

# A GENERAL MODEL FOR THE RELIABILITY ANALYSIS OF SYSTEMS UNDER VARIOUS PREVENTIVE MAINTENANCE POLICIES<sup>1</sup>

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**Summary.** The problem treated is that of predicting the reliability characteristics of a complex system from data on individual components. A general model for systems maintained over a period of time is proposed, based on the idea that every system failure is induced by a component failure and corrected by the replacement of a single component. Moreover, it is assumed that components are sometimes replaced even when the system is operating correctly, in order to prevent unscheduled interruptions in operation. The assumptions which define the general model cover a number of different preventive maintenance policies, among them the following:

(a) *Block Changes:* All components of a given type are replaced simultaneously, at times determined by a renewal process.

(b) *Individual Component Replacement on the Basis of Age:* If a component reaches some given age without failing, it is preventively replaced.

(c) *System Check-Outs:* If a component is used only intermittently and it fails while it is not being used, it does not induce a system failure until it is called into use. At regular intervals, those components which have failed without inducing system failure are located and replaced.

(d) *Marginal Testing:* At regular intervals, a test is conducted to locate those components which are still operating satisfactorily but which are expected to fail in the near future. All components located by this test are replaced.

It is assumed that preventive removals are regeneration points and that the performance of a component may be described by a distribution function  $F(x: y)$ , the probability that a component is removed by time  $x$ , given that it enters the system at  $y$ , where  $x$  and  $y$  are both measured from the time of the last preventive removal.  $F(x: y)$  is the sum of  $A(x: y)$  and  $B(x: y)$ , where  $A(x: y)$  is the probability that the component is preventively removed by  $x$  and  $B(x: y)$  is the probability that the component induces a system failure by  $x$ . The integral equations which determine the following measures of system performance from  $F(x: y)$ ,  $A(x: y)$ , and  $B(x: y)$  are developed:

- (1) the expected number of failures in a given time interval
- (2) the expected number of preventive removals in a given time interval
- (3) the reliability function; i.e., the probability of no failure in a given interval following a given system age.

Results from Renewal Theory and the Theory of Regenerative Stochastic

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Processes, developed by W. L. Smith, are applied to the problem of exploring the asymptotic behavior of these quantities.

Conditions sufficient for maintenance policies a, b, c, and d to meet the assumptions of the general model are precisely formulated, and the analysis necessary to derive  $F(x: y)$ ,  $A(x: y)$ , and  $B(x: y)$  is carried out for each policy.

**1. Introduction.** In modern technology there is a pressing need for methods of predicting the reliability characteristics of complex systems from data on individual components. In general, the techniques that are currently available fail to take into account adequately a number of important factors, such as the change in component survival probability with age, the effects of preventive maintenance procedures, and the effects of intermittent component usage. The simplest and most common prediction model is based on the assumptions that all components fail independently, that every component failure induces a system failure, and that the distribution functions of time to component failure are exponential. In this model, no preventive maintenance is considered. Under these assumptions, the times between system failures are exponentially distributed with a mean equal to the inverse of the sum over all component positions of the inverses of the mean times to component failure. A number of papers which consider one or more modifications of this model appear in the literature. When the assumption of exponentiality is eliminated, the times between failure in a single component position are treated as a renewal process. The expected number of failures and the survival probability for a given interval as a function of system age, both exact and asymptotic values, are obtained by applying well-known results in Renewal Theory. [6], [7], [8], [14], [15] Several authors have considered the effects of removing components which survive to some predetermined age. [4], [5], [17] However, no satisfactory model which includes the effects of marginal testing or of intermittent component usage has, as yet, been devised.

In Section 2, the precise assumptions that define the general model proposed in this paper are formulated and, in Section 3, the integral equations which determine the significant measures of performance are developed. Section 4 discusses the application of asymptotic results from Renewal Theory and Smith's analysis of equilibrium and cumulative processes [13, 14] to yield the asymptotic values of these performance measures. Then, in Section 5, it is demonstrated that, with appropriate assumptions, systems subject to maintenance policies a, b, c, and d are special cases of the general model.

**2. Assumptions—General Model.** In this section, the assumptions which define the general model of a system are tabulated. In Section 5, it will be demonstrated that the four maintenance policies mentioned in the summary meet these assumptions.

(1) The system is an assemblage of a finite number of components which performs some function.

(2) The components fail independently and permanently.

(3) The system operates continuously except for interruptions due to failures and preventive maintenance procedures. All maintenance time is neglected.

(4) When a system failure occurs, repair is effected by the replacement of one failed component. This is termed a "failure removal."

(5) Components may be replaced at times when no system failures occur. These are termed "preventive removals." With probability one, only a finite number of preventive removals occur in any finite interval.

(6) When a component is removed, it is replaced by a new one from the same population.

(7) For each component position, the preventive removals are regeneration points, i.e., if it is known that a preventive removal occurs at some time  $y$ , then knowledge of failures or maintenance procedures at times less than  $y$  has no predictive value. Associated with each component there is a distribution function  $F(x: y)$  of time of removal for a component which enters the system at  $y$ , where  $x$  and  $y$  are both measured from the time of the last preventive removal or, if no preventive removal has occurred, from the initial use of the system. (Assumption 5 justifies the term "last preventive removal.")

(8)  $F(x: y)$  is the sum of  $A(x: y)$  and  $B(x: y)$ , where  $A(x: y)$  is the probability that the component is preventively removed by  $x$  and  $B(x: y)$  is the probability that the component induces a system failure by  $x$ .

(9)  $F(x: y)$ ,  $A(x: y)$ , and  $B(x: y)$  have the following properties:

$$(a) \quad F(x: y) = 0 \quad \text{for } x \leq y,$$

$$(b) \quad \lim_{x \rightarrow y \rightarrow \infty} F(x: y) = 1,$$

(c) There exists a constant  $\alpha > 0$ , such that for any  $y$ ,

$$\alpha \leq \lim_{x \rightarrow \infty} A(x: y) \leq 1, \quad 0 \leq \lim_{x \rightarrow \infty} B(x: y) \leq 1 - \alpha.$$

It is intuitively clear that Assumptions 1 through 6 are approximately valid for many systems under a variety of maintenance policies. Assumptions 7, 8, and 9 will be more fully justified in Section 5 where it is demonstrated that a distribution function  $F(x: y)$  which has the required properties may be formulated for each of the maintenance policies mentioned in the summary. Heuristically, we note that Assumption 7 implies that, whenever a preventive removal occurs, the prediction of future behavior in that component position proceeds exactly as if the system were new at the time of the preventive replacement. When a failure removal occurs, on the other hand, the time to the next maintenance procedure is, in general, different from the time between maintenance points, so that a failure removal does not constitute a regeneration point and  $F(x: y)$  is not necessarily equal to  $F(x - y: 0)$ . Assumption 9 states that, with probability one, a component is not removed immediately after it enters the system, but that it will be removed at some time. Furthermore, the probability that a component is preventively removed before it causes a system failure is bounded away from zero, over all system entry time,  $y$ .

