APPLICATION OF THE GEOMETRY OF QUADRICS FOR CONSTRUCTING PBIB DESIGNS

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0. Summary. The properties of linear flats in finite projective geometry are greatly used by many authors to construct incomplete block designs. In this paper the incidence properties of linear flats contained in quadrics in a finite projective geometry are exploited to construct two associate class partially balanced incomplete block designs.

1. Introduction. Partially balanced incomplete block (PBIB) designs with two associate classes were introduced by Bose and Nair [3]. Bose and Shimamato [4] have rephrased the definition so as to stress the distinction between the association scheme and the design. The Bose and Shimamato definition for the PBIB design with m associate classes is substantially as follows:

A PBIB design with m associate classes is an arrangement of v treatments in b blocks such that

(1) Each of the v treatments is replicated r times in b blocks each of size k and no treatment occurs more than once in any block.

(2) There exists a relationship of association between every pair of the v treatments satisfying the following conditions:

(2a) Any two treatments are either first associates, second associates, $\cdots$, or mth associates.

(2b) Each treatment has n_1 first associates, n_2 second associates, $\cdots$, and n_m mth associates.

(2c) Given any two treatments which are ith associates, the number p^i_{jk} of treatments which are jth associates of the first and kth associates of the second is independent of the pair of treatments with which we start. Furthermore $p^i_{jk} = p^i_{kj}$, for $i, j, k = 1, 2, \cdots, m$.

(3) Any pair of treatments which are ith associates occurs together in exactly λ_i blocks for $i = 1, 2, \cdots, m$.

Bose and Clatworthy [2] have shown that for a PBIB design with two associate classes the condition (2c) is equivalent to (2c′) given below:

(2c′) For any pair of the v treatments which are ith associate the number p^i_{11} of treatments common to the first associates of the first and first associates of the second is independent of the pair of treatments with which we start, $i = 1, 2$.

Since it is easier to check the condition 2c′, we shall use this instead of using 2c for PBIB designs with 2 associate classes. The following relations hold between

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the parameters of a PBIB design and are useful for computing some parameters when others are known.

$$vr = bk, \qquad v = n_1 + n_2 + \cdots + n_m + 1$$

$$r(k-1) = \lambda_1 n_1 + \lambda_2 n_2 + \cdots + \lambda_m n_m,$$

$$(1.1) \qquad \sum_{k=1}^{m} p_{jk}^{i} = \begin{cases} n_i - 1, & \text{for } i = j \\ n_j, & \text{for } i \neq j \end{cases} \qquad i, j = 1, 2, \cdots, m,$$

$$n_i p_{jk}^{i} = n_j p_{ik}^{j}, \qquad i, j, k = 1, 2, \cdots, m.$$

The theory of linear spaces in finite projective geometry has been used very profitably by several authors in constructing BIB and PBIB designs. Bose [1] first used the properties of quadric surfaces in finite projective geometry of two and three dimensions for constructing experimental designs. In [8] we have derived several properties of quadrics in finite projective spaces.

In the present paper we use the geometry of quadrics to construct several series of PBIB designs with two associate classes.

Let **B** denote the class consisting of the sets $B_1, B_2, \cdots, B_b$, where B_j, $j = 1, 2, \cdots, b$, is a set of points in $PG(n, s)$. Let **V** denote another class consisting of the sets $V_1, V_2, \cdots, V_v$, where $V_i, i = 1, 2, \cdots v$, is a set of points in $PG(n, s)$. These two classes generate a design $D(\mathbf{B}, \mathbf{V})$ with the following incidence matrix:

$$\underset{v \times b}{N} = \underset{v \times b}{((n_{ij}))},$$

where $n_{ij} = 1(0)$ as $V_i \cap B_j \neq \phi (= \phi)$. Then we have the following relationships:

$$r_i = \sum_{j=1}^{b} n_{ij} = \text{number of sets of the class } B \text{ which intersect } V_i,$$

$$k_j = \sum_{i=1}^{v} n_{ij} = \text{number of sets of the class } V \text{ which intersect } B_j,$$

$$\lambda_{ii'} = \sum_{j=1}^{b} n_{ij} n_{i'j} = \text{number of sets of the class } B \text{ which intersect both } V_i \text{ and } V_{i'}\ i \neq i' = 1, 2, \cdots, v.$$

