ON THE GENERAL TIME DEPENDENT QUEUE WITH A SINGLE SERVER

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1. Introduction. In a previous paper [1] we discussed a multi-dimensional phase space model of queuing processes. The approach was developed in detail there for a time dependent queue subject to Poisson arrival and general service time distributions. The present paper extends this approach to the study of a time dependent queue with a single server subject to general arrival and general service time distributions. For this case we are able to carry out the analysis in detail in terms of the state densities introduced in [1]. The problem leads to simultaneous Wiener-Hopf equations with an analytic side condition. We resolve the problem by establishing its equivalence to a Hilbert problem (in the sense of [2]), for which we can give an explicit solution.

The analysis of the time dependent problem is valid whether the system tends to an equilibrium state or not. Thus we are able to derive expressions for the system regeneration time and server occupation time distributions which are valid for unstable as well as stable queues (Section 7).

A brief outline of the paper follows. In Sections 2 and 3 we describe the appropriate phase space for the system and develop the corresponding differential equations, boundary conditions and initial conditions for the general time dependent problem. The solution of this problem is based on the analysis of an associated "first passage" problem which we formulate in Section 4. In Section 5 we take advantage of the essential Wiener-Hopf character of the problem in order to reduce it to an integral equation. In Sections 6 and 7 we show how to formulate an equivalent homogeneous Hilbert problem for which we give an explicit solution. Next in Section 8 we generalize the first passage problem for arbitrary initial conditions thereby obtaining an associated inhomogeneous Hilbert problem whose solution is shown to be intimately related to that of the preceding homogeneous problem. Finally in Section 9 we show how the results from the first passage problems can be used to obtain the complete solution of our original general queuing problem of Section 3. In the concluding Section 10, we point out some connections between this work and that of Lindley [3] and Pollaczek [4].

A few remarks are in order here concerning the nature of the arguments we present in deriving the basic equations describing our process in Sections 2, 3, and 4. The basic quantities we work with are the *densities* of the time dependent probability distributions over the state space. It is not *a priori* evident that these densities exist, let alone possess the requisite smoothness properties for the derivations in Sections 3 and 4. Our justification for the arguments in these

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sections is twofold. First because the analysis proves constructive, we are able to demonstrate that for sufficiently smooth initial distributions (C^1), our integro-differential-difference equations indeed possess a solution (existence of the state densities), the solution is unique, and it inherits the smoothness of the initial conditions, thereby providing an *a posteriori* justification of the derivations. However, we also wish to accommodate initial distributions which are "concentrated", that is initial probability distributions with delta "function" densities. In this case our state densities together with their partial derivatives exist only as generalized functions (in Schwartz' sense of distributions [5]). We would then take as our *starting point* the basic pair of integral equations (4.18), (4.19) with $f(t - x)$ in place of $\delta(x - t)$ which on the one hand could be formulated directly thereby avoiding the differential argument with a certain sacrifice of facility, and on the other hand having a limiting validity in the distribution theoretic sense of Schwartz when $f(t - x)$ tends to $\delta(x - t)$, or more generally when the initial conditions (3.9) tend to distributions in the sense of [5].

2. Phase space. Following [1] we observe that the instantaneous state of the non-vacant queuing system is completely characterized by the triple (m, x, y) where m is the queue length, x is the elapsed time since entrance of the item into service, and y is the elapsed time since the last arrival to the queue. For $m = 1, 2, \cdots$, i.e., when there is a queue, we denote the probability density that the system is in the state (m, x, y) at time t by $\mathbf{W}_m(x, y, t)$.

The motion on the set of states $\{0, x, y\}$ requires separate consideration. When a "customer" arrives at an idle server, the system coordinates x and y remain equal until a completion or subsequent arrival occurs. It is convenient to describe the set of states of the system during this phase by $\{x\}$ and the associated density at time t by $\mathbf{F}(x, t)$. The density of states $\{0, x, y\}$ accessible to the system following the depletion of a queue will be denoted by $\mathbf{W}_0(x, y, t)$. Whenever a system is in such a state, moreover, $y > x$ so that $\mathbf{W}_0(x, y, t) = 0$ for $y \leqq x$. Finally there is the sets of states $\{y\}$ in which the server is idle and a period of time y has elapsed since the last arrival whose density at time t will be denoted by $\mathbf{E}(y, t)$.

The set of mutually exclusive and totally exhaustive states $\{x\}$, $\{y\}$, $\{m, x, y\}$ constitutes a phase space Γ in which the temporal evolution of the system can be discussed.

3. Derivation of equations. If $A(y)$ denotes the density function for the inter-arrival time distribution and $D(x)$ that for the service time distribution, then continuity of flow in Γ during a time interval Δ [1] requires

$$\mathbf{W}_m(x + \Delta, y + \Delta, t + \Delta) = \mathbf{W}_m(x, y, t)[1 - \lambda(y)\Delta][1 - \eta(x)\Delta], \qquad m = 0, 1, \cdots, \tag{3.1}$$

to first order terms in Δ. $\lambda(y)\Delta$ and $\eta(x)\Delta$ are simply the first order probabilities of an arrival and a service completion, respectively, occurring in the intervals $(y, y + \Delta)$ and $(x, x + \Delta)$, respectively, conditioned on the system's having the coordinates x and y, respectively. The significant relationships for λ and η are

$$A(y) = \lambda(y) \exp\left[-\int_0^y \lambda(y')\,dy'\right] \tag{3.2a}$$

and

$$D(x) = \eta(x) \exp\left[-\int_0^x \eta(x')\,dx'\right]. \tag{3.2b}$$

By rearranging terms in (3.1), dividing by Δ and taking the limit as $\Delta \to 0$, we obtain

$$\frac{\partial \mathbf{W}_m}{\partial t} + \frac{\partial \mathbf{W}_m}{\partial x} + \frac{\partial \mathbf{W}_m}{\partial y} + [\lambda(y) + \eta(x)]\mathbf{W}_m = 0, \qquad m = 0, 1, \cdots \tag{3.3}$$

on the interior of Γ. By a similar continuity argument

$$\frac{\partial \mathbf{E}}{\partial t} + \frac{\partial \mathbf{E}}{\partial y} + \lambda(y)\mathbf{E} = \int_0^\infty \mathbf{W}_0(x, y, t)\eta(x)\,dx + \eta(y)\mathbf{F}(y, t), \tag{3.4}$$

$$\frac{\partial \mathbf{F}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + [\lambda(x) + \eta(x)]\mathbf{F} = 0. \tag{3.5}$$

In order to complete our mathematical description, it is necessary to specify the boundary conditions on Γ together with some initial conditions. The derivation of the boundary conditions requires a consideration of the motion of the system in Γ when arrivals and completions occur. Omitting the argument, which is essentially that in [1], we find that the appropriate boundary conditions on $0 \leqq x < \infty$ are

$$\mathbf{W}_0(x, 0, t) = 0, \tag{3.6a}$$

$$\mathbf{W}_1(x, 0, t) = \lambda(x)\mathbf{F}(x, t) + \int_0^\infty \mathbf{W}_0(x, y, t)\lambda(y)\,dy, \tag{3.6b}$$

$$\mathbf{W}_m(x, 0, t) = \int_0^\infty \mathbf{W}_{m-1}(x, y, t)\lambda(y)\,dy, \qquad m = 2, 3, \cdots, \tag{3.6c}$$

and on $0 \leqq y < \infty$

$$\mathbf{W}_m(0, y, t) = \int_0^\infty \mathbf{W}_{m+1}(x, y, t)\eta(x)\,dx, \qquad m = 0, 1, \cdots \tag{3.7}$$

and

$$\mathbf{E}(0, t) = 0 \quad \text{and} \quad \mathbf{F}(0, t) = \int_0^\infty \mathbf{E}(y, t)\lambda(y)\,dy. \tag{3.8}$$

Finally the most general initial conditions we could consider are

$$\mathbf{E}(y, 0) = \mathbf{E}_0(y), \ \mathbf{F}(x, 0) = \mathbf{F}_0(x), \ \mathbf{W}_m(x, y, 0) = \mathbf{W}_{0m}(x, y), \quad m = 0, 1, \cdots, \tag{3.9}$$

subject to the normalization

$$\int_0^\infty \mathbf{E}_0(y)\, dy + \int_0^\infty \mathbf{F}_0(x)\, dx + \sum_{m=0}^{\infty} \int_0^\infty \int_0^\infty \mathbf{W}_{0m}(x, y)\, dx\, dy = 1. \tag{3.10}$$

4. The basic first passage problem. The mathematical character and analysis of the system (3.3)–(3.8) is closely related to that of an associated "first passage" problem which we define below. The first passage problem, moreover is not without interest in its own right in providing a natural framework for the discussion of server occupation time and system regeneration time distributions. The *basic* first passage problem is defined as follows: at time $t = 0$ an item to be serviced arrives at an idle system and enters service directly; we are then concerned with (a) the joint probability densities $W_m(x, y, t)$, $F(x, t)$, $E(y, t)$ that the system at time t is in the state (m, x, y), (x), (y), respectively, and that the system has not returned to the initial state, $m = 0$, $x = 0$, $y = 0$ which we term the regeneration state, and (b) the probability distribution $R(t)$ that the system has regenerated by time t. The server occupation time distribution $S(t)$ is directly available from the first passage density $E(y, t)$.