**3. General analysis.** Three significant measures of performance for systems subject to preventive maintenance have been chosen, and the analysis necessary

to predict these quantities from component data will now be developed. These three measures are:

- (1) the expected number of system failures as a function of time,
- (2) the expected number of preventive removals of components of each type as a function of time,
- (3) the system reliability function, which is the probability of no system failure in a given interval following a given system age.

Since we have assumed that to each system failure there corresponds one failure removal (Assumption 4), the expected number of system failures is the sum over all component positions of the expected number of failure removals. Correspondingly, the expected number of preventive removals of a component of a given type may be obtained by summing over all positions which contain components of that type. Finally, because of the assumption of independence (Assumption 2), the system reliability function is the product, over all component positions, of the probabilities of no failure removals in a given interval following a given system age. Thus, all three measures of system performance may be obtained from elementary operations on the corresponding functions for all component positions. The analysis that follows is therefore concerned with a single component position, for which functions  $U_p(t)$ ,  $U_f(t)$ , and  $R(t; x)$  defined below will be derived. In these definitions, it is assumed that time is measured from the initial use of the system.

(1)  $U_p(t) = E\{N_p(t)\}$ , where  $N_p(t)$  is the number of preventive removals in the interval  $(0, t]$ ,

(2)  $U_f(t) = E\{N_f(t)\}$ , where  $N_f(t)$  is the number of failure removals in the interval  $(0, t]$ ,

(3)  $R(t; x)$  is the probability of no failure removal in the interval  $(x, x + t]$ .

Since it has been assumed that a preventive removal constitutes a regeneration point, it is clear that the number of failures between preventive removals and the time between preventive removals are two sequences of independent, identically distributed random variables. In accordance with this line of reasoning,  $U_p(t)$  and  $U_f(t)$  will be derived by the following sequence of operations:

(1)  $\mathfrak{B}(t)$ , the expected number of successive failures in the interval  $(0, t]$  and before the first preventive removal, will be expressed in terms of an integral equation involving  $B(x; y)$ .

(2)  $G(t)$ , the distribution function of the time between preventive removals, will be expressed in terms of  $\mathfrak{B}(t)$  and  $F(x; y)$ .

(3)  $U_p(t)$  is then the renewal function determined by  $G(t)$ .

(4)  $U_f(t)$  will be expressed in terms of  $U_p(t)$  and  $\mathfrak{B}(t)$ .

Finally a method of obtaining  $R(t; x)$  from functions previously defined will be presented.

**THEOREM 3.1.** *The function  $\mathfrak{B}(t)$  defined above is the unique solution of*

$$(3.1) \quad \mathfrak{B}(t) = B(t; 0) + \int_0^t B(t; y) d\mathfrak{B}(y).$$

Moreover, the expected number of failures between preventive removals is finite, i.e.,  $\nu_1 \equiv \mathfrak{B}(\infty) < \infty$ .

PROOF. Let  $B^{(k)}(t)$  be the probability of at least  $k$  failures in  $(0, t]$  and before the first preventive removal. Then

$$B^{(1)}(t) = B(t: 0) \quad \text{and} \quad B^{(k+1)}(t) = \int_0^t B(t: y) dB^{(k)}(y).$$

Furthermore,

$$\mathfrak{B}(t) = \sum_{k=1}^{\infty} B^{(k)}(t) = B(t: 0) + \sum_{k=1}^{\infty} \int_0^t B(t: y) dB^{(k)}(y).$$

Since, by Assumption 9c,  $B(t: y) \leq 1 - \alpha < 1$ , for all  $t$  and  $y$ ,

$$B^{(1)}(t) \leq 1 - \alpha, \quad B^{(k)}(t) \leq (1 - \alpha)B^{(k-1)}(t) \leq (1 - \alpha)^k,$$

and therefore  $\mathfrak{B}(t) \leq (1 - \alpha)/\alpha$  for all  $t$ . Since  $\mathfrak{B}(t)$  is non-decreasing and bounded, the theorem follows.

THEOREM 3.2.

$$(3.2) \quad G(t) = F(t: 0) - \int_0^t [1 - F(t: y)] d\mathfrak{B}(y),$$

and  $G(\infty) = 1$ , i.e.,  $G(t)$  is a proper distribution function.

PROOF:  $G(t)$  is the probability that the component which enters the system at 0 is preventively removed in the interval  $(0, t]$  plus the probability that one or more failures occur in  $(0, t]$  and some replacement is preventively removed at or before  $t$ . Thus,  $G(t) = A(t: 0) + \int_0^t A(t: y) d\mathfrak{B}(y)$ . Upon substituting  $A(t: y) = F(t: y) - B(t: y)$  and using (3.1) one obtains (3.2).

To prove that  $G$  is a proper distribution function, we note that  $F(\infty: 0) = 1$  and  $\lim_{t \rightarrow \infty} \int_0^t [1 - F(t: y)] d\mathfrak{B}(y) = 0$ . This is a consequence of the Lebesgue Dominated Convergence Theorem since  $B(\infty) < \infty$  by Theorem 1 and  $1 - F(\infty: y) = 0$  for all  $y$  by Assumption 9b.

THEOREM 3.3.  $U_p(t)$  is the renewal function associated with  $G(t)$ , and so satisfies

$$(3.3) \quad U_p(t) = G(t) + \int_0^t U_p(t - \tau) dG(\tau).$$

This is the well-known renewal equation and no proof is required.

THEOREM 3.4.

$$(3.4) \quad U_f(t) = \mathfrak{B}(t) + \int_0^t \mathfrak{B}(t - \tau) dU_p(\tau).$$

PROOF: This equation follows from the fact that the total number of failures in  $(0, t]$  is equal to the number of failures in  $(0, t]$  before the first preventive removal plus the number of failures in  $(0, t]$  that follow preventive removals, summed over all the preventive removals that take place by  $t$ .

THEOREM 3.5.  $R(t; x)$ , the probability of no failure in  $(x, x + t]$ , may be ex-

pressed as follows:

$$(3.5) \quad R(t; x) = \Psi_t(x) + \int_0^x \Psi_t(x - \xi) dU_p(\xi),$$

where

$$(3.6) \quad \Psi_t(x) = 1 - F(x + t; 0) + \int_0^x [1 - F(x + t; y)] d\mathcal{R}(y) \\ + \int_0^t [1 - F(t - \tau; 0)] d\tau \mathcal{G}(\tau, x),$$

and  $\mathcal{G}(\tau, x)$  is determined by the following integral equation:

$$(3.7) \quad \mathcal{G}(\tau, x) = A(x + \tau; 0) - A(x; 0) + \int_0^x [A(x + \tau; y) - A(x; y)] d\mathcal{R}(y) \\ + \int_0^\tau A(\tau - u; 0) d\mathcal{G}(u, x).$$

PROOF. For any fixed  $t$ , define a stochastic process  $\{Z_t(x) : x \geq 0\}$  by

$$Z_t(x) = \begin{cases} 1 & \text{if no failure occurs in } (x, x + t] \\ 0 & \text{if any failure occurs in } (x, x + t]. \end{cases}$$

Clearly,  $R(t; x) = \Pr \{Z_t(x) = 1\}$ . According to the assumptions defining this model,  $Z_t(x)$  is an equilibrium process over  $x$ , as defined by Smith [13]; i.e., if  $S_{N_p(x)}$  is the time of the last preventive removal in  $(0, x]$  (with probability one, this is a well-defined point because of Assumption 5), we have

$$(3.8) \quad \Pr \{Z_t(x) = 1 \mid N_p(x) > 0; S_{N_p(x)}\} = \frac{\Psi_t[x - S_{N_p(x)}]}{1 - G[x - S_{N_p(x)}]},$$

where  $\Psi_t(x) = \Pr \{Z_t(x) = 1, N_p(x) = 0\}$ , i.e.,  $\Psi_t(x)$  is the probability of no failure in  $(x, x + t]$  and no preventive removal in  $(0, x]$ . Then, as Smith pointed out ([13], p. 15, (3.4.3)),  $R(t; x) = \Psi_t(x) + \int_0^x \Psi_t(x - \xi) dU_p(\xi)$ .