Let $C_1, C_2, \cdots, C_m$ be m disjoint classes of sets in $PG(n, s)$ such that $V_i \cap V_{i'} \varepsilon C_j$, for some $j, j = 1, 2, \cdots, m$. The sets V_i and $V_{i'}$ are said to be jth associates if $V_i \cap V_{i'} \varepsilon C_j$, $j = 1, 2, \cdots, m$. Let $p_{kl}^{t}(V_i, V_{i'})$ denote the number of sets of the class V which are kth associates of V_i and lth associates of $V_{i'}$, $i \neq i'$, $i, i' = 1, 2, \cdots, v$; $k, l = 1, 2, \cdots, m$; $t, u, v = 1, 2, \cdots, m$. The following result, stated as a lemma for the sake of reference, follows easily if we interpret the sets B_i as blocks and the sets V_j as treatments, $i = 1, 2, \cdots, b$, $j = 1, 2, \cdots, v$.

LEMMA 1. *The design $D(\mathbf{B}, \mathbf{V})$ is a* PBIB *design with m associate classes if the following are true*:

(1) *Any two sets are either first associates, second associates,* $\cdots$, *or* mth *associates.*

(2) *Each set* $V_i (i = 1, 2, \cdots, v)$ *has* n_1 *first associates,* n_2 *second associates,* $\cdots$, *and* n_m mth *associates.*

(3) $p_{kl}^t(V_i, V_{i'})$ *is independent of the* ith *associate pair of sets* $(V_i, V_{i'})$ *and* $p_{kl}^t = p_{lk}^t$, $t, k, l = 1, 2, \cdots, m$.

(4) $r_i = r$, $i = 1, 2, \cdots, v$ *and* $k_j = k$, $j = 1, 2, \cdots, b$.

(5) *For any pair of* tth *associate sets* $(V_i, V_{i'})$, $\lambda_{ii'} = \lambda_t$, $t = 1, 2, \cdots, m$.

Using the condition 2c′ for PBIB designs with two associate classes we get the following result.

LEMMA 2. *The design* $D(\mathbf{B}, \mathbf{V})$ *is a* PBIB *design with two associate classes if the conditions* (1), (2), (4) *and* (5) *are satisfied for* $m = 2$, *and the following condition* (3)′ *is satisfied instead of* (3).

(3)′. *The number* $p_{11}^t(V_i, V_{i'})$ *is equal for every pair of* tth *associate sets* $(V_i, V_{i'})$, $t = 1, 2; i \neq i'; i, i' = 1, 2, \cdots, v$.

2. Preliminaries on quadrics in finite geometry. Let $\alpha_0, \alpha_1, \cdots, \alpha_{s-1}$ be the s elements of a Galois field GF(s) where s is a prime power. $PG(n, s)$, the finite projective geometry of n dimensions, consists of points and m flats, $m = 1, 2, \cdots, n - 1$. Points are also called 0 flats. The points can be taken to be $(n + 1)$-tuples $x = (x_0, x_1, \cdots, x_n)$, where each x_i is an element of GF(s), $i = 0, 1, \cdots, n$. The null $(n + 1)$-tuple $0 = (0, 0, \cdots, 0)$ is excluded from consideration. The $(n + 1)$ tuple $x = (\rho x_0, \rho x_1, \cdots, \rho x_n)$ is assumed to represent the same point as x for every non-null element ρ of GF(s). An m-flat Σ_m consists of the points x satisfying the matrix equation $x A' = 0$, where A is an $(n - m) \times (n + 1)$ matrix of rank $n - m$ with elements from GF(s).