The following slightly modified set of equations (cf., Section 3) govern the problem

$$\frac{\partial W_m}{\partial t} + \frac{\partial W_m}{\partial x} + \frac{\partial W_m}{\partial y} + [\lambda(y) + \eta(x)]W_m = 0, \qquad m = 0, 1, \cdots, \tag{4.1}$$

$$\frac{\partial E}{\partial t} + \frac{\partial E}{\partial y} + \lambda(y)E = \int_0^\infty W_0(x, y, t)\eta(x)\, dx + \eta(y)F(y, t), \tag{4.2}$$

$$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} + [\lambda(x) + \eta(x)]F = 0, \tag{4.3}$$

$$\frac{dR}{dt} = \int_0^\infty E(y, t)\lambda(y)\, dy, \tag{4.4a}$$

$$S(t) = R(t) + \int_0^\infty E(y, t)\, dy, \tag{4.4b}$$

$$W_0(x, 0, t) = 0, \tag{4.5a}$$

$$W_1(x, 0, t) = \lambda(x)F(x, t) + \int_0^\infty W_0(x, y, t)\lambda(y)\, dy, \tag{4.5b}$$

$$W_m(x, 0, t) = \int_0^\infty W_{m-1}(x, y, t)\lambda(y)\, dy, \qquad m = 2, 3, \cdots, \tag{4.5c}$$

$$W_m(0, y, t) = \int_0^\infty W_{m+1}(x, y, t)\eta(x)dx, \qquad m = 0, 1, \cdots, \tag{4.6}$$

$$E(0, t) = 0 \quad \text{and} \quad F(0, t) = 0, \tag{4.7}$$

with the initial conditions

$$W_m(x, y, 0) = 0, \quad m = 0, 1, \cdots, \quad E(y, 0) = 0, \tag{4.8}$$
$$F(x, 0) = \delta(x - 0), \quad \text{and} \quad R(0) = 0,$$

where δ is the delta distribution function. While the modifications (4.7) together with the initial conditions (4.8) achieve some simplification over the preceding system, the essential mathematical character of the equations (4.1)–(4.8) is the same; we shall show in Section 9 how the analysis of the first passage problem serves to generate the complete time dependent state densities of our original problem.

We first make a formal reduction of (4.1)–(4.8) by introducing the following transformations suggested by the form of the equations:

$$W_m(x, y, t) = \exp(-[L(y) + N(x)])\mathcal{W}_m(x, y, t), \tag{4.9a}$$

$$E(y, t) = e^{-L(y)}\,\mathcal{E}(y, t), \tag{4.9b}$$

$$F(x, t) = \exp(-[L(x) + N(x)])\mathcal{F}(x, t), \tag{4.9c}$$

where

$$L(y) = \int_0^y \lambda(y')\,dy' \quad \text{and} \quad N(x) = \int_0^x \eta(x')\,dx'.$$

Under these transformations (4.1)–(4.8) become, respectively,

$$\frac{\partial \mathcal{W}_m}{\partial t} + \frac{\partial \mathcal{W}_m}{\partial x} + \frac{\partial \mathcal{W}_m}{\partial y} = 0, \qquad m = 0, 1, \cdots, \tag{4.10}$$

$$\frac{\partial \mathcal{E}}{\partial t} + \frac{\partial \mathcal{E}}{\partial y} = \mathcal{F}(y, t)D(y) + \int_0^\infty \mathcal{W}_0(x, y, t)D(x)\,dx, \tag{4.11}$$

$$\frac{\partial \mathcal{F}}{\partial t} + \frac{\partial \mathcal{F}}{\partial x} = 0 \Rightarrow \mathcal{F} = f(t - x), \tag{4.12}$$

$$\mathcal{W}_0(x, 0, t) = 0, \tag{4.13a}$$

$$\mathcal{W}_1(x, 0, t) = f(x - t)A(x) + \int_0^\infty \mathcal{W}_0(x, y, t)A(y)\,dy, \tag{4.13b}$$

$$\mathcal{W}_m(x, 0, t) = \int_0^\infty \mathcal{W}_{m-1}(x, y, t)A(y)\,dy, \qquad m = 2, 3, \cdots, \tag{4.13c}$$

$$\mathcal{W}_m(0, y, t) = \int_0^\infty \mathcal{W}_{m+1}(x, y, t)D(x)\,dx, \qquad m = 0, 1, \cdots, \tag{4.14}$$

$$\mathcal{E}(0, t) = 0 \quad \text{and} \quad f(t) = 0, \qquad t > 0, \tag{4.15}$$

and finally the initial conditions

$$(4.16)\qquad \mathcal{W}_m(x, y, 0) = 0, \quad \mathcal{E}(y, 0) = 0, \quad \mathcal{F}(x, 0) = f(-x) = \delta(x - 0).$$

Continuing the formal reduction, we introduce the generating function

$$G(s, x, y, t) = \sum_{m=0}^{\infty} s^m \mathcal{W}_m(x, y, t),$$

in terms of which (4.10) becomes

$$(4.17)\qquad \frac{\partial G}{\partial t} + \frac{\partial G}{\partial x} + \frac{\partial G}{\partial y} = 0,$$

(4.13) and (4.16) yield

$$(4.18)\qquad G(s, x, 0, t) = sA(x)\delta(x - t) + s\int_0^\infty G(s, x, y, t)A(y)\,dy;$$

while (4.14) gives

$$(4.19)\qquad sG(s, 0, y, t) = \int_0^\infty G(s, x, y, t)D(x)\,dx - \int_0^\infty G(0, x, y, t)D(x)\,dx.$$

The general solution of (4.17) subject to the initial condition $G(s, x, y, 0) = 0$ for $x, y \geqq 0$ can be represented in the form

$$(4.20)\qquad G(s, x, y, t) = \mathcal{G}_1(s, x - y, t - y) + \mathcal{G}_2(s, x - y, t - x), \quad \text{for } x, y, t \geqq 0,$$

where $\mathcal{G}_1$ vanishes for $x < y$ *or* $t < y$ and $\mathcal{G}_2$ vanishes for $x > y$ *or* $t < x$. Equations (4.18) and (4.19) then yield the basic pair of simultaneous non-homogeneous integral equations

$$(4.21)\qquad \begin{aligned} \mathcal{G}_1(s, x, t) = {} & sA(x)\delta(x - t) + s\int_0^\infty \mathcal{G}_1(s, x - y, t - y)A(y)\,dy \\ & + s\int_0^\infty \mathcal{G}_2(s, x - y, t - x)A(y)\,dy, \qquad \text{for } x, t \geqq 0, \end{aligned}$$

$$(4.22)\qquad \begin{aligned} s\mathcal{G}_2(s, -y, t) = {} & \int_0^\infty \mathcal{G}_1(s, x - y, t - y)D(x)\,dx \\ & + \int_0^\infty \Big[\mathcal{G}_2(s, x - y, t - x) - \mathcal{G}_2(0, x - y, t - x)\Big] D(x)\,dx, \\ & \text{for } y, t \geqq 0, \end{aligned}$$

where we have used the fact that $\mathcal{G}_1(0, x - y, t - y) = 0$, a consequence of (4.13).

5. Method of analysis. The basic character of the system (4.21)–(4.22) is, as will become clearer presently, of the Wiener-Hopf type. Our method of analysis, accordingly, will be to take transforms with respect to x, y and t. In the process of resolving the transformed equations, an auxiliary analyticity condition arises which serves to uniquely determine the "non-homogeneous" term involving the basic function $\mathcal{G}_0(x - y, t - x) = \mathcal{G}_2(0, x - y, t - x)$.

Since $\mathcal{G}_1(s, x, t)$ and $\mathcal{G}_2(s, -y, t)$ are zero for negative t, if we assume that we may take *Laplace* transforms in (4.21) and (4.22) with respect t for $t \geqq 0$, there then results the set of equations (denoting the transform by tilde) for $x \geqq 0$

$$\tilde{\mathcal{G}}_1(s, x, p) = sA(x)e^{-px} + s\int_0^\infty \tilde{\mathcal{G}}_1(s, x - y, p)e^{-py}A(y)\,dy + s\int_0^\infty \tilde{\mathcal{G}}_2(s, x - y, p)e^{-px}A(y)\,dy, \tag{5.1}$$

and for $y \geqq 0$

$$s\tilde{\mathcal{G}}_2(s, -y, p) = \int_0^\infty \tilde{\mathcal{G}}_1(s, x - y, p)e^{-py}D(x)\,dx + \int_0^\infty \tilde{\mathcal{G}}_2(s, x - y, p)e^{-px}D(x)\,dx - \int_0^\infty \tilde{\mathcal{G}}_0(x - y, p)e^{-px}D(x)\,dx, \tag{5.2}$$

where we have set $\mathcal{G}_2(0, x - y, t - x) \doteq \mathcal{G}_0(x - y, t - x)$.

In order to emphasize the Wiener-Hopf character of the above system of equations, we introduce the following convention: to any function $f(x)$ on $-\infty < x < \infty$ we shall associate its positive and negative half-line components defined as follows

$$f^+(x) = \begin{cases} f(x), & x \geqq 0 \\ 0, & x < 0 \end{cases} \qquad f^-(x) = \begin{cases} 0, & x \geqq 0 \\ f(x), & x < 0 \end{cases} \tag{5.3}$$

so that $f(x) = f^+(x) + f^-(x)$ on $-\infty < x < \infty$. In addition we shall employ the $+$ and $-$ indices as *superscripts* for such decompositions of *unknown* functions and as *subscripts* for *known* functions.