Now  $\Psi_t(x)$  is equal to the probability of no removal of either kind in  $(0, x + t]$  plus the probability of one or more successive failures in  $(0, x]$ , with the last replacement remaining in the system past  $x + t$ , plus the probability of no preventive removal in  $(0, x]$ , no failure in  $(x, x + t]$ , but one or more preventive removals in  $(x, x + t]$ . Thus, by utilizing this breakdown, we obtain (3.6), with  $\mathcal{G}(\tau, x)$  defined by

$$(3.9) \quad \mathcal{G}(\tau, x) = \sum_{j=1}^{\infty} \mathcal{G}^{(j)}(\tau, x),$$

where  $\mathcal{G}^{(j)}(\tau, x)$  is the probability of at least  $j$  preventive removals in  $(x, x + \tau)$ , no failures between  $x$  and the  $j$ th preventive removal and no preventive removals in  $(0, x]$ .

Clearly,  $\mathcal{G}^{(1)}(\tau, x)$  is the probability that the initial component is preventively removed in  $(x, x + \tau]$  plus the probability of one or more successive failures in

$(0, x]$  followed by a preventive removal in  $(x, x + \tau]$ . Thus, we have,

$$(3.10) \quad \begin{aligned} \mathfrak{Q}^{(1)}(\tau, x) &= A(x + \tau: 0) - A(x: 0) \\ &+ \int_0^x [A(x + \tau: y) - A(x: y)] d\mathfrak{G}(y). \end{aligned}$$

Furthermore, for  $j > 1$ ,

$$(3.11) \quad \mathfrak{Q}^{(j)}(\tau, x) = \int_0^\tau A(\tau - u: 0) d\mathfrak{Q}^{(j-1)}(u, x).$$

From (3.9) and (3.11), we obtain

$$\mathfrak{Q}(\tau, x) = \mathfrak{Q}^{(1)}(\tau, x) + \int_0^\tau A(\tau - u: 0) d\mathfrak{Q}(u, x).$$

Substituting (3.10) yields (3.7) and completes the proof.

**4. Asymptotic results.** In the previous section, methods were developed for determining the expected number of failures, the expected number of preventive removals, and the reliability function for a single component position for finite time. Now the asymptotic properties of these functions will be established by using well-known results from Renewal Theory. It has already been pointed out that the times between consecutive preventive removals constitute a renewal process and that the random variable  $Z_i(x)$ , which is equal to one when no failure occurs in an interval  $t$  following system age  $x$  is an equilibrium process in the sense of Smith [13]. Furthermore, it will be shown that the number of failures by system age  $x$  is a cumulative process, as defined by Smith ([14], p. 262). Using these ideas, the limiting values of the significant statistical measures of performance will be expressed in terms of  $\nu_1$  defined in Theorem (3.1) and the moments of  $G(t)$ .

First, we shall consider the asymptotic properties of  $U_p(t)$  and other probabilistic measures of  $N_p(t)$ . The pertinent theorems fall into two categories, depending upon whether or not the time between preventive removals is a lattice random variable. For a system model which corresponds to a policy of preventive maintenance such as marginal test, system check-out, or block change at fixed intervals,  $G(t)$  will of course be a lattice distribution function. If, on the other hand, the times between these maintenance procedures are random variables with a continuous distribution function,  $G(t)$  will be non-lattice. Well-known results in Renewal Theory, as summarized in [14], [15], lead to the following asymptotic properties of  $N_p(t)$  and  $U_p(t)$ . We make the convention that  $\mu_1^{-1} = 0$  when  $\mu_1 = \infty$ .

$$(a) \quad \lim_{t \rightarrow \infty} N_p(t)/t = \mu_1^{-1} \text{ a.s.}$$

$$(b) \quad \lim_{t \rightarrow \infty} U_p(t)/t = \mu_1^{-1}.$$

(c) If  $G(t)$  is a lattice function with period  $T$ ,

$$\lim_{t \rightarrow \infty} \{U_p(t + jT) - U_p(t)\} = jT\mu_1^{-1}.$$

If  $G(t)$  is non-lattice,

$$\lim_{t \rightarrow \infty} \{U_p(t + \Delta t) - U_p(t)\} = \Delta t \mu_1^{-1}.$$

(d) For  $G(t)$  lattice with period  $T$  and  $\mu_2 < \infty$ ,

$$\lim_{j \rightarrow \infty} \{U_p(jT) - jT \mu_1^{-1}\} = \frac{\mu_2 + \mu_1 T}{2\mu_1^2} - 1.$$

For  $G(t)$  non-lattice and  $\mu_2 < \infty$ ,

$$\lim_{t \rightarrow \infty} \{U_p(t) - t \mu_1^{-1}\} = \{\mu_2 / (2\mu_1^2)\} - 1.$$

(e) If  $G(t)$  is lattice with period  $T$  and  $u_j$  is the probability of a preventive removal at  $jT$ ,  $\lim_{j \rightarrow \infty} u_j = T \mu_1^{-1}$ .

(f) Let  $\sigma_p^2(t) = \text{var } N_p(t)$  and  $\sigma_1^2 = \mu_2 - \mu_1^2 < \infty$ . Then

$$\lim_{t \rightarrow \infty} \{\sigma_p^2(t) / t\} = \sigma_1^2 / \mu_1^3.$$

(g) If  $\sigma_1^2 < \infty$ ,

$$\lim_{t \rightarrow \infty} \Pr \{N_p(t) \geq t \mu_1^{-1} - \alpha \sigma_1 \mu_1^{-1} (t \mu_1^{-1})^{\frac{1}{2}}\} = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\alpha} e^{-\frac{1}{2}y^2} dy.$$

In order to obtain corresponding asymptotic results about  $N_f(t)$  and  $U_f(t)$ , we shall prove that  $N_f(t)$  is a cumulative process as defined by Smith [14] and that all the moments of the number of failures between preventive removals are finite. Results of Smith may then be applied directly.

**THEOREM 4.1.**  $N_f(t)$  is a "cumulative process" with 0 and the times of preventive removal as regeneration points, i.e., if  $T_1, T_2, \dots$  are the times at which preventive removals occur, the following conditions hold:

(1)  $N_f(T_1) = 0, N_f(T_2) - N_f(T_1), \dots$  is a sequence of independent, identically distributed random variables.