A quadric Q in $PG(n, s)$ is the set of all points $x = (x_0, x_1, \cdots, x_n)$ which satisfies an equation of the form

$$x A x' = 0, \tag{2.1}$$

where A is an $(n + 1) \times (n + 1)$ triangular matrix with elements from the field GF(s) and x' is the transpose of the row vector x. The expression $x A x'$ is called the form of the quadric Q. A quadric Q in $PG(n, s)$ is said to be degenerate if there exists a nonsingular transformation $x = By$ which transforms the form of Q to $\sum_{i=0}^{r} \sum_{j \geq i} c_{ij} y_i y_j$, $r < n + 1$. A quadric which is not degenerate will be said to be nondegenerate. Two points $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_n)$ and $\beta = (\beta_0, \beta_1, \cdots, \beta_n)$ are said to be mutually conjugate with respect to (w.r.t.) Q if $\alpha(A + A')\beta' = 0$. The polar space $\tau(\alpha)$ of a point α is the set of all points β which are conjugate to α w.r.t. Q. The polar of α is the linear space determined by the equation

$$\alpha(A + A')x' = 0 \tag{2.2}$$

where A' is the transpose of A.

If α is a point of Q, $\tau(\alpha)$ is called the tangent space of Q at the point α. If α and β are two points of Q which are mutually conjugate, then the line $\alpha\beta$ determined by the two points α and β is contained in Q. A line contained in Q is called a generator of Q. Every nondegenerate quadric in $PG(2k, s)$ contains linear spaces of dimensionality $k - 1$ and does not contain any linear space of higher dimensionality.

Every nondegenerate quadric in $PG(2k-1, s)$ contains $(k-2)$-flats (linear spaces of dimensionality $k-2$). If it contains also $(k-1)$-flats, it is called hyperbolic. Otherwise it is elliptic. $\phi(p, k)$, $\phi_1(p, k)$ and $\phi_2(p, k)$ respectively denote the number of p-flats contained in a nondegenerate quadric in $PG(2k, s)$, an elliptic quadric in $PG(2k-1, s)$ and a hyperbolic quadric in $PG(2k-1, s)$. Formulas for these functions are obtained in Theorem 3.2 of [8]. It is proved in Theorem 4.1 of [8] that for every nondegenerate quadric in $PG(2k, s)$ there is a point S such that every line through S intersects the quadric in a single point. This point S is called the nucleus of polarity of the quadric.

Now we prove a few lemmas which will be useful later. Lemma 3 gives simple equations for hyperbolic and elliptic quadrics in $PG(2k-1, s)$. These simple equations are useful for writing down the blocks of the designs which are obtained from the incidence properties of these quadrics.

LEMMA 3. *Let* GF(s) *be a Galois field with characteristic not equal to* 2. *Let* α *be a non-zero element of* GF(s) *such that* $-\alpha$ *is a square and* β *be a non-zero element such that* $-\beta$ *is not a square. Let* λ *be an element of* GF(2^m) *such that* $\lambda(x_1^2 + x_2^2) + x_1x_2$ *is irreducible in* GF(2^m). *Then*

(1) *The quadric* Q_{2k-1} *in* $PG(2k-1, s)$, $s \neq 2^m$, *with the equation*

$$x_1^2 + \alpha x_2^2 + x_3^2 + \alpha x_4^2 + \cdots + x_{2k-1}^2 + \alpha x_{2k}^2 = 0$$

is a hyperbolic nondegenerate quadric.

(2) *The quadric* Q_{2k-1} *in* $PG(2k-1, s)$, $s \neq 2^m$, *with the equation*

$$x_1^2 + \beta x_2^2 + x_3^2 + \alpha x_4^2 + \cdots + x_{2k-1}^2 + \alpha x_{2k}^2 = 0$$

is an elliptic nondegenerate quadric.

(3) *The quadric* Q_{2k-1} *in* $PG(2k-1, 2^m)$ *with the equation*

$$x_1x_2 + x_3x_4 + \cdots + x_{2k-1}x_{2k} = 0$$

is a hyperbolic nondegenerate quadric.

(4) *The quadric* Q_{2k-1} *in* $PG(2k-1, 2^m)$ *with the equation*

$$\lambda(x_1^2 + x_2^2) + x_1x_2 + x_3x_4 + \cdots + x_{2k-1}x_{2k} = 0$$

is an elliptic nondegenerate quadric.

PROOF.