In terms of this convention the equations (5.1), (5.2) can be expressed in the form for $x \geqq 0$:

$$\tilde{\mathcal{G}}_1^+(s, x, p) = sA_+(x)e^{-px} + s\int_0^\infty \tilde{\mathcal{G}}_2^-(s, x - y, p)e^{-p(x-y)}e^{-py}A_+(y)\,dy + s\int_0^\infty \tilde{\mathcal{G}}_1^+(s, x - y, p)e^{-py}A_+(y)\,dy, \tag{5.4}$$

for $y \geqq 0$

$$s\tilde{\mathcal{G}}_2^-(s, -y, p) = \int_0^\infty \tilde{\mathcal{G}}_1^+(s, x - y, p)e^{p(x-y)}e^{-px}D_+(x)\,dx - \int_0^\infty \tilde{\mathcal{G}}_0^-(x - y, p)e^{-px}D_+(x)\,dx + \int_0^\infty \tilde{\mathcal{G}}_2^-(s, x - y, p)e^{-px}D_+(x)\,dx. \tag{5.5}$$

Our next step is to formally extend these equations to hold on $(-\infty, \infty)$ by introducing unknown functions $L^-(s, x, p)$ and $sR^+(s, -y, p)$ (noting that (5.5) is $O(s)$) into (5.4) and (5.5), respectively. Now we assume that equations (5.4), (5.5) so modified are Fourier transformable with respect to x and $-y$, respectively. It is crucial to point out here that this assumption requires that $\mathcal{G}_2^-(s, x, p)e^{-px}$ and $\mathcal{G}_1^+(s, x, p)e^{-px}$ be Fourier transformable. We shall see in the following section that the problem indeed permits such transformability when $\mathrm{Re}(p) > 0$. Denoting the Fourier transform of a capital or script-lettered function by the corresponding lower case letter, we obtain

$$g_1^+(s, \omega, p)[1 - sa_+(\omega + ip)] = sa_+(\omega + ip) + sg_2^-(s, \omega + ip, p)a_+(\omega + ip) + l^-(s, \omega, p) \tag{5.6}$$

and

$$g_2^-(s, \omega, p)[s - d_-(\omega - ip)] = d_-(\omega - ip)g_1^+(s, \omega - ip, p) - d_-(\omega - ip)g_0^-(\omega, p) + sr^+(s, \omega, p), \tag{5.7}$$

on $-\infty < \omega < \infty$, where

$$g_1^+(s, \omega, p) = \int_{-\infty}^{\infty} \tilde{\mathcal{G}}_1^+(s, x, p)e^{i\omega x}\, dx = \int_0^{\infty} \mathcal{G}_1^+(s, x, p)e^{i\omega x}\, dx, \tag{5.8a}$$

$$g_2^-(s, \omega, p) = \int_{-\infty}^{\infty} \tilde{\mathcal{G}}_2^-(s, -y, p)e^{-i\omega y}\, dy = \int_{-\infty}^{0} \tilde{\mathcal{G}}_2^-(s, y, p)e^{i\omega y}\, dy, \tag{5.8b}$$

$$d_-(\omega - ip) = \int_{-\infty}^{\infty} D_+(y)e^{-py}e^{-i\omega y}\, dy = \int_0^{\infty} D_+(y)e^{-i(\omega - ip)y}\, dy, \tag{5.8c}$$

and where l^- and r^+ are arbitrary except that they must be the Fourier transforms of an L^- and R^+ function, respectively. In general the Fourier transform of a "+" function is a bounded analytic function in the upper half plane of the transform variable while that of a "−" function is bounded analytic in the lower half plane. Thus in (5.6) and (5.7) the + and − indices acquire the significance of denoting the upper and lower ω-half planes, respectively, of bounded analyticity. Since we take the Fourier transform of (5.5) with respect to $-y$, however, in order to obtain the appropriate form for $\mathcal{G}_2^-$ (see (5.8b)), we find D_+ transforms into d_- (i.e., a transform which is bounded analytic in the *lower* half-plane, see (5.8c)).

To bring (5.7) into a form compatible with (5.6), we make the transformation of variable $\omega \to \omega + ip$, which leads to

$$g_2^-(s, \omega + ip, p)[s - d_-(\omega)] = d_-(\omega)g_1^+(s, \omega, p) - d_-(\omega)g_0^-(\omega + ip, p) + sr^+(s, \omega + ip, p). \tag{5.7a}$$

Denoting $g_2^-(s, \omega + ip, p)$ and $g_0^-(\omega + ip, p)$ by $h_2^-(s, \omega, p)$ and $h_0^-(\omega, p)$, rerespectively, in keeping with the assumed behavior of $\mathcal{G}_1^+$ and $\mathcal{G}_2^-$, we may rewrite equations (5.6), (5.7a) in the form

$$(5.9)\quad [g_1^+(s, \omega, p) + h_2^-(s, \omega, p) + 1]\phi_+(s, \omega, p) = \phi^-(s, \omega, p),$$

$$(5.10)\quad \begin{aligned}[g_1^+(s, \omega, p) + h_2^-(s, \omega, p) + 1]\psi_-(s, \omega) &= \psi^+(s, \omega, p)\\ &\quad - d_-(\omega)[1 + h_0^-(\omega, p)],\end{aligned}$$

respectively, where

$$(5.11a)\quad \phi_+(s, \omega, p) = 1 - sa_+(\omega + ip),$$

$$(5.11b)\quad \phi^-(s, \omega, p) = 1 + h_2^-(s, \omega, p) + l^-(s, \omega, p),$$

$$(5.11c)\quad \psi_-(s, \omega) = s - d_-(\omega),$$

$$(5.11d)\quad \psi^+(s, \omega, p) = s[1 + g_1^+(s, \omega, p) + r^+(s, \omega + ip, p)].$$

Now eliminating the common term between (5.9) and (5.10), we reduce the problem to the single equation on $-\infty < \omega < \infty$

$$(5.12)\quad \begin{aligned}\phi_+(s, \omega, p)\psi^+(s, \omega, p) - \phi^-(s, \omega, p)\psi_-(s, \omega)&\\ = \phi_+(s, \omega, p)d_-(\omega)[1 + h_0^-(\omega, p)]&\end{aligned}$$

To emphasize the basic character of the problem with respect to its ω dependence, we rewrite (5.12) as

$$(5.13)\quad X^+(\sigma) - X^-(\sigma) = \kappa(\sigma), \qquad -\infty < \sigma < \infty,$$

where

$$(5.14)\quad \begin{aligned}\operatorname{Re}(\omega) = \sigma, X^+ = \phi_+\psi^+, \qquad X^- &= \phi^-\psi_- + d_-(1 + h_0^-)\\ \text{and} \qquad \kappa &= -sa_+d_-(1 + h_0^-).\end{aligned}$$

In general the functions X^+, X^- and κ involved here depend on s and p as well. Whether this dependence is suppressed or not, any conditions or equations involving analyticity, integrability, etc., with respect to ω are meant to hold uniformly for $0 \leqq s \leqq 1$ and $\operatorname{Re}(p) > 0$.

In this form we may characterize our problem as follows: it is required to find a sectionally holomorphic function $X(\omega)$, bounded at infinity and satisfying the boundary condition (5.13), where $X^+(\sigma)$ (respectively $X^-(\sigma)$) signify the limiting values assumed by $X(\omega)$ along curves such that $\operatorname{Im}(\omega) \to 0+$ (respectively $\operatorname{Im}(\omega) \to 0-$), and where κ (though here an unknown) possesses a well defined Cauchy integral. X is given directly in terms of κ by the Plemelj formula [2] as

$$(5.15)\quad X(\omega) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\kappa(\sigma)}{\sigma - \omega}\, d\sigma + P(\omega)$$

with the additive polynomial in ω, P, reflecting the prescribed behavior of X at infinity. In our case, (5.14) together with (5.11) require that $P(\omega) = s$. We will assume throughout the following that

$$a_+(\sigma) \to 0 \quad \text{or} \quad d_-(\sigma) \to 0 \quad \text{(or both)}, \tag{5.16}$$

as $\sigma \to \pm\infty$. The basic role of this assumption will be in evidence at several points of the analysis. At this point, in particular, it serves to assure the existence of the Cauchy integral in (5.15). A further discussion of the significance of these conditions on the arrival and service distributions will be given in Section 7.

The analyticity condition. We have in particular, from (5.15) that for $\mathrm{Im}(\omega) < 0$

$$X^-(\omega) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\kappa(\sigma)}{\sigma - \omega}\, d\sigma + s, \tag{5.17}$$

or, substituting from (5.14) and (5.11)

$$\begin{aligned}[s - d_-(\omega)]\phi^-(s, \omega, p) &= s - d_-(\omega)[1 + h_0^-(\omega, p)] \\ &\quad - \frac{s}{2\pi i}\int_{-\infty}^{\infty} \frac{a_+(\sigma + ip)d_-(\sigma)[1 + h_0^-(\sigma, p)]}{\sigma - \omega}\, d\sigma.\end{aligned} \tag{5.18}$$

We now observe that to each value of s such that $0 \leqq s \leqq 1$, there exists a (unique) pure imaginary value of ω lying in the lower half plane such that $s - d_-(\omega) = 0$. ϕ^-, consequently, can be holomorphic in the ω-lower half plane *only* if the right hand side of (5.18) also vanishes at the above described pairs (s, ω) for which $s = d_-(\omega)$. Applying this analyticity condition, we are directly led to the following integral equation for the quantity $1 + h_0^-$,

$$1 + h_0^-(\omega, p) = 1 - \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{a_+(\sigma + ip)d_-(\sigma)[1 + h_0^-(\sigma, p)]}{\sigma - \omega}\, d\sigma. \tag{5.19}$$

Moreover comparison of equations (5.18) and (5.19) yields the basic identification

$$\phi^-(s, \omega, p) = 1 + h_0^-(\omega, p). \tag{5.20}$$

We have thus achieved a formal reduction of our original set of simultaneous Wiener-Hopf integral equations to a single integral equation in the transform variables. In the next section we shall take advantage of (5.20) to show that ϕ^- also satisfies a Hilbert problem [2] for which we can give an explicit solution.