(2)  $N_f(t)$  is, with probability one, of bounded variation in every finite  $t$ -interval.

(Smith's third condition is redundant in the present context since  $N_f(t)$  is non-decreasing.)

**PROOF.** Condition 1 follows from Assumption 7. To establish Condition 2, we note first that the number of failures between two preventive removals is clearly finite with probability one, since

$$\lim_{n \rightarrow \infty} \Pr \{N_f(T_i) - N_f(T_{i-1}) \geq n\} = \lim_{n \rightarrow \infty} B^{(n)}(\infty) \leq \lim_{n \rightarrow \infty} (1 - \alpha)^n = 0.$$

Furthermore, in Assumption 5 we have assumed that, with probability one, only a finite number of preventive removals occur in any finite interval. Together, these two ideas validate Condition 2.

**THEOREM 4.2.** Let  $\nu_r$  be the  $r$ th moment of  $N_f(T_i) - N_f(T_{i-1})$ , the number of failures between preventive removals. Then, for  $r \geq 1, \nu_r < \infty$ .

**PROOF:** Since the probability of exactly  $n$  failures between preventive removals



is equal to  $B^{(n)}(\infty) - B^{(n+1)}(\infty)$ , we have

$$\nu_r = \sum_{n=1}^{\infty} n^r \{B^{(n)}(\infty) - B^{(n+1)}(\infty)\}.$$

Furthermore,

$$\sum_{n=1}^{\infty} n^r B^{(n)}(\infty) \leq \sum_{n=1}^{\infty} n^r (1 - \alpha)^n < \infty$$

so that

$$\nu_r = \sum_{k=1}^r (-1)^{k-1} \binom{r}{k} \sum_{n=1}^{\infty} n^{r-k} B^{(n)}(\infty) < \infty.$$

Now we have all the preliminaries necessary to stating the asymptotic properties of  $U_f(t)$  and  $N_f(t)$ .

THEOREM 4.3.

(a)  $\lim_{t \rightarrow \infty} (N_f(t))/t = \nu_1 \mu_1^{-1}$  a.s.

(b)  $\lim_{t \rightarrow \infty} (U_f(t))/t = \nu_1 \mu_1^{-1}$ .

(c) Let  $\sigma_f^2(t) = \text{var } N_f(t)$  and  $\gamma = \sigma_2^2 - 2\rho\sigma_1\sigma_2\nu_1\mu_1^{-1} + \sigma_1^2\nu_1^2\mu_1^{-2}$ , where  $\sigma_1^2 = \mu_2 - \mu_1^2 < \infty$ ,  $\sigma_2^2 = \nu_2 - \nu_1^2$ , and

$$\rho\sigma_1\sigma_2 = \sum_{n=1}^{\infty} \int_0^{\infty} nt \, d_t \left[ \int_0^t A(t:y) \, dB^{(n)}(y) \right] - \nu_1 \mu_1.$$

Then  $\lim_{t \rightarrow \infty} \{\sigma_f^2(t)/t\} = \gamma\mu_1^{-1}$ .

(d) If  $\mu_2 < \infty$ ,

$$\lim_{t \rightarrow \infty} \Pr \{N_f(t) - \nu_1 \mu_1^{-1} t \leq \alpha(\gamma t \mu_1^{-1})^{\frac{1}{2}}\} = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\alpha} e^{-\frac{1}{2}y^2} \, dy.$$

PROOF: Statements a and b are direct applications of Smith's Theorems 7 and 8i. [13]

Statements c and d are given by Smith's Theorems 8ii and 10 where  $\rho$  is the correlation between the random variables  $T_i - T_{i-1}$  and  $N_f(T_i) - N_f(T_{i-1})$ . Clearly,

$$E\{[T_i - T_{i-1}][N_f(T_i) - N_f(T_{i-1})]\} = \sum_{n=1}^{\infty} \int_0^{\infty} nt \, d_t P(n, t),$$

where  $P(n, t) = \Pr \{N_f(T_i) - N_f(T_{i-1}) = n, T_i - T_{i-1} \leq t\}$ . Thus,  $P(n, t)$  is equal to the probability of  $n$  successive failures followed by a preventive removal, all in the interval  $(0, t]$ , namely

$$P(n, t) = \int_0^t A(t:y) \, dB^{(n)}(y),$$

from which the expression for  $\rho\sigma_1\sigma_2$  follows.

The final limit results of interest in this study concern the existence of an asymptotic reliability, the limiting probability of no failure in an interval of length  $t$  as the system age goes to infinity.

**THEOREM 4.4.**

(a) Suppose  $G$  is a lattice distribution function with period  $T$ . Then, if  $\mu_1 < \infty$ , for all  $y < T$ ,

$$\lim_{k \rightarrow \infty} R(t; kT + y) = \mu_1^{-1} \sum_{j=0}^{\infty} \Psi_i(jT + y),$$

where  $\Psi_i(x)$  is defined in (3.6) and (3.7).

(b) Suppose  $G$  is a non-lattice distribution function. Then, if  $\mu_1 < \infty$ ,

$$\lim_{x \rightarrow \infty} R(t; x) = \mu_1^{-1} \int_0^{\infty} \Psi_i(x) dx.$$

**PROOF.**

(a) It was pointed out in the proof of Theorem 3.5 that  $Z_t(x)$ , the random variable which is equal to one when no failure occurs in  $(x, x + t]$  is an equilibrium process over  $x$ . Furthermore, from (3.6) and (3.2),

$$\sum_{j=0}^{\infty} \Psi_i(jT + y) \leq \sum_{j=0}^{\infty} [1 - G(jT + y + t)] \leq \mu_1$$

so that, if  $\mu_1 < \infty$ ,  $\sum_{j=0}^{\infty} \Psi_i(jT + y)$  converges. Therefore, Smith's Theorem 3 [13] yields (a).

(b) Since  $Z_t(x)$  is an equilibrium process,  $G(\infty) = 1$ , and  $\Psi_i(x)$  is of bounded variation in every finite  $x$ -interval, Smith's Theorem 2 [13] yields (b).

An interesting interpretation of (b) is obtained if it is noted that  $\int_0^{\infty} \Psi_i(x) dx$  is the expected time between preventive removals for which  $Z_t(x) = 1$ , i.e., the interval to a failure is greater than  $t$ . Thus, the asymptotic reliability function is expressed as the ratio of two expectations.

**5. Applications.** It will now be shown that a number of different preventive maintenance policies fit the assumptions of the general model, and the appropriate  $F(x; y)$ ,  $A(x; y)$ , and  $B(x; y)$  will be derived for each.

(a) *Block changes.* Under this policy, all components of a given type are replaced simultaneously at times independent of the failure history of the system. It is assumed that the times between replacements are independent, identically distributed random variables. This includes replacement at fixed intervals as a degenerate case. It is further assumed that Assumptions 1 through 6 are valid. To justify Assumptions 7, 8, and 9 we note first that, clearly, for each component position, a preventive removal constitutes a regeneration point. Now we need only derive  $F(x; y)$ ,  $A(x; y)$ , and  $B(x; y)$  and show that they have the correct properties.