(1) It is obvious that Q_{2k-1} is nondegenerate. We shall show that Q_{2k-1} is hyperbolic by finite induction on k. First we prove the result for $k = 1$. Since $-\alpha$ is a square element of GF(s), there exists an element λ of GF(s) such that $-\alpha = \lambda^2$. The equation of Q_1 is $x_1^2 + \alpha x_2^2 = 0$. It can be easily seen that Q_1 contains the two points $(\lambda, 1)$ and $(-\lambda, 1)$. Since Q_1 contains linear space of dimen-

sionality $O(=k-1)$, i.e., points, Q_1 is hyperbolic. Assume that Q_{2k-3} is hyperbolic. Consider the nonsingular transformation

$$\begin{aligned} y_i &= x_i, & i = 1, 2, \cdots, 2k-2, \\ y_{2k-1} &= x_{2k-1} + \lambda x_{2k}, \\ y_{2k} &= x_{2k-1} - \lambda x_{2k}. \end{aligned}$$

It is easy to see that under this transformation Q_{2k-1} transforms to Q'_{2k-1} with the equation

$$y_1^2 + \alpha y_2^2 + y_3^2 + \alpha y_4^2 + \cdots + y_{2k-3}^2 + \alpha y_{2k-2}^2 + y_{2k-1}y_{2k} = 0.$$

Since the incidence properties in a projective geometry remain invariant over nonsingular transformations, it is sufficient to show that Q'_{2k-1} is hyperbolic. Consider the point $P = (0\,0\,\cdots\,0\,1\,0)$ of Q'_{2k-1}. The equation of $\tau(P)$ is obviously $y_{2k} = 0$. Let π be the $(n-1)$-flat with the equation $y_{2k-1} = 0$. Then $Q'_{2k-1} \cap \tau \cap \pi$ has the equation

$$y_1^2 + \alpha y_2^2 + y_3^2 + \alpha y_4^2 + \cdots + y_{2k-3}^2 + \alpha y_{2k-2}^2 = 0.$$

By assumption $Q'_{2k-1} \cap \tau \cap \pi$ is hyperbolic and hence contains a $(k-2)$-flat Σ_{k-2}. The point P and the $(k-2)$-flat Σ_{k-2} are both contained in the quadric Q'_{2k-1} and are mutually conjugate. Hence by Lemma 2.3 of [8] the $(k-1)$-flat determined by P and Σ_{k-2} is contained in Q'_{2k-1}. Hence Q'_{2k-1} is hyperbolic.

(2) We shall prove the result by induction of k. First we prove the result for $k = 1$. The quadric Q_1 in $PG(1, s)$ with the equation $x_1^2 + \beta x_2^2 = 0$ will be elliptic if Q_1 does not contain any point. If possible, suppose Q_1 contains a point (x'_1, x'_2). Then $x'_2 \neq 0$. We can easily get $x_1'^2/x_2'^2 = -\beta$, which contradicts the assumption that $-\beta$ is a non-square element. Hence Q_1 is elliptic.

Assume that the result is true for $k-1$ so that Q_{2k-3} is a nondegenerate elliptic quadric. Applying the nonsingular transformation used in part 1, we can transform Q_{2k-1} to Q'_{2k-1} with the equation

$$y_1^2 + \beta y_2^2 + y_3^2 + \alpha y_4^2 + \cdots + y_{2k-3}^2 + \alpha y_{2k-2}^2 + y_{2k-1}y_{2k} = 0.$$

As before it will be sufficient to show that Q'_{2k-1} is elliptic. If possible, suppose Q'_{2k-1} is hyperbolic. Then Q_{2k-1} contains $(k-1)$-flats. Consider the point $P = (0\,0\,\cdots\,0\,1\,0)$. The equation of $\tau(P)$ is $y_{2k} = 0$. Let π be the $(n-1)$-flat with the equation $y_{2k-1} = 0$. Then the equation of $Q'_{2k-1} \cap \tau \cap \pi$ is

$$y_1^2 + \beta y_2^2 + y_3^2 + \alpha y_4^2 + \cdots + y_{2k-3}^2 + \alpha y_{2k-2}^2 = 0,$$