6. The associated Hilbert problem. Substituting from (5.20) into (5.12), we obtain on $-\infty < \sigma < \infty$,

$$[1 - sa_+(\sigma + ip)][\psi^+(s, \sigma, p)/s] = [1 - a_+(\sigma + ip)d_-(\sigma)]\phi^-(\sigma, p). \tag{6.1}$$

We shall solve this boundary value problem for the unknowns ψ^+/s and ϕ^- in two steps. First we shall consider the special case of (6.1) for $s = 1$; namely,

$$\begin{aligned}&[1 - a_+(\sigma + ip)]\psi^+(1, \sigma, p) \\ &\quad = [1 - a_+(\sigma + ip)d_-(\sigma)]\phi^-(\sigma, p), \quad -\infty < \sigma < \infty.\end{aligned} \tag{6.2}$$

Since it was previously established in (5.20) that ϕ^- is independent of s, the solution of (6.2) provides part of the solution of (6.1), ϕ^- to be precise, directly; $\psi^+(s, \omega, p)$ may also be obtained from the solution of (6.2) by means of the relation

$$\psi^+(s, \omega, p) = s\,\frac{1 - a_+(\omega + ip)}{1 - sa_+(\omega + ip)}\,\psi^+(1, \omega, p), \qquad \operatorname{Im}(\omega) \geqq 0. \tag{6.3}$$

We now rewrite (6.2) as

$$\Lambda_p^+(\sigma) = K_p(\sigma)\Lambda_p^-(\sigma), \qquad -\infty < \sigma < \infty, \tag{6.4}$$

where $\Lambda_p^+(\sigma) = [1 - a_+(\sigma + ip)]\psi^+(1, \sigma, p)$, $\Lambda_p^-(\sigma) = \phi^-(\sigma, p)$ and $K_p(\sigma) = 1 - a_+(\sigma + ip)d_-(\sigma)$. In the form (6.4) we are able to identify our problem as a Hilbert problem (in the sense of [2]), indexed by a parameter p, for the sectionally holomorphic function $\Lambda_p(\omega)$. It only remains to determine those values of the parameter p for which the "kernel" K_p in (6.4) constitutes a well posed Hilbert problem. There are several conditions required on K_p which, as we shall show, are satisfied when $\operatorname{Re}(p) > 0$. First it is required that K_p be Hölder continuous on $-\infty < \sigma < \infty$, which follows immediately for $\operatorname{Re}(p) \geqq 0$ if we assume, in addition to (5.16), the existence of first moments for the arrival and service time distributions. Similarly for $\operatorname{Re}(p) > 0$ it is immediate that $K_p(\sigma) \neq 0$ on $-\infty < \sigma < \infty$, and that K_p has a finite index on $-\infty < \sigma < \infty$, i.e., that the increment of arg[log $K_p(\sigma)$] as σ goes from $-\infty$ to $+\infty$ is finite. Since $\operatorname{Re}(K_p(\sigma)) > 0$ on $-\infty < \sigma < \infty$ for $\operatorname{Re}(p) > 0$, it easily follows that the index is zero.

Thus under the restriction that $\operatorname{Re}(p) > 0$, we can take advantage of the extensive results available concerning Hilbert problems. The unique sectionally holomorphic solution of (6.4) satisfying the boundary condition $\Lambda_p(\omega) \to 1$ as $\omega \to \infty$ is given by

$$\Lambda_p(\omega) = \exp\left[\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\log K_p(\sigma)}{\sigma - \omega}\, d\sigma\right], \tag{6.5}$$

so that in terms of the variables of interest,

$$\phi^-(\omega, p) = \exp\left[\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\log\,[1 - a_+(\sigma + ip)d_-(\sigma)]}{\sigma - \omega}\, d\sigma\right] \tag{6.6}$$

for $\operatorname{Im}(\omega) \leqq 0$, $\operatorname{Re}(p) > 0$.

Our analysis requires for consistency not only that $\phi^-(\omega, p)$ be holomorphic in $\operatorname{Im}(\omega) < 0$, but also, in view of (5.20), that the nature of its dependence on ω be such that $\phi^-(\omega, p) - 1$ have the character of a Fourier transform, hereafter called F-character, which is immediate for $\operatorname{Re}(p) > 0$. In view of (5.7a) and (5.20), moreover, $\phi^-(\omega - ip, p) - 1 = g_0^-(\omega, p)$ which is required to have F-character in ω and L (Laplace)-character in p. The F-character of $g_0^-(\omega, p)$ for $\operatorname{Re}(p) > 0$ follows from the structure of (6.6) and the L-character of $a_+(\sigma + ip)$.

To conclude this section, we shall show how the rest of our basic variables, g_1^+, h_2^- and g_2^-, may also be obtained directly in terms of ϕ^-, and that our assumptions regarding the F- and L-character of these functions are verified. From (5.9) we have

$$g_1^+(s, \sigma, p) + h_2^-(s, \sigma, p) = \frac{\phi^-(\sigma, p) - 1 + sa_+(\sigma + ip)}{1 - sa_+(\sigma + ip)}, \tag{6.7}$$

for $-\infty < \sigma < \infty$ and $\text{Re}(p) > 0$.

Clearly we may consider (6.7) as a boundary value problem for the unknowns g_1^+ and h_2^-, for which the Plemelj formulae yield the solution

$$h_2^-(s, \omega, p) = \frac{-1}{2\pi i} \int_{-\infty}^{\infty} \left[\frac{\phi^-(\sigma, p) - 1 + sa_+(\sigma + ip)}{1 - sa_+(\sigma + ip)} \cdot \frac{1}{\sigma - \omega} \right] d\sigma, \quad \text{Im}(\omega) \leqq 0,$$

or by contour integration and (5.20)

$$h_2^-(s, \omega, p) = \frac{-1}{2\pi i} \int_{-\infty}^{\infty} \left[\frac{h_0^-(\sigma, p)}{1 - sa_+(\sigma + ip)} \cdot \frac{1}{\sigma - \omega} \right] d\sigma; \tag{6.8}$$

and similarly,

$$\begin{aligned} g_1^+(s, \omega, p) &= \frac{sa_+(\omega + ip)}{1 - sa_+(\omega + ip)} \\ &\quad + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left[\frac{h_0^-(\sigma, p)}{1 - sa_+(\sigma + ip)} \cdot \frac{1}{\sigma - \omega} \right] d\sigma. \end{aligned} \tag{6.9}$$

Finally by (6.8) and the fact that $g_2^-(s, \omega, p) = h_2^-(s, \omega - ip, p)$, we have

$$g_2^-(s, \omega, p) = \frac{-1}{2\pi i} \int_{-\infty}^{\infty} \left[\frac{g_0^-(\sigma, p)}{1 - sa_+(\sigma)} \cdot \frac{1}{\sigma - \omega} \right] d\sigma. \tag{6.10}$$

(6.9) may be correspondingly rewritten as

$$\begin{aligned} g_1^+(s, \omega, p) &= \frac{sa_+(\omega + ip)}{1 - sa^+(\omega + ip)} \\ &\quad + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left[\frac{g_0^-(\sigma, p)}{1 - sa_+(\sigma)} \cdot \frac{1}{\sigma - (\omega + ip)} \right] d\sigma. \end{aligned} \tag{6.11}$$

That expressions (6.8)–(6.11) for $\text{Re}(p) > 0$ do, indeed, possess the requisite F- and L-character assumed previously of g_1^+, g_2^- and h_2^- is immediate.

7. The regeneration times and occupation times. The Laplace transform of (4.11) with respect to t yields

$$\frac{\partial \tilde{\mathfrak{G}}}{\partial y} + p\tilde{\mathfrak{G}} = D_+(y)e^{-py} + \int_0^{\infty} \tilde{\mathfrak{G}}_0^-(x - y, p)e^{-px}D_+(x)\, dx, \tag{7.1}$$

where we have utilized (4.16). The Fourier transform of (7.1) with respect to

$-y$ (noting that (7.1) holds for $-\infty < y < \infty$) then yields

$$\begin{aligned} i(\omega - ip)e^{-}(\omega, p) \\ = (1 + g_0^{-}(\omega, p))d_{-}(\omega - ip) = \phi^{-}(\omega - ip, p)d_{-}(\omega - ip), \end{aligned} \tag{7.2}$$

where we have taken account of (5.20) to bring in ϕ^{-}.