**THEOREM 5.1.** Let preventive removals occur at times  $T_1, T_2, \dots$  where  $T_1, T_2 - T_1, T_3 - T_2, \dots$  constitute a renewal process with distribution function

$H(t)$ , with  $H(0) = 0$  and  $H(\infty) = 1$ . Let failure removals occur immediately upon component failure and the age at component failure have a distribution function  $\Phi(t)$ , with  $\Phi(0) = 0$ ,  $\Phi(\infty) = 1$ , and satisfying

$$\left\{ \int_0^\infty \Phi(t) dH(t+y) \right\} / \{1 - H(y)\} \leq 1 - \alpha \text{ for all } y, \quad \text{where } 0 < \alpha < 1.$$

Then, under assumptions 1-6, assumptions 7-9 hold with

$$(5.1) \quad F(x:y) = \{\Phi(x-y)[1 - H(x)] + H(x) - H(y)\} / \{1 - H(y)\},$$

$$(5.2) \quad A(x:y) = \left\{ \int_0^{x-y} [1 - \Phi(t)] dH(t+y) \right\} / \{1 - H(y)\},$$

$$(5.3) \quad B(x:y) = \left\{ \int_0^{x-y} [1 - H(t+y)] d\Phi(t) \right\} / \{1 - H(y)\}.$$

PROOF. To justify (5.1) we express  $F(x:y)$  as follows:

$$F(x:y) = \frac{\Pr \left\{ \begin{array}{l} \text{component fails by age } x-y \text{ and no preventive removal occurs} \\ \text{in } (0, x] \text{ or first preventive removal occurs in } (y, x] \end{array} \right\}}{\Pr \{ \text{no preventive removal in } (0, y] \}}$$

which directly yields (5.1). Moreover,

$$A(x:y) = \frac{\Pr \left\{ \begin{array}{l} \text{first preventive removal occurs in } (y, x] \text{ and component} \\ \text{does not fail before this preventive removal} \end{array} \right\}}{\Pr \{ \text{no preventive removal in } (0, y] \}}.$$

This justifies (5.2), and (5.3) follows by subtraction. Assumptions 7, 8, and 9 are clearly satisfied.

For the case in which block changes occur at fixed intervals of length  $T$ , we have

$$H(t) = \begin{cases} 0 & t < T \\ 1 & t \geq T \end{cases},$$

so that

$$(5.4) \quad F(x:y) = \begin{cases} \Phi(x-y) & x < T \\ 1 & x \geq T \end{cases},$$

$$(5.5) \quad A(x:y) = \begin{cases} 0 & x < T \\ 1 - \Phi(T-y) & x \geq T \end{cases},$$

$$(5.6) \quad B(x:y) = \begin{cases} \Phi(x-y) & x < T \\ \Phi(T-y) & x \geq T \end{cases}.$$

The problem of block changes has been studied in detail by Welker, Drenick, and Barlow and Hunter [18], [5], [2] among others. In particular, the problem of finding that replacement interval which minimizes total maintenance costs has received attention.

(b) *Individual component replacement on the basis of age.* Under this policy, a

record is kept of the time of installation of each component. When a component enters the system, as a preventive replacement or as the replacement for a failed component, the time when it will be preventively removed is scheduled. Thus,  $T_p$ , the component age for a scheduled preventive removal is a well-defined random variable. Moreover, a failure removal occurs immediately upon component failure and  $T_f$ , the component age at failure, is a second random variable. If  $T_p < T_f$ , the component is preventively removed at age  $T_p$ , while, if  $T_f < T_p$ , the component fails at age  $T_f$ . In the degenerate case,  $T_p$  may have a fixed value. It is assumed that Assumptions 1 through 6 are valid. Here again, it is clear that each preventive removal, and, in this special case, each failure removal, constitutes a regeneration point. We shall derive  $F(x: y)$ ,  $A(x: y)$ , and  $B(x: y)$  and shows that Assumptions 7, 8, and 9 are satisfied.

**THEOREM 5.3.** *Let the time to scheduled preventive removal  $T_p$ , measured from system-entry time, have distribution function,  $H(t)$ , with  $H(0) = 0$  and  $H(\infty) = 1$ . Let the time to failure,  $T_f$ , measured from system-entry time, have distribution function  $\Phi(t)$ , with  $\Phi(0) = 0$ ,  $\Phi(\infty) = 1$ , and  $\int_0^\infty \Phi(t) dH(t) < 1$ . Then, under Assumptions 1-6, Assumptions 7-9 hold with*

$$(5.7) \quad F(x: y) = \Phi(x - y) + H(x - y) - \Phi(x - y)H(x - y),$$

$$(5.8) \quad A(x: y) = \int_0^{x-y} [1 - \Phi(t)] dH(t),$$

$$(5.9) \quad B(x: y) = \int_0^{x-y} [1 - H(t)] d\Phi(t).$$

**PROOF.** In this case,  $F(x: y)$  is simply the probability that either  $T_p \leq x - y$  or  $T_f \leq x - y$ . Similarly,  $A(x: y)$  is the probability that  $T_p \leq x - y$  and  $T_f > T_p$ . Conversely,  $B(x: y)$  is the probability that  $T_f \leq x - y$  and  $T_p > T_f$ . Assumptions 7, 8, and 9 are clearly valid.

If preventive removals are scheduled at a fixed time  $T$  after system entry, i.e.,

$$H(t) = \begin{cases} 0 & t < T \\ 1 & t \geq T \end{cases},$$

we have

$$(5.10) \quad F(x: y) = \begin{cases} \Phi(x - y) & x - y < T \\ 1 & x - y \geq T \end{cases},$$

$$(5.11) \quad A(x: y) = \begin{cases} 0 & x - y < T \\ 1 - \Phi(T) & x - y \geq T \end{cases},$$

$$(5.12) \quad B(x: y) = \begin{cases} \Phi(x - y) & x - y < T \\ \Phi(T) & x - y \geq T \end{cases}.$$

The policy of individual component replacement on the basis of age was studied in detail by Brender. [4] Based on a linear cost model, the maintenance cost functions, both exact and asymptotic, were derived.

(c) *System check-outs.* In a complex system, it is often true that a large number of components are used only intermittently. In this case, a component may fail while it is not being used, but it will not induce a system failure until it is called into use once more. Furthermore, the probability that the component fails during an interval of non-use may be different from the probability that it fails during a usage interval of the same length. In a system in which a large fraction of the components have relatively long periods of non-use, it should be practical to prevent system failure during operation by conducting periodic system check-outs in order to locate and replace those components which have failed but which have not yet induced system failure. We shall now formulate a model to represent the essential features of this policy.

ASSUMPTIONS.

(c1) Assumptions 1-6 of the general model.

(c2) In each component position, the state of being in use alternates with the state of non-use. The intervals of non-use have distribution function  $H_0(t) = 1 - e^{-\lambda t}$ . The intervals of use have distribution function  $H_1(t) = 1 - e^{-\mu t}$ .

(c3) At system age zero, the component is in the non-use state.