which is elliptic by induction assumption. By Theorem 3.2 of [8] the number of p-flats passing through a point P of a nondegenerate quadric and contained in the quadric is equal for every point P of the quadric. Since Q'_{2k-1} contains $(k-1)$-flats, it follows that there exists a $(k-1)$-flat Σ_{k-1} contained in Q'_{2k-1} and passing through P. By Theorem 2.2 of [8] Σ_{k-1} is contained in $Q'_{2k-1} \cap \tau(P)$. Hence Σ_{k-1} intersects $Q'_{2k-1} \cap \tau(P) \cap \pi$ in a $(k-2)$-flat Σ_{k-2}. So $Q'_{2k-1} \cap \tau(P) \cap \pi$ contains a $(k-2)$-flat Σ_{k-2}. This contradicts the assumption that $Q'_{2k-1} \cap \tau(P) \cap \pi$ is elliptic. This completes the proof of part (2). Parts (3) and (4) of the theorem can be proved by arguments exactly similar to arguments used in parts (1) and (2) respectively.

3. PBIB designs from the configuration of generators for blocks and points for treatments of the quadric.

DEFINITION. *Generator.* A line which is contained in the quadric is called a generator of the quadric.

LEMMA 4. *Let P_1 and P_2 be two points of a nondegenerate quadric Q_n in $PG(n, s)$ such that the line P_1P_2 is not a generator. The number of points P such that both the lines PP_1 and PP_2 are generators of the quadric is $N(0, n-2)$ where $N(p, n)$ denotes the number of p-flats contained in a non-degenerate quadric of the type of Q_n (elliptic or hyperbolic) in $PG(n, s)$.*

PROOF. Since the line P_1P_2 is not a generator, by Lemma 2.3 of [8] the points P_1 and P_2 are not mutually conjugate. Let τ_1 and τ_2 denote the tangent spaces at P_1 and P_2 respectively. Let P be a point of Q_n such that both PP_1 and PP_2 are generators of Q_n. Since PP_1 is a generator, by Lemma 2.3 of [8] P must be conjugate to P_1. Hence P must be a point of τ_1. Similarly, P must be a point of τ_2. Hence P is a point of $Q_n \cap \tau_1 \cap \tau_2$. Conversely if P is a point of $Q_n \cap \tau_1 \cap \tau_2$, P is a conjugate to both P_1 and P_2 and hence both PP_1 and PP_2 are generators of Q_n. It follows from the above argument that the required number is equal to the number of points in $Q_n \cap \tau_1 \cap \tau_2$.

Since P_1 and P_2 are mutually not conjugate τ_2 does not pass through τ_1. So τ_1 is a tangent space at P_1 and τ_2 is an $(n-1)$-flat not passing through P_1. By Theorem 2.1 of [8] $Q_n \cap \tau_1 \cap \tau_2$ is a nondegenerate quadric Q_{n-2} in $PG(n-2, s)$ which is elliptic or hyperbolic according Q_n is elliptic or hyperbolic. Hence the lemma follows.

LEMMA 5. *Let P_1 and P_2 be two points of Q_n in $PG(n, s)$ such that P_1P_2 is a generator of Q_n. Then the number of points P other than P_1 and P_2 such that both PP_1 and PP_2 are generators of Q_n is $(s-1) + s^2N(0, n-4)$.*

PROOF. Let τ_1 and τ_2 be the tangent spaces at P_1 and P_2 respectively. Let Σ_{n-2} be an $(n-2)$-flat not intersecting the line Σ_1 determined by P_1 and P_2. Let P be a point of Q_n other than P_1 and P_2. As in the proof of Lemma 4, we can show that both PP_1 and PP_2 are generators of Q_n if and only if P is a point of $Q_n \cap \tau_1 \cap \tau_2$. Hence the required number of points is equal to the number of points of $Q_n \cap \tau_1 \cap \tau_2$ other than P_1 and P_2. By Lemma 2.2 of [8], $\tau_1 \cap \tau_2 = \tau(\Sigma_1)$, the polar of Σ_1. By Theorem 2.1 of [8], $Q_n \cap \tau(\Sigma_1) \cap \Sigma_{n-2}$ is Q_{n-4}, a nondegenerate quadric in $PG(n-4, s)$ and $Q_n \cap \tau(\Sigma_1)$ is a cone with Σ_1 as the vertex and $Q_n \cap \tau(\Sigma_1) \cap \Sigma_{n-2}$ as the base.