We are now in a position to determine the system regeneration time distribution R. Taking the Laplace transform of (4.4a), we have, in view of the initial condition $R(0) = 0$,

$$p\tilde{R}(p) = \int_0^\infty \tilde{\mathcal{E}}(y, p)A(y)\, dy.$$

Equivalently in terms of our basic transform variables, we have

$$p\tilde{R}(p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-}(\sigma, p)a_{+}(\sigma)\, d\sigma, \tag{7.3}$$

or substituting for e^{-} from (7.2),

$$p\tilde{R}(p) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{a_{+}(\sigma)d_{-}(\sigma - ip)\phi^{-}(\sigma - ip, p)}{\sigma - ip}\, d\sigma. \tag{7.4}$$

If we make the transformation of variable $\sigma \to \sigma + ip$ for $\operatorname{Re}(p) > 0$ in (7.4), we obtain

$$p\tilde{R}(p) = \frac{1}{2\pi i}\int_{-\infty - ip}^{\infty - ip} \frac{a_{+}(\sigma + ip)d_{-}(\sigma)\phi^{-}(\sigma, p)}{\sigma}\, d\sigma. \tag{7.4a}$$

In view of the half planes of analyticity of the functions in the integrand of (7.4a), we have

$$p\tilde{R}(p) = \frac{1}{2\pi i}\int_{\mathfrak{C}} \frac{a_{+}(\sigma + ip)d_{-}(\sigma)\phi^{-}(\sigma, p)}{\sigma}\, d\sigma, \tag{7.4b}$$

where the contour $\mathfrak{C}$ is the real axis indented into the *lower* half plane at $\sigma = 0$.

We next establish a relationship between the integral in (7.4b) and ψ^{+}. Applying the Plemelj formulae once more to (5.12) and taking account of (5.20), we find

$$\begin{aligned} [1 - a_{+}(\omega + ip)]\psi^{+}(1, \omega, p) \\ = 1 + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{[1 - a_{+}(\sigma - ip)]d_{-}(\sigma)\phi^{-}(\sigma, p)}{\sigma - \omega}\, d\sigma \end{aligned} \tag{7.5}$$

for $\operatorname{Im}(\omega) \geqq 0$ and $\operatorname{Re}(p) > 0$. Since ω lies in the upper half plane, however,

$$\int_{-\infty}^{\infty} \frac{d_{-}(\sigma)\phi^{-}(\sigma, p)}{\sigma - \omega}\, d\sigma = 0,$$

so that taking the limit of (7.5) as $\omega \to 0$ we obtain the relationship

$$[1 - a_{+}(ip)]\psi^{+}(1, 0, p) = 1 - \frac{1}{2\pi i}\int_{\mathfrak{C}} \frac{a_{+}(\sigma + ip)d_{-}(\sigma)\phi^{-}(\sigma, p)}{\sigma}\, d\sigma,$$

where $\mathcal{C}$ is the contour described above. Hence

$$p\tilde{R}(p) = 1 - [1 - a_+(ip)]\psi^+(1, 0, p). \tag{7.6}$$

Next we determine the occupation time distribution S. The Laplace transform of (4.4b) yields

$$\tilde{S}(p) = \tilde{R}(p) + \int_0^\infty \tilde{\mathcal{E}}(y, p)e^{-L(y)}\, dy,$$

whence

$$\tilde{S}(p) = \tilde{R}(p) + \frac{1}{2\pi i}\int_{-\infty}^{\infty} e^-(\sigma, p)\left[\frac{a_+(\sigma) - 1}{\sigma}\right] d\sigma. \tag{7.7}$$

It follows from (7.7), (7.4), (7.2), (5.20) and (5.19) that

$$p\tilde{S}(p) = 1 - [1 - d_-(-ip)]\phi^-(-ip, p), \qquad \text{Re}(p) > 0. \tag{7.8}$$

We observe that the expressions obtained above for $\tilde{R}$ and $\tilde{S}$ possess the required L-character for $\text{Re}(p) > 0$ as a consequence of the fact that $a_+(\sigma + ip)$ and $\phi^-(\sigma, p)$ do uniformly with respect to σ such that $-\infty < \sigma < \infty$. Hence for $\text{Re}(p) > 0$, these expressions are valid and meaningful with no further restrictions.

As we shall see next, however, we must consider a "stability" condition, i.e., a condition between the mean arrival and mean service times, in order to study the limit of $R(t)$ and $S(t)$ as $t \to \infty$, or equivalently, the limit of $p\tilde{R}(p)$ and $p\tilde{S}(p)$ as $p \to 0+$. We see from (7.6) that $\lim_{p\to 0+} p\tilde{R}(p) = 1$ is assured provided $\lim_{p\to 0+} \psi^+(1, 0, p)$ exists and is finite. Returning to (6.2), we have

$$\psi^+(1, \sigma, p) = \left[\frac{1 - a_+(\sigma + ip)d_-(\sigma)}{1 - a_+(\sigma + ip)}\right]\phi^-(\sigma, p), \tag{7.9}$$

which constitutes a well posed Hilbert problem for ψ^+, ϕ^- for $\text{Re}(p) > 0$. To see this we need only observe that the kernel involved in (7.9) does not vanish on $-\infty < \sigma < \infty$ for $\text{Re}(p) > 0$, and that since the real parts of the numerator and denominator of the kernel are positive, its index is zero. The formal limit of the boundary value problem (7.9) as $\text{Re}(p) \to 0+$ is

$$\psi^+(1, \sigma, 0) = \left[\frac{1 - a_+(\sigma)d_-(\sigma)}{1 - a_+(\sigma)}\right]\phi^-(\sigma, 0), \qquad -\infty < \sigma < \infty. \tag{7.10}$$

The kernel of this limiting problem is also well defined and non-vanishing for $\sigma \neq 0$ provided $|a_+(\sigma)| < 1$ for $\sigma \neq 0$. Furthermore we have

$$\lim_{\sigma\to 0} \frac{1 - a_+(\sigma)d_-(\sigma)}{1 - a_+(\sigma)} = 1 - \frac{x_D}{x_A}$$

where x_D and x_A are the mean service and mean interarrival times, respectively; hence if the usual stability condition prevails, i.e., if $x_D < x_A$, the kernel of (7.10) does not vanish on $-\infty < \sigma < \infty$, and the index continues to be zero.

Under these conditions it is easily shown that the solutions $\psi^+(1, \omega, p), \phi^-(\omega, p)$ of (6.2) for $\text{Re}(p) > 0$ converge to the solution of (7.10) as $\text{Re}(p) \to 0+$. Thus in view of (7.6) we have the result that our system always regenerates under the stated conditions on $a_+(\sigma)$ and x_D, x_A. This analysis shows, moreover that under these same conditions $\lim_{\text{Re}(p)\to 0+} \phi(\omega, p)$ exists and satisfies the limiting Hilbert problem. It follows, in particular that $\lim_{\text{Re}(p)\to 0+} \phi^-(-ip, p)$ exists and is finite, whence from (7.8) we have that $\lim_{p\to 0+} pS(p) = 1$ as well.

The assumption that $|a_+(\sigma)| < 1$ for $\sigma \neq 0$ in the discussion of the limits is already contained in the previous assumption (Section 5) that $a_+(\sigma) \to 0$ as $\sigma \to \pm\infty$. Indeed, if there is a real value of $\sigma = \sigma_0$ such that

$$\int_0^\infty A(y)[e^{i\sigma_0 y} - 1]\, dy = 0,$$

then $A(y)$ must have a lattice support of the form

$$A(y) = \sum_{n=1}^{\infty} C_n \delta\left[y - \frac{2\pi n}{\sigma_0}\right] \tag{7.11}$$

where the $C_n \geqq 0$, $n = 1, 2, \cdots$, and $\sum_{n=1}^{\infty} C_n = 1$. We shall refer to such arrivals as "synchronous". The special case of deterministic arrivals, $A(y) = \delta(y - T)$, is the most familiar example.

In the case of synchronous arrivals, the regeneration time density, dR/dt, may be directly seen to be of synchronous form also. In such cases it can be shown that the system does not approach a time independent limit as $t \to \infty$, even when the stability condition prevails, but rather tends to a periodic behavior. Because of the character of a_+ at infinity in the synchronous case, these problems cannot be discussed directly within the framework of the analysis presented above. Between arrivals, the service time completely determines state transitions, however, so that the problem is amenable to analysis in a lower dimensional phase space. When the service time density has a lattice support and the arrival density does not, the problem *can* be analyzed within the Hilbert framework even though $d_-(\sigma)$ does not tend to zero as $\sigma \to \pm\infty$. This unsymmetric character of the Hilbert problem with respect to the admissability of arrivals and departures with lattice support is due entirely to the unsymmetric manner in which a_+ and d_- enter in (6.2). That the previously obtained solution forms for the Hilbert problem continue to be valid in the case of lattice supported service times may be seen directly by considering the limit of suitable approximating service time distributions for which the previous analysis holds.