(c4) The probability of component failure by the end of time  $y_0$  of non-use and time  $y_1$  of use depends only on  $y_0$  and  $y_1$  and is independent of the times at which changes of state occur. Let this probability be called  $K(y_0, y_1)$ , and let  $K(0, 0) = 0$ ,  $K(\infty, y_1) = 1$  for all  $y_1$ ,  $K(y_0, \infty) = 1$  for all  $y_0$ .

(c5) When a component fails during a use interval, it causes system failure and is removed immediately. When a component failure occurs during a non-use interval, system failure is delayed until the next change of state.

(c6) At the end points of fixed intervals of length  $T$ , checks are conducted, and all components which have failed without causing system failure are replaced. These checks require zero time and do not interrupt the use non-use pattern.

To shed further light on Assumption c4, consider the case for which  $K(y_0, y_1)$  is absolutely continuous in both variables and let  $k_i(y_0, y_1) = \partial K(y_0, y_1) / \partial y_i$ , i.e.,  $k_i(y_0, y_1)$  is the failure density in state  $i$  after time  $y_0$  in the off state and time  $y_1$  in the on state. Then Assumption c4 implies that  $\partial k_0 / \partial y_1 = \partial k_1 / \partial y_0$ . For example, suppose  $k_0(y_0, y_1) = ck_1(y_0, y_1)$ , where  $0 < c < 1$ , i.e., roughly speaking, the component fails only  $c$  times as fast in the off state as in the on state. Then  $K(y_0, y_1) = \bar{K}(cy_0 + y_1)$ , where  $\bar{K}$  is a distribution function of one variable.

In order to derive  $F(x: y)$ ,  $A(x: y)$ , and  $B(x: y)$ , we pursue roughly the following line of reasoning: We note first that because of Assumption (c5) every failure removal must occur during a period of use, and, considering (c5) and (c6), every preventive removal must occur during a period of non-use. Thus,  $F(x: 0)$  will be the probability of removal by component age  $x$ , given non-use at age 0 and given check points at component age  $T, 2T, \dots$ . If  $kT < y < (k+1)T$ ,  $F(x: y)$  will be the probability of removal by component age  $x - y$ , given use at age 0 and given check points at component age  $(k+1)T - y, (k+2)T - y, \dots$ . Because of the assumption of exponential intervals, it is not necessary to

take into account the length of time that the use or non-use state has been going on when a replacement occurs.  $A(x: y)$  and  $B(x: y)$  will be derived from  $F(x: y)$  by noting that all removals at check points,  $T, 2T, \dots$ , are preventive removals while all those in the open intervals  $(0, T), (T, 2T), \dots$  are failure removals.

We start by defining and formulating the distributions of total use time which will be necessary to this analysis. Let state 0 be the state of non-use and state 1 be the state of being in use.

**DEFINITION.**

(i)  $V_j(u_1, t)$  is the probability that, in an interval of length  $t$ , the total use time is equal to or less than  $u_1$  and the component is in use at the end of the interval, given state  $j$  at the beginning of the interval, where  $j = 0, 1$ , and  $0 \leq u_1 \leq t$ .

(ii)  $W_j(u_0, u_1, t)$  is the probability that, in an interval of length  $t$ , the total use time is equal to or less than  $u_1$ , there is at least one transition into state 1 during the interval  $(0, t]$ , the component is in state 0 at the end of the interval, and the total non-use time before the last transition into state 1 is equal to or less than  $u_0$ , given state  $j$  at the beginning of the interval where  $j = 0, 1$ , and  $0 \leq u_0 \leq t - u_1, 0 \leq u_1 \leq t$ .

**THEOREM 5.5.** *On the basis of Assumption (c2),*

(a)  $V_j(u_1, t)$  is absolutely continuous in  $u_1$  in the interval  $(0, t]$  with density functions  $v_j(u_1, t) = (\partial/\partial u_1)V_j(u_1, t)$  given by

$$(5.13) \quad v_0(u_1, t) = \lambda e^{-\lambda(t-u_1)} e^{-\mu u_1} \sum_{n=0}^{\infty} \frac{[\lambda(t-u_1)]^n [\mu u_1]^n}{(n!)^2}.$$

$$(5.14) \quad v_1(u_1, t) = \lambda e^{-\lambda(t-u_1)} e^{-\mu u_1} \sum_{n=0}^{\infty} \frac{[\lambda(t-u_1)]^n [\mu u_1]^{n+1}}{n!(n+1)!},$$

or by the equivalent representations

$$(5.13') \quad v_0(u_1, t) = \lambda e^{-\lambda(t-u_1)} e^{-\mu u_1} I_0\{2[\lambda(t-u_1)(\mu u_1)]^{\frac{1}{2}}\},$$

$$(5.14') \quad v_1(u_1, t) = \lambda e^{-\lambda(t-u_1)} e^{-\mu u_1} \left[ \frac{\mu u_1}{\lambda(t-u_1)} \right]^{\frac{1}{2}} I_1\{2[\lambda(t-u_1)(\mu u_1)]^{\frac{1}{2}}\}.$$

(b)  $W_j(u_0, u_1, t)$  is jointly absolutely continuous in  $(u_0, u_1)$  for  $0 \leq u_0 \leq t - u_1, 0 \leq u_1 \leq t$ , with density functions

$$w_j(u_0, u_1, t) = (\partial^2/\partial u_0 \partial u_1) W_j(u_0, u_1, t)$$

given by:

$$(5.15) \quad w_0(u_0, u_1, t) = \lambda \mu e^{-\lambda(t-u_1)} e^{-\mu u_1} \sum_{n=0}^{\infty} \frac{[\lambda u_0]^n [\mu u_1]^n}{(n!)^2},$$

$$(5.16) \quad w_1(u_0, u_1, t) = \lambda \mu e^{-\lambda(t-u_1)} e^{-\mu u_1} \sum_{n=0}^{\infty} \frac{[\lambda u_0]^n [\mu u_1]^{n+1}}{n!(n+1)!},$$

or by the equivalent representations

$$(5.15') \quad w_0(u_0, u_1, t) = \lambda \mu e^{-\lambda(t-u_1)} e^{-\mu u_1} I_0\{2[(\lambda u_0)(\mu u_1)]^{\frac{1}{2}}\},$$

$$(5.16') \quad w_1(u_0, u_1, t) = \lambda \mu e^{-\lambda(t-u_1)} e^{-\mu u_1} [\mu u_1 / \lambda u_0]^{\frac{1}{2}} I_1\{2[(\lambda u_0)(\mu u_1)]^{\frac{1}{2}}\}.$$

where  $I_0$  and  $I_1$  are modified Bessel functions of the first kind.