To count the required number of points, we notice that for every point P of $Q_n \cap \tau(\Sigma_1) \cap \Sigma_{n-2}$ the plane determined by P and Σ_1 is contained in $Q_n \cap \tau(\Sigma_1)$. The number of points of such a plane which do not lie on Σ_1 is s^2. Hence it follows easily that the required number of points is $(s-1) + s^2N(0, n-4)$.

THEOREM 1. *Let* **B** *be the class of generators of* Q_n, *a nondegenerate quadric in* $PG(n, s)$ *and* **V** *be the class of points of* Q_n, *each point being regarded as a pointset. The* $D(\mathbf{B}, \mathbf{V})$ *is a* PBIB *design with two associate classes with the following parameters*: $v = N(0, n)$, $b = N(1, n)$, $k = s + 1$, $r = N(0, n - 2)$, $\lambda_1 = 1$, $\lambda_2 = 0$, $n_1 = sN(0, n - 2)$, $p_{11}^1 = (s - 1) + s^2N(0, n - 4)$ *and* $p_{11}^2 = N(0, n - 2)$.

PROOF. It will be sufficient to check the conditions of Lemma 2. It follows easily that b = number of generators of $Q_n = N(1, n)$, k = number of points on a generator $= s + 1$, v = number of points of $Q_n = N(0, n)$ and r = number of generators passing through a point $= N(0, n - 2)$ (Theorem 3.3 of [8]). The association scheme is defined as follows. Two points P_1 and P_2 of Q_n are first associates of each other if the line P_1P_2 is a generator and are second associates of each other if the line P_1P_2 is not a generator. Since there can be at most one generator passing through two points of Q_n, we have $\lambda_1 = 1$ and $\lambda_2 = 0$. Let P_1 be a particular point of Q_n. The number of points P which are first associates to P_1 is equal to the number of points P such that PP_1 is a generator and hence equal to the number of points lying on the generators passing through P_1. From the above argument, we have $n_1 = sN(0, n - 2)$. Let P_1 and P_2 be two first associate points. Then P_1P_2 is a generator of Q_n. The number of points P which are first associates of both P_1 and P_2 is equal to the number of points P other than P_1 and P_2 such that both PP_1 and PP_2 are generators. By Lemma 5 this number is equal to $(s - 1) + s^2N(0, n - 4)$. From the above argument it follows that $p_{11}^1 = (s - 1) + s^2N(0, n - 4)$. Let P_1 and P_2 be two second associate points. Then the line P_1P_2 is not a generator. The number of points P which are first associates to both P_1 and P_2 is equal to the number of points P which are such that both PP_1 and PP_2 are generators. By Lemma 4 this number does not depend on the particular pair of second associate points and is equal to $N(0, n - 2)$. From the above argument it follows that $p_{11}^2 = N(0, n - 2)$. This completes the proof of the theorem.

Specializing the quadric Q_n of Theorem 1, we shall get a number of series of PBIB designs. Taking $n = 2t$, we get the following series:

$$v = \frac{s^{2t} - 1}{s - 1} \qquad r = \frac{s^{2t-2} - 1}{s - 1}, \qquad k = s + 1,$$

$$b = \frac{(s^{2t} - 1)(s^{2t-2} - 1)}{(s - 1)^2(s + 1)}, \qquad \lambda_1 = 1, \qquad \lambda_2 = 0,$$

$$n_1 = \frac{s(s^{2t-2} - 1)}{(s - 1)}, \qquad p_{11}^1 = (s - 1) + \frac{s^2(s^{2t-4} - 1)}{(s - 1)}, \quad \text{and} \quad p_{11}^2 = \frac{s^{2t-2} - 1}{s - 1}.$$

Putting $t = 2$, we get the symmetric series with

$$v = b = s^3 + s^2 + s + 1, \qquad r = k = s + 1, \qquad \lambda_1 = 1, \qquad \lambda_2 = 0$$

$$n_1 = s^2 + s, \qquad p_{11}^1 = s - 1 \quad \text{and} \quad p_{11}^2 = s + 1.$$