Regeneration times in the absence of stability. We shall briefly consider the problem of determining the limit as $t \to \infty$ of the distribution of regeneration times for the system in the absence of the usual stability criterion. Intuitively it is clear that in this case there should be a non-trivial probability of the system's *not* regenerating. More precisely, in terms of (7.6) we should expect that $\lim_{p\to 0} [1 - a_+(ip)]\psi^+(1, 0, p)$ exist, but be non-zero so that $\lim_{t\to\infty} R(t) = R(\infty) < 1$ in this case. We shall show that this case can be completely characterized in terms of a different Hilbert problem for the quantities $[1 - a_+(\sigma + ip)]\psi^+(1, \sigma, p)$ and $\phi^-(\sigma, p)$ rather than ψ^+ and ϕ^-. The existence and non-triviality of $\lim_{p\to0} [1 - a_+(ip)]\psi^+(1, 0, p)$ would thus be an immediate consequence of the fact that the Hilbert problem in question; namely,

$$(7.12)\quad [1 - a_+(\sigma + ip)]\psi^+(1, \sigma, p) = [1 - a_+(\sigma + ip)d_-(\sigma)]\phi^-(\sigma, p),$$

remain meaningful in the limit as $p \to 0$. As (7.12) stands, however, the limit of its kernel is $[1 - a_+(\sigma)d_-(\sigma)]$ which obviously is unsuitable since it vanishes at $\sigma = 0$. In order to avoid this difficulty of the kernel vanishing on $-\infty < \sigma < \infty$ in the limit, as well as to insure that the index (see Section 6) in the limit is zero, we consider the modified problem

$$(7.13)\quad \begin{aligned}&\{[1 - a_+(\sigma + ip)]\psi^+(1, \sigma, p)\}^+ \\ &\qquad = \left[\frac{1 - a_+(\sigma + ip)d_-(\sigma)}{1 - a_+(ip)d_-(\sigma)}\right] \{[1 - a_+(ip)d_-(\sigma)]\phi^-(\sigma, p)\}^-\end{aligned}$$

for the quantities $\{\ \}^+$ and $\{\ \}^-$, which satisfies all the requirements (as well as leaving the boundary condition at infinity intact) provided $x_D > x_A$, i.e., provided we are in the *unstable* case. Indeed, we have

$$\lim_{\sigma\to0}\lim_{p\to0+}\left[\frac{1 - a_+(\sigma + ip)d_-(\sigma)}{1 - a_+(ip)d_-(\sigma)}\right] = 1 - \frac{x_A}{x_D},$$

which insures a zero index for the limiting Hilbert problem provided $x_D > x_A$. We thus have the result that in the unstable case, the probability of the system's *not* regenerating, i.e., not emptying is given by

$$(7.14)\quad 1 - \lim_{p\to0+} p\tilde{R}(p) = \lim_{\omega\to0} \exp\left[\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\log\left[\dfrac{1 - a_+(\sigma)d_-(\sigma)}{1 - d_-(\sigma)}\right]}{\sigma - \omega}\, d\sigma\right],$$

where the limiting path lies in the upper half plane.

Special cases: E/G *and* G/E. We shall consider further the two special cases of a general Erlangian distribution of interarrival times with general service times and a general distribution of interarrival times with general Erlangian service times. These two cases have the analytical property of permitting a complete characterization of the root structure of the quantity $1 - a_+(\omega + ip)d_-(\omega)$ in a half plane. When this situation prevails the Hilbert problem can be resolved by direct factorization.

(i) When the interarrival times have a general Erlangian distribution

$$A\,(y) = \sum_{n=0}^{N} P_n(y)e^{-\lambda_n y},$$

where the $P_n(y)$ are taken to be real, positive valued polynomials (some obvious generalizations are possible). Then the expression $1 - a_+(\omega + ip)d_-(\omega)$ will have singularities $\zeta_m(p) = -i(\lambda_m + p)$ and roots $\rho_n(p)$ in the *lower* ω half plane. Since $1 - a_+(\omega + ip)d_-(\omega)$ has zero index and vanishes at infinity in the closed lower half plane, $\zeta_m(p)$ and $\rho_n(p)$, when weighted by their respective multiplicities, will be in one to one correspondence, so that the (unique) solution of (6.2) is given by

$$[1 - a_+(\omega + ip)]\psi^+(1, \omega, p) = \prod_{m,n} \frac{[\omega - \rho_n(p)]^{\alpha_n}}{[\omega - \zeta_m(p)]^{\beta_m}}, \tag{7.15}$$

where α_n and β_m are the associated multiplicities. (The expression for $\phi^-(\omega, p)$ follows directly from (6.2) and (7.15).)

(ii) When the service time distribution is general Erlangian,

$$D(x) = \sum_{n=0}^{N} Q_n(x)e^{-\eta_n x}$$

has singularities $\zeta_n = i\eta_n$ and roots $\rho_m(p)$ in the upper ω half plane which weighted by their respective multiplicities γ_n and δ_m are again in one to one correspondence, whence

$$\phi^-(\omega, p) = \prod_{m,n} \frac{(\omega - \zeta_n)^{\gamma_n}}{[\omega - \rho_m(p)]^{\delta_m}}. \tag{7.16}$$

The simplest and most familiar example of (i) is the case of exponentially distributed arrivals and general service times. In this case (7.15) becomes

$$[1 - a_+(\omega + ip)]\psi^+(1, \omega, p) = \frac{\omega - \rho_0(p)}{\omega + i(\lambda + p)}, \tag{7.17a}$$

where $\rho_0(p)$ is defined by the equation $\rho_0 + ip + i\lambda[1 - d_-(\rho_0)] = 0$; and hence,

$$\phi^-(\omega, p) = \frac{\omega - \rho_0(p)}{\omega + ip + i\lambda[1 - d_-(\omega)]}. \tag{7.17b}$$

It follows from (7.6) and (7.8) that

$$p\tilde{R}(p) = [\lambda/(\lambda + p)]d_-[\rho_0(p)] \quad \text{and} \quad p\tilde{S}(p) = d_-[\rho_0(p)], \tag{7.18}$$

respectively, for unstable as well as stable systems.

Similarly, the most familiar example of (ii) is that of general interarrival times and exponentially distributed service times. In this case (7.16) yields

$$\phi^-(\omega, p) = \frac{\omega - i\eta}{\omega - \rho_*(p)}, \tag{7.19a}$$

where $\rho_*(p)$ is defined by the equation $\rho_* - i\eta[1 - a_+(\rho_* + ip)] = 0$; and hence,

$$[1 - a_+(\omega + ip)]\psi^+(1, \omega, p) = \frac{\omega - i\eta[1 - a_+(\omega + ip)]}{\omega - \rho_*(p)}. \tag{7.19b}$$

Again from (7.6) and (7.8), we have in this case

$$(7.20)\quad p\tilde{R}(p) = \frac{\rho_*(p) - i\eta[1 - a_+(ip)]}{\rho_*(p)}, \qquad \text{and} \qquad p\tilde{S}(p) = \frac{\rho_*(p)}{\rho_*(p) + ip},$$

respectively, regardless of stability. Hence for example in the *unstable* case, we have from (7.20) that the system still regenerates with probability x_A/x_D.

Finally to conclude the discussion of these special cases, we shall see in Sections 8 and 9 that knowledge of the expressions (7.17) and (7.19) effectively generate the complete solution of their respective full queuing problems (Section 3). The complete time dependent queuing problem corresponding to case (i) recovers the results previously obtained in [1] for the M/G queue, while that corresponding to (ii) appears to be entirely new.

8. The general first passage problem. In the next section we shall show how the complete time dependent behavior of our original queuing problem with arbitrary initial conditions can be directly obtained in terms of the solution of a first passage problem subject to the same initial conditions. In this section we show that the solution of the first passage problem subject to *any* given initial conditions is itself, in turn, directly obtainable in terms of the solution of the basic *first* passage problem discussed above. The mathematical relationship between these two latter problems is simply that between an inhomogeneous Hilbert problem and its associated homogeneous problem.

The general first passage problem, i.e., the first passage problem subject to general initial conditions, must, of course, also satisfy equations (4.1) thru (4.7) with the general initial conditions

$$(8.1)\quad \begin{aligned} &W_m(x, y, 0) = W_{0m}(x, y), m = 0, 1, \cdots; \qquad E(y, 0) = E_0(y); \\ &F(x, 0) = F_0(x); \qquad \text{and} \qquad R(0) = 0; \end{aligned}$$

replacing (4.8). It involves no essential restriction of the discussion to consider the conditions

$$(8.2)\quad \begin{aligned} &W_{0m}(x, y) = C_W\delta_{mN}\delta(x - x_0, y - y_0), \qquad m = 0, 1, \cdots, \\ &E_0(y) = C_E\delta(y - y_0), \quad F_0(x) = C_F\delta(x - x_0), \quad \text{and} \quad R(0) = 0, \end{aligned}$$

where C_W, C_E, C_F are non-negative constants such that $C_W + C_E + C_F = 1$, δ_{mN} is Kronecker's delta, and δ the appropriate dimensional "delta function".

