3}n\sigma^2v^2)\,dv\right)$$

which is $o(n^{-K})$ for all K. In the range $v_0 < |v| \leqq \pi$, $|\beta_1(iv)| \leqq \rho$, say, $(0 < \rho < 1)$ and so

$$\int_{v_0<|v|\leqq\pi} B_n(v)\, dv = O(\rho^n).$$

Combining these we find finally

$$r_{jk}^{(n)} \Pr (T_n = na \,|\, K_0 = j, K_n = k) = (2\pi n\sigma^2)^{-\frac{1}{2}} x_k^* y_k^* \{1 + O(n^{-1})\}$$

and the result (i) now follows from (5.4). The result (ii) follows from (5.5) by summing with respect to s $(s = 0, 1, 2, \cdots)$.

6. Acknowledgments. I should like to thank Dr. P. Whittle for helpful suggestions and to acknowledge a grant from the South African Council for Scientific and Industrial Research.

REFERENCES

[1] David Blackwell and J. L. Hodges, Jr., "The probability in the extreme tail of a convolution," *Ann. Math. Stat.*, Vol. 30 (1959), pp. 1113–1120.
[2] Salomon Bochner and William Ted Martin, *Several Complex Variables*, Princeton University Press, Princeton, N. J., 1948.
[3] Gerard Debreu and I. N. Herstein, "Non-negative square matrices," *Econometrica*, Vol. 21 (1953), pp. 597–607.
[4] I. S. Volkov, "On the distribution of sums of random variables defined on a homogeneous Markov chain with a finite number of states," *Teor. Veroyat. i ee Prim.*, Vol. 3 (1958), pp. 413–429 (in Russian).
[5] I. S. Volkov, "On probabilities for extreme values of sums of random variables defined on a homogeneous Markov chain with a finite number of states", *Teor. Veroyat. i ee Prim.*, Vol. 5 (1960), pp. 338–352 (in Russian).