PROOF.  $V_0(u_1, t)$  is the sum over  $n$  of the probability that the sum of  $n + 1$  intervals in state 0 and  $n$  intervals in state 1 is less than or equal to  $t$ , but that the sum of  $n + 1$  intervals in state 0 and  $n + 1$  intervals in state 1 is greater than  $t$ , where  $n = 0, 1, \dots$ , and that the total time in state 1 is equal to or less than  $u_1$ . Thus,

$$V_0(u_1, t) = \sum_{n=0}^{\infty} \int_{t-u_1}^t [H_1^{(n)}(t-x) - H_1^{(n+1)}(t-x)] dH_0^{(n+1)}(x).$$

where  $H_0^{(n)}$  or  $H_1^{(n)}$  denotes the  $n$ th convolution of  $H_0$  or  $H_1$  with itself and  $H_0^{(0)}$  and  $H_1^{(0)}$  are unit step functions. By analogous argument,

$$V_1(u_1, t) = \sum_{n=0}^{\infty} \int_{t-u_1}^t [H_1^{(n)}(t-x) - H_1^{(n+1)}(t-x)] dH_0^{(n)}(x).$$

Substituting the values of  $H_1$  and  $H_2$  from Assumption (c2), we obtain (5.13) and (5.14).

$W_0(u_0, u_1, t)$  is the sum over  $n$  of the probability that the sum of  $n$  intervals in state 0 and  $n$  intervals in state 1 is less than or equal to  $t$ , but that the sum of  $n + 1$  intervals in state 0 and  $n$  intervals in state 1 is greater than  $t$ , where  $n = 1, 2, \dots$ , that the sum of the state-1 intervals is less than or equal to  $u_1$  and the sum of the state-0 intervals before the last transition into state 1 is less than or equal to  $u_0$ . Thus,

$$W_0(u_0, u_1, t) = \sum_{n=1}^{\infty} \int_0^{u_1} \int_0^{u_0} [1 - H_0(t-x-y)] dH_0^{(n)}(x) dH_1^{(n)}(y).$$

By analogous argument,

$$W_1(u_0, u_1, t) = \sum_{n=1}^{\infty} \int_0^{u_1} \int_0^{u_0} [1 - H_0(t-x-y)] dH_0^{(n)}(x) dH_1^{(n+1)}(y).$$

Substituting the expressions for  $H_0$  and  $H_1$  from Assumption (c2) yields (5.15) and (5.16).

The reader should notice that these arguments apply to general distribution functions  $H_0$  and  $H_1$  if it is assumed that the component enters state  $j$  at the beginning of the interval of length  $t$ . The assumption of exponentiality is critical only in that the definitions of  $V_j$  and  $W_j$  require that these functions be independent of the length of time spent in state  $j$  prior to the beginning of the interval. This analysis is similar to that developed by Takács [16] to express the distribution of sojourn times in each state of a two-state Semi-Markov process.

We now proceed to define and formulate expressions for the joint distributions

of time to component failure and to component-induced system failure. Let  $\Phi_j(t_1, t_2)$  be the probability that a component which enters the system at 0 fails in the interval  $(0, t_1]$  and induces a system failure in the interval  $(0, t_2]$ , given state  $j$  at time 0.

THEOREM 5.6. For all  $0 \leq t_1 \leq t_2 < \infty$ ,

$$\begin{aligned} \Phi_0(t_1, t_2) = & \int_0^{t_1} K(t_1 - u_1, u_1) v_0(u_1, t_1) du_1 \\ (5.17) \quad & + \int_0^{t_1} \int_0^{t_1 - u_1} \{e^{-\lambda(t_2 - t_1)} K(u_0, u_1) \\ & + [1 - e^{-\lambda(t_2 - t_1)}] K(t_1 - u_1, u_1)\} w_0(u_0, u_1, t_1) du_0 du_1 \\ & + K(t_1, 0)[e^{-\lambda t_1} - e^{-\lambda t_2}], \end{aligned}$$

$$\begin{aligned} \Phi_1(t_1, t_2) = & K(0, t_1) e^{-\mu t_1} + \int_0^{t_1} K(t_1 - u_1, u_1) v_1(u_1, t_1) du_1 \\ (5.18) \quad & + \int_0^{t_1} \int_0^{t_1 - u_1} \{e^{-\lambda(t_2 - t_1)} K(u_0, u_1) \\ & + [1 - e^{-\lambda(t_2 - t_1)}] K(t_1 - u_1, u_1)\} w_1(u_0, u_1, t_1) du_0 du_1 \\ & + \int_0^{t_1} \{e^{-\lambda(t_2 - t_1)} K(0, u_1) \\ & + [1 - e^{-\lambda(t_2 - t_1)}] K(t_1 - u_1, u_1)\} \mu e^{-\mu u_1} e^{-\lambda(t_1 - u_1)} du_1, \end{aligned}$$

where  $K(y_0, y_1)$  is defined in Assumption (c4) and  $v_j(u_1, t)$  and  $w_j(u_0, u_1, t)$  are expressed in Theorem 5.5. For  $0 \leq t_2 < t_1$ ,  $\Phi_i(t_1, t_2) = \Phi_i(t_2, t_2)$ .

PROOF. The probability that a component fails in  $(0, t_1]$  and induces a system failure in  $(0, t_2]$  is the sum of the probabilities of three mutually exclusive events, i.e.,

A: Component failure in  $(0, t_1]$  and state 1 at  $t_1$ . System failure is induced in  $(0, t_1]$ .

B: Component-induced system failure in  $(0, t_1]$  and state 0 throughout  $[t_1, t_2]$ . In this case, the component is in use at some time during  $(0, t_1]$  and fails before the last transition into state 0.

C: Component failure in  $(0, t_1]$ , state 0 at  $t_1$ , and a transition into state 1 in  $(t_1, t_2]$ .

Under the condition that the component is initially in state 0, these events occur with the following probabilities:

$$\Pr_0(A) = \int_0^{t_1} K(t_1 - u_1, u_1) v_0(u_1, t_1) du_1,$$

$$\Pr_0(B) = e^{-\lambda(t_2 - t_1)} \int_0^{t_1} \int_0^{t_1 - u_1} K(u_0, u_1) w_0(u_0, u_1, t_1) du_0 du_1,$$



$$\Pr_0(C) = [1 - e^{-\lambda(t_2-t_1)}] \left[ K(t_1, 0)e^{-\lambda t_1} + \int_0^{t_1} \int_0^{t_1-u_1} K(t_1 - u_1, u_1)w_0(u_0, u_1, t_1) du_0 du_1 \right].$$

If the component is initially in state 1, the corresponding probabilities are as follows:

$$\Pr_1(A) = K(0, t_1)e^{-\mu t_1} + \int_0^{t_1} K(t_1 - u_1, u_1)v_1(u_1, t_1) du_1,$$

$$\Pr_1(B) = e^{-\lambda(t_2-t_1)} \left[ \int_0^{t_1} K(0, u_1)\mu e^{-\mu u_1} e^{-\lambda(t_1-u_1)} du_1 + \int_0^{t_1} \int_0^{t_1-u_1} K(u_0, u_1)w_1(u_0, u_1, t_1) du_0 du_1 \right],$$

$$\Pr_1(C) = [1 - e^{-\lambda(t_2-t_1)}] \left[ \int_0^{t_1} K(t_1 - u_1, u_1)\mu e^{-\mu u_1} e^{-\lambda(t_1-u_1)} du_1 + \int_0^{t_1} \int_0^{t_1-u_1} K(t_1 - u_1, u_1)w_1(u_0, u_1, t_1) du_0 du_1 \right].$$

Now we can express  $F(x: y)$  in terms of  $\Phi_0$  and  $\Phi_1$ .