The analysis of the above problem is so closely related to that of the basic first passage problem considered in Sections 4 thru 7, that we shall content ourselves with indicating only those aspects of the analysis differing significantly from the preceding, leaving the details to the reader. The initial formal reductions are identical down to the point (equation (4.16)) where the initial conditions are brought in; in the present case we have

$$(8.3)\quad \begin{aligned} &\mathcal{W}_m(x, y, 0) = C_W\delta_{mN}e^{L(y)+N(x)}\delta(x - x_0, y - y_0), \quad m = 0, 1, \cdots, \\ &\mathcal{E}(y, 0) = C_E e^{L(y)}\delta(y - y_0), \\ &\mathcal{F}(x, 0) = f(x) = C_F e^{L(x)+N(x)}\delta(x - x_0). \end{aligned}$$

As a consequence of the fact that the $\mathcal{W}_m(x, y, t)$ and hence $G(s, x, y, t)$ now carry nontrivial initial data, we choose the general representation for G (cf., (4.20))

$$G(s, x, y, t) = \mathcal{G}_1(s, x - y, t - y) + \mathcal{G}_2(s, x - y, t - x) + \mathcal{G}_3(s, x - t, y - t) \tag{8.4}$$

for $x, y, t \geqq 0$ where $\mathcal{G}_1$ and $\mathcal{G}_2$ have the same "half line" character as before in their dependence on the differences involved, and $\mathcal{G}_3$ vanishes when $x < t$ *or* $y < t$. $\mathcal{G}_3$ is thus seen to carry the initial data associated with the $\mathcal{W}_m$; indeed, we have

$$\mathcal{G}_3(s, x, y) = C_W s^N e^{L(y)+N(x)} \delta(x - x_0, y - y_0), \qquad x, y \geqq 0. \tag{8.5}$$

Substituting (8.4), (8.5) into the integral equation for G corresponding to (4.18), (4.19), and then taking Laplace and Fourier transforms as before, we obtain the basic pair of equations (cf. (5.6), (5.7))

$$\begin{aligned} g_1^+(s, \omega, p)[1 - sa_+(\omega + ip)] = {} & sC_F e^{L(x_0)+N(x_0)} e^{px_0} a_+(\omega + ip) \\ & + sg_2^-(s, \omega + ip) a_+(\omega + ip) \\ & + s^{N+1} C_W e^{-L(y_0)+N(x_0)} e^{-i\omega x_0} \\ & \cdot a_+(\omega + ip; y_0) l^-(s, \omega, p), \end{aligned} \tag{8.6}$$

and

$$\begin{aligned} g_2^-(s, \omega, p)[s - d_-(\omega - ip)] = {} & d_-(\omega - ip) g_1^+(s, \omega - ip, p) \\ & - d_-(\omega - ip) g_0^-(\omega, p) + s^N C_W \\ & \cdot e^{L(y_0)+N(x_0)} e^{i\omega y_0} d_-(\omega - ip; x_0) \\ & + sr^+(s, \omega, p) \end{aligned} \tag{8.7}$$

on $-\infty < \omega < \infty$, where g_1^+, g_2^-, l^-, and r^+ have the same significance as previously, and

$$a_+(\omega + ip; y_0) = \int_0^\infty A(y + y_0) e^{i(\omega+ip)y}\, dy,$$

$$d_-(\omega - ip; x_0) = \int_0^\infty D(x + x_0) e^{-i(\omega-ip)x}\, dx.$$

At this point and precisely for the same reasons discussed above, we restrict p to $\operatorname{Re}(p) > 0$ and replace $g_2^-(s, \omega + ip, p)$ and $g_0^-(\omega + ip, p)$ by $h_2(s, \omega, p)$ and $h_0^-(\omega, p)$, respectively. It then follows from (8.6), (8.7) that g_1^+ and h_2^- satisfy the equations on $-\infty < \omega < \infty$ (cf. (5.9), (5.10))

$$[g_1^+ + h_2^- + 1]\phi_+(s, \omega) = \Phi^-(s, \omega, p) + sK_1(s, \omega, p), \tag{8.8}$$

$$[g_1^+ + h_2^- + 1]\psi_-(s, \omega, p) = \Psi^+(s, \omega, p) - d_-(\omega)[h_0^-(\omega, p) + 1] + sK_2(s, \omega, p), \tag{8.9}$$

where ϕ_+ and ψ_- are the same (known) functions previously involved, Φ^- and Ψ^+ are structurally the same as ϕ^- and ψ^+ in their dependence on unknowns h_1^+, g_2^-, r^+ and l^- (cf., (5.11)), and

$$sK_1(s, \omega, p) = sa_+(\omega + ip)[C_F e^{L(x_0)+N(x_0)} e^{px_0} - 1] + sa_+(\omega + ip; y_0) s^N C_W e^{L(x_0)+N(x_0)} e^{-i\omega x_0}, \tag{8.10}$$

$$sK_2(s, \omega, p) = d_-(\omega; x_0) s^N C_W e^{L(y_0)+N(x_0)} e^{i\omega y_0} e^{-py_0}. \tag{8.11}$$

Thus we see that the general first passage problem described by (8.8), (8.9) differs from the basic first passage problem (5.9), (5.10) only by the presence of the additional *known* inhomogeneous terms K_1 and K_2 in (8.8) and (8.9), respectively. Eliminating, as before, the common term $[g_1^+ + h_2^- + 1]$ between (8.8) and (8.9), we obtain (cf., (5.12))

$$\phi_+(s, \omega)\Psi^+(s, \omega, p) - \Phi^-(s, \omega, p)\psi_-(s, \omega, p) = \phi_+(s, \omega)d_-(\omega)[1 + h_0^-(\omega, p)] + sK_3(s, \omega, p), \tag{8.12}$$

where

$$K_3(s, \omega, p) = \psi_-(s, \omega, p)K_1(s, \omega, p) - \phi_+(s, \omega)K_2(s, \omega, p). \tag{8.13}$$

Next applying the Plemelj formulae to (8.12), and appealing to the analyticity condition as discussed in Section 5, we obtain the following (nonhomogeneous) integral equation for the quantity $1 + h_0^-$ (cf., (5.19))

$$1 + h_0^-(\omega, p) = 1 - \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{a_+(\sigma + ip)d_-(\sigma)[1 + h_0^-(\sigma, p)]}{\sigma - \omega}\, d\sigma + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{K_3(d_-(\omega), \sigma, p)}{\sigma - \omega}\, d\sigma, \qquad \text{for } \mathrm{Im}(\omega) \leqq 0. \tag{8.14}$$

This relationship, in turn, leads to (cf., (5.20))

$$\Phi^-(s, \omega, p) = 1 + h_0^-(\omega, p) + sK_4(s, \omega, p), \tag{8.15}$$

where

$$K_4(s, \omega, p) = \frac{1}{2\pi i[s - d_-(\omega)]}\int_{-\infty}^{\infty} \frac{K_3(s, \sigma, p) - K_3(d_-(\omega), \sigma, p)}{\sigma - \omega}\, d\sigma \tag{8.16}$$

As in Section 6, (8.15) may be used in (8.12) to obtain the nonhomogeneous Hilbert problem on $-\infty < \sigma < \infty$,

$$[1 + sa_+(\sigma + ip)]\frac{\Psi^+(s, \sigma, p)}{s} = [1 - a_+(\sigma + ip)d_-(\sigma)]\Phi^-(s, \sigma, p) + K_5(s, \sigma, p), \tag{8.17}$$

where

$$K_5(s, \sigma, p) = K_3(s, \sigma, p) - \phi_+(s, \sigma)d_-(\sigma)K_4(s, \sigma, p). \tag{8.18}$$

The solution of this nonhomogeneous Hilbert problem can be obtained directly in terms of its associated homogeneous problem [2]:

$$(8.19)\quad [1 - sa_+(\sigma + ip)]\frac{\psi^+(s, \sigma, p)}{s} = [1 - a_+(\sigma + ip)d_-(\sigma)]\phi^-(s, \sigma, p), \qquad -\infty < \sigma < \infty$$

which is precisely the boundary value problem we arrived at in Sections 5 and 6 for the basic first passage problem (cf., (6.1)). It follows, in particular, that

$$(8.20)\qquad \Phi^-(s, \omega, p) = \phi^-(\omega, p)\left[1 + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{K_5(s, \sigma, p)\, d\sigma}{\psi^+(s, \sigma, p)(\sigma - \omega)}\right],$$

thereby exhibiting the fundamental role of the *basic* first passage problem in generating the solution of the *general* first passage problem. We shall not pursue the general first passage problem any further here since state densities, regeneration times and occupation times are obtainable explicitly in terms of Φ^-, as we have already seen (Sections 6, 7).

9. The general queue with arbitrary initial conditions. In this section we shall reconsider the original queuing problem described in Section 3, subject to arbitrary initial conditions, with the point of view of establishing its relationship to the first passage problems discussed above. In particular, we shall assume that we already have available the solution of the first passage problem corresponding to the *same* initial conditions as the problem at hand (see Section 8).

If we compare the mathematical description of the original queuing problem ((3.3)–(3.8)) with that of the first passage problem ((4.1)–(4.7)) with the corresponding initial conditions, we see that the basic unknowns $\mathbf{W}_m$, $\mathbf{E}$ and $\mathbf{F}$ are subject to the same equations as W_m, E and F (of the corresponding problem) with the single exception that

$$(3.8)\qquad \mathbf{F}(0, t) = \int_0^\infty \mathbf{E}(y, t)\lambda(y)\, dy,$$

replaces

$$(4.7)\qquad F(0, t) = 0.$$

The immediate consequence of this exception is that

$$\mathbf{F}(x, t) = e^{-[L(x)+N(x)]}\mathbf{f}^+(t - x), \qquad x, t \geqq 0,$$

where $\mathbf{f}^+(t) = \int_0^\infty \mathbf{E}(y, t)\lambda(y)dy$ must be carried in the analysis as an auxiliary unknown, whereas previously we had

$$F(x, t) = e^{-[L(x)+N(x)]}f(t - x),$$

with $f^+(t) = 0$. The consequence of carrying along this additional unknown in the analysis is that we obtain the following basic pair of equations in the transform variables:

$$[\mathbf{g}_1^+ + \mathbf{h}_2^- + 1 + \tilde{\mathbf{f}}]\phi_+ = \mathbf{\Phi}^- + sK_1, \tag{9.1}$$

$$[\mathbf{g}_1^+ + \mathbf{h}_2^- + 1 + \tilde{\mathbf{f}}]\psi_- = \mathbf{\Psi}^+ - d_-[\mathbf{h}_0^- + 1 + \tilde{\mathbf{f}}] + sK_2, \tag{9.2}$$

where ϕ_+, ψ_-, K_1 and K_2 are defined as before (see (5.11), (8.10), (8.11)),

$$\mathbf{\Phi}^- = \mathbf{h}_2 + \mathbf{l}^- + 1 + \tilde{\mathbf{f}}, \qquad \mathbf{\Psi}^+ = s[\mathbf{g}_1^+ + \mathbf{r}^+ + 1 + \tilde{\mathbf{f}}],$$

and

$$\tilde{\mathbf{f}}(p) = \int_0^\infty \mathbf{f}^+(t)e^{-pt}\, dt,$$

in place of (8.8), (8.9). Following the previous analysis again, we obtain (cf., (8.15))

$$\mathbf{\Phi}^- = 1 + \tilde{\mathbf{f}} + \mathbf{h}_0^- + sK_4, \tag{9.3}$$

where K_4 is defined as before by (8.16). It thus follows that $\mathbf{\Phi}^-$ satisfies the same Hilbert problem as Φ^- (see (8.17)) except that $(1 + \tilde{\mathbf{f}})$ replaces 1 as the boundary condition at infinity. It follows that

$$\mathbf{\Phi}^-(s, \omega, p) = [1 + \tilde{\mathbf{f}}(p)]\phi^-(\omega, p)\left[1 + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{K_5(s, \sigma, p)}{[1 + f(p)]\psi^+(s, \sigma, p)(\sigma - \omega)}\right]$$

whence

$$\mathbf{\Phi}^-(s, \omega, p) = \Phi^-(s, \omega, p) + \tilde{\mathbf{f}}(p)\phi^-(\omega, p), \tag{9.4}$$

gives the unique solution of our problem as a function of $\tilde{\mathbf{f}}$, which, itself, remains to be determined from (3.4) and (3.8).

Expressing (3.4) in terms of $\mathbf{e}$, $\mathbf{f}$ and $\mathbf{g}_0$ (see (4.9)), taking the appropriate transforms (noting that (3.4) holds on $-\infty < y < \infty$), we find

$$\begin{aligned}(p + i\omega)\mathbf{e}^-(\omega, p) \\ = C_E e^{L(y_0)}e^{-i\omega y_0} + [C_F e^{L(x_0)+N(x_0)}e^{px_0} + \tilde{\mathbf{f}}(p) + \mathbf{g}_0^-(\omega, p)]d_-(\omega - ip),\end{aligned} \tag{9.5}$$

or, in view of (9.3) and (9.4),

$$\begin{aligned} i(\omega - ip)\mathbf{e}^-(\omega, p) \\ = [\Phi^-(s, \omega - ip, p) + \tilde{\mathbf{f}}(p)\phi^-(\omega - ip, p)]d_-(\omega - ip) + K_6(s, \omega - ip, p),\end{aligned} \tag{9.6}$$

where

$$\begin{aligned} K_6(s, \omega - ip, p) \\ = C_E e^{L(y_0)}e^{-i(\omega - ip)y_0}e^{py_0} + [C_F e^{L(x_0)+N(x_0)}e^{px_0} - 1 - sK_4(s, \omega - ip, p)]d_-(\omega - ip).\end{aligned}$$

On the other hand, from (3.8) we have

$$\tilde{\mathbf{f}}(p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{e}^-(\sigma, p)a_+(\sigma)\, d\sigma, \tag{9.7}$$

or, substituting from (9.6)

$$\tilde{\mathfrak{f}}(p) = \frac{\tilde{\mathfrak{f}}(p)}{2\pi i} \int_{-\infty}^{\infty} \frac{a_+(\sigma)d_-(\sigma - ip)\phi^-(\sigma - ip, p)}{\sigma - ip} \, d\sigma$$

(9.8)

$$+ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{a_+(\sigma)[d_-(\sigma - ip)\Phi^-(s, \sigma - ip) + K_6(s, \sigma - ip, p)]}{\sigma - ip} \, d\sigma.$$

Now the first integral on the right side of (9.8) is equal to p times the transform of the regeneration time distribution for the *basic* first passage problem (see (7.4)), while the second integral is the corresponding entity for the first passage problem associated with the initial conditions under consideration; hence (9.8) becomes simply

(9.9) $$\tilde{\mathfrak{f}}(p) = \frac{p\tilde{R}_A(p)}{1 - p\tilde{R}(p)},$$

where R_A denotes the transform of the regeneration time distribution for the associated first passage problem and R is given by (7.4).

In terms of our original variables, equation (9.9) takes the form

(9.10) $$F(0, t) = \frac{dR_A}{dt} + \int_0^{\infty} F(0, t - t') \frac{dR}{dt'} \, dt',$$

i.e., $F(0, t)$, the probability per unit time of the system regenerating at time t, is the probability per unit time that the system is regenerating for the *first* time at time t starting from the given initial conditions plus the probability per unit time of its regenerating at time t having last regenerated at time t'. (9.10) could well have been posed directly and its solution (9.9) obtained immediately without recourse to the analysis of the full problem. We have preferred to obtain (9.9), however, as a by product of the analysis of the complete queuing problem posed in Section 3 in order to show clearly how the additional difficulties of the full problem may be dealt with within the framework of the general formalism.

In concluding this discussion of the complete queuing problem, we remark that on the basis of the direct relationships established above between the complete problem and the first passage problem, the questions of stable and unstable solutions, synchronous solutions, and the like for the complete problem may be analyzed directly in terms of the corresponding considerations given for the first passage problem (Section 7).

10. Some relationships with previous literature. In this final section we wish to indicate briefly the relationship of the foregoing to that of F. Pollaczek [3] and D. V. Lindley [4]. In [3] Pollaczek has considered the problem of determining the server occupation time distribution for a simple queue subject to general interarrival and general service time distributions. Under the assumption that both the arrival and service distributions fall off exponentially, he was able to derive (in an independent way) the following integral equation (cf., (5.19)) for a basic quantity I (translated to our setting):

$$(10.1)\qquad I^-(\omega) = 1 + \frac{s}{2\pi i}\int_C \frac{a_+(\sigma)d_-(\sigma - ip)I^-(\sigma)\,d\sigma}{\sigma - \omega}$$

for which he was able to give (again by an independent argument) the solution (cf., (6.6)):

$$(10.2)\qquad I^-(s, \omega, p) = \exp\left[\frac{1}{2\pi i}\int_C \frac{\log[1 - sa_+(\sigma)d_-(\sigma - ip)]}{\sigma - \omega}\right] d\sigma.$$

Pollaczek's analysis here, though of obvious importance in its own right, contains no consideration of the significance for the general queuing problem of the integral equation nor its solution beyond the immediate problem of determining the distribution of the server occupation times.

In [4] Lindley has considered the problem of determining the distribution, F, of waiting times for entry into service of arrivals to a simple queue in equilibrium subject to general interarrival time and general service time distributions, with a first-come first-served discipline. Under these conditions, he has derived the following integral equation for F:

$$(10.3)\qquad F(x) = \int_0^\infty F(y)\,dG(x - y), \qquad x \geqq 0,$$

where G is a (cumulative) distribution function composed of the given interarrival and service time distributions. In the event they possess densities $A(y)$ and $D(x)$, respectively, it follows that the density of F exists on $0 < x < \infty$ and its component there satisfies the integral equation

$$(10.5)\qquad f^+(x) = F_0\, g(x) + \int_0^\infty f^+(y)g(x - y)\,dy, \qquad x > 0,$$

where g is the density of G on $(-\infty, \infty)$ and $F_0 = \lim_{x\to 0+} F(x) > 0$ is the probability of the server's being vacant and the queue empty. Adding an unknown function $l^-(x)$ to (10.5) and taking a Fourier transform with respect to $-x$, we obtain

$$(10.6)\qquad \psi^+(\sigma) - \phi^-(\sigma) = -a_+(\sigma)d_-(\sigma)\phi^-(\sigma),$$

where

$$\psi^+(\sigma) = F_0 + \int_{-\infty}^\infty l^-(x)e^{-i\sigma x}\,dx,$$

$$\phi^-(\sigma) = F_0 + \int_{-\infty}^\infty f^+(x)e^{-i\sigma x}\,dx,$$

$$a_+(\sigma)d_-(\sigma) = \int_{-\infty}^\infty g(x)e^{i\sigma x}\,dx.$$

The Plemelj formulae then yield for $\text{Im}(\omega) \leqq 0$

$$\phi^-(\omega) = F_0 - \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{a_+(\sigma)d_-(\sigma)\phi^-(\sigma)}{\sigma - \omega}\, d\sigma, \tag{10.7}$$

which is the $p \to 0$ limit of (5.19) to within the multiplicative constant F_0, the boundary condition at infinity. We thus have the result that the entire state densities for the queue *in equilibrium* are available once the solution of Lindley's integral equation for the waiting time distribution is available. In place of considering (10.6) as a Plemelj problem leading to the integral equation (10.7), it may also be analyzed as a Hilbert problem and resolved directly.

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