**THEOREM 5.7.** For  $jT \leq x < (j + 1)T$ ,

$$(5.19) \quad F(x: 0) = \Phi_0(jT, \infty) + \Phi_0(x, x) - \Phi_0(jT, x),$$

and for  $y > 0$ ,

$$(5.20) \quad F(x: y) = \Phi_1(jT - y, \infty) + \Phi_1(x - y, x - y) - \Phi_1(jT - y, x - y).$$

**PROOF.** A component is removed by  $x$  if it fails by the last check point ( $jT$ ) before  $x$  or if it causes a system failure between the last check point and  $x$ . A component inserted as a preventive replacement is initially in an interval of non-use so that  $F(x: 0)$  is expressed in terms of  $\Phi_0$ , while a replacement for a failure is initially in an interval of use, so that, for  $y > 0$ ,  $F(x: y)$  is expressed in terms of  $\Phi_1$ .

**THEOREM 5.8.** For  $jT \leq x < (j + 1)T$ ,

$$(5.21) \quad A(x: 0) = \Phi_0(jT, \infty) - \sum_{k=1}^j [\Phi_0(kT-, kT-) - \Phi_0(kT - T, kT-)]$$

and, for  $y > 0$ ,

$$(5.22) \quad A(x: y) = \Phi_1(jT - y, \infty) - \sum_{k=\lfloor y/T \rfloor + 1}^j [\Phi_1(kT - y-, kT - y-) - \Phi_1(kT - T - y, kT - y-)].$$

**PROOF.** The probability that a component inserted at zero is preventively removed at  $kT$  is  $F(kT: 0) - F(kT-: 0)$ . Thus, the probability of preventive re-

moval by  $x$ , where  $jT \leq x < (j+1)T$  is

$$A(x:0) = \sum_{k=1}^j \{F(kT:0) - F(kT-:0)\}.$$

Substituting (5.19) yields (5.21). Correspondingly, for  $y > 0$ ,

$$A(x:y) = \sum_{k=\lfloor y/T \rfloor + 1}^j \{F(kT:y) - F(kT-:y)\},$$

where  $(\lfloor y/T \rfloor + 1)T$  is the time of the first check point after  $y$ . Substituting (5.20) gives (5.22).

It is clear that Assumptions (7), (8), (9a), and (9b) of the general model are satisfied by this particular model. In order to satisfy Assumption (9c), it is necessary to place a restriction on  $K(y_0, y_1)$ , i.e.,

ASSUMPTION c7.  $K(y_0, y_1)$  is such that there exists an  $\alpha > 0$ , for which

$$\sum_{k=1}^{\infty} \varphi_0(kT-, kT-) - \varphi_0(kT - T, kT-) \leq 1 - \alpha,$$

$$\sum_{k=1}^{\infty} [\varphi_1(kT - y-, kT - y-) - \varphi_1(kT - T - y, kT - y-)] \leq 1 - \alpha,$$

for all  $y < T$ , where  $\varphi_0$  and  $\varphi_1$  are given by (5.17) and (5.18).

The basic ideas underlying this model were presented by the author in [9]. However, in that paper it was not recognized that preventive replacements must occur during non-use intervals and failures must occur during use intervals, so that the analysis contains certain errors. Furthermore, prior to the present work, no attempt has been made to provide for different failure probabilities during intervals of use and non-use.

(d) *Marginal testing.* Marginal testing is an important part of the preventive maintenance technique used for many electronic systems, particularly for electronic computers. The standard procedure consists of performing some test at regular intervals with the object of locating and replacing components which are still operating satisfactorily, but which are likely to induce system failure shortly after the test point. A model for a system subject to this policy which meets the assumptions of the general model will now be postulated. We assume that the system meets general assumptions 1 to 3 plus the following special conditions:

ASSUMPTIONS.

(d1) At any time, a component is in one of three states: A (good), B (marginal), or C (failed). In state A, a component operates satisfactorily in the system and passes the marginal test if it is performed. In state B, it operates satisfactorily but fails the test if it is conducted. In state C, it does not perform its function in the system and fails the test if it can be performed.

(d2) Transitions take place from A to B, A to C, and B to C.

(d3) Component behavior may be characterized by a continuous joint dis-

tribution function of time in state A, and time to entering state C, both times measured from system entry, i.e.,

$$\begin{aligned} \Phi(t_1, t_2) &= \Pr \{ \text{time in A} \leq t_1, \text{ time to entering C} \leq t_2 \}, \text{ with} \\ \Phi(0, t_2) &= 0 \text{ for all } t_2 \text{ and } \Phi(\infty, \infty) = 1. \end{aligned}$$

(d4) When a component enters state C, it immediately causes system failure and is replaced by a new one from the same population.

(d5) At times  $T, 2T, \dots$ , measured from the initial use of the system, the marginal test is performed. If the component is found to be in state A, it is allowed to remain in position. If it is found in state B, it is replaced by a new one from the same population. (Because of Assumption (d4), it cannot be found in state C.) The performance of the test has no effect on the component.

(d6) There exists an  $\alpha > 0$ , such that for all  $y < T$ ,

$$\sum_{k=1}^{\infty} [\varphi(kT - y, kT - y) - \varphi(kT - T - y, kT - y)] \leq 1 - \alpha$$

$F(x: y)$  and  $A(x: y)$  may be expressed for this model by reasoning analogous to that used in Theorems 5.7 and 5.8 for the preceding cases.

**THEOREM 5.9.** For all  $y \leq x, jT \leq x < (j + 1)T, j = 1, 2, \dots$ , one has

$$\begin{aligned} (5.23) \quad F(x: y) &= \Phi(jT - y, \infty) \\ &\quad + \Phi(x - y, x - y) - \Phi(jT - y, x - y), \end{aligned}$$

$$\begin{aligned} (5.24) \quad A(x: y) &= \Phi(jT - y, \infty) \\ &\quad - \sum_{k=\lfloor x/T \rfloor + 1}^j [\Phi(kT - y, kT - y) - \Phi(kT - T - y, kT - y)]. \end{aligned}$$

**PROOF.** (5.23) is an expression of the fact that, if  $jT \leq x < (j + 1)T$ , the probability of removal by  $x$  is the probability that the component leaves state A by  $jT$  or enters state C between  $jT$  and  $x$ . (5.24) arises from the fact that all preventive removals are performed at test points  $T, 2T, \dots$ .

The analogy between this theorem and Theorems 5.7 and 5.8 is clear if one notes that in state A of this context or successful component operation of the preceding one, no removal occurs. If the component is in state B or there is component failure without induced system failure when a lattice point occurs, the component is removed. When a component enters state C or a system failure is induced, the component is replaced immediately. Thus the event of leaving state A is analogous to component failure, while entering state C is analogous to inducing system failure.

It is easily seen that this model satisfies all the assumptions of the general model, so that the finite-time and asymptotic values of the expected numbers of removals of the two kinds and the reliability function may be computed by the methods developed in Sections 2-4.

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