This series was obtained by Clatworthy [6] by a different method. Taking $n = 2t - 1$, $t \geqq 3$, and Q_n elliptic, we get the series with

$$v = \frac{s^{2t-1} - s^t + s^{t-1} - 1}{s - 1}, \qquad r = \frac{s^{2t-3} - s^{t-1} + s^{t-2} - 1}{s - 1}$$

$$k = s + 1, \qquad b = \frac{(s^{2t-1} - s^t + s^{t-1} - 1)(s^{2t-3} - s^{t-1} + s^{t-2} - 1)}{(s - 1)^2(s + 1)}$$

$$\lambda_1 = 1, \qquad \lambda_2 = 0, \qquad n_1 = \frac{s(s^{2t-3} - s^{t-1} + s^{t-2} - 1)}{s - 1},$$

$$p_{11}^1 = (s - 1) + \frac{s^2(s^{2t-5} - s^{t-2} + s^{t-3} - 1)}{s - 1}, \qquad \text{and}$$

$$p_{11}^2 = \frac{s^{2t-3} - s^{t-1} + s^{t-1} - 1}{s - 1}.$$

Taking $n = 2t - 1$, $t \geqq 2$ and Q_n hyperbolic, we get the series with

$$v = \frac{s^{2t-1} + s^t - s^{t-1} - 1}{s - 1}, \qquad r = \frac{s^{2t-3} + s^{t-1} - s^{t-2} - 1}{s - 1},$$

$$k = s + 1, \qquad b = \frac{(s^{2t-1} + s^t - s^{t-1} - 1)(s^{2t-3} + s^{t-1} - s^{t-2} - 1)}{(s - 1)^2(s + 1)},$$

$$\lambda_1 = 1, \qquad \lambda_2 = 0, \qquad n_1 = \frac{s(s^{2t-3} + s^{t-1} - s^{t-2} - 1)}{s - 1},$$

$$p_{11}^1 = (s - 1) + \frac{s^2(s^{2t-5} + s^{t-2} - s^{t-3} - 1)}{s - 1}, \qquad \text{and}$$

$$p_{11}^2 = \frac{s^{2t-3} + s^{t-1} - s^{t-2} - 1}{s - 1}.$$

Putting $t = 2$, in the above series, we get the family of simple lattice designs.

4. PBIB designs from the configuration of points of a quadric for blocks and generators of a quadric for treatments. In this section we consider a nondegenerate quadric Q_n in $PG(n, s)$ which contains lines but does not contain planes. Therefore Q_n can be a nondegenerate quadric in $PG(4, s)$ or an elliptic quadric in $PG(5, s)$ or a hyperbolic nondegenerate quadric in $PG(3, s)$.

LEMMA 6. *Let l_1 and l_2 be two intersecting generators of Q_n. Then the number of generators other than l_1 and l_2 which intersect both l_1 and l_2 is $N(0, n - 2) - 2$.*

PROOF. Suppose the two generators in l_1 and l_2 intersect at the point P. Then there cannot be any generator which intersects both l_1 and l_2 at points other than P. If possible, suppose there exists a generator which intersects l_1 at P_1 and l_2 at P_2. Then the three points P, P_1 and P_2 are mutually conjugate and are points of Q_n. By Lemma 2.3 of [8], the plane determined by them is contained in Q_n. This contradicts the assumption that Q_n does not contain any planes. So it follows that the required generators are those which intersect both l_1 and l_2 at the point P. By Theorem 3.3 of [8] the number of generators passing through P is $N(0, n-2)$ of which l_1 and l_2 are two generators. Hence the lemma follows.

LEMMA 7. *Let l_1 and l_2 be two nonintersecting generators of Q_n. Then the number of generators which intersect both l_1 and l_2 is $(s+1)$.*

PROOF. Let P be a point of the generator l_2. Consider the generators which intersect l_2 at P and also intersect l_1. Any such generator will lie in the plane Σ_2 determined by l_1 and the point P. The plane Σ_2 is not contained in Q_n and contains a generator l_1 of Q_n and a point P of Q_n not lying on l_1. From these facts it follows easily that Σ_2 intersects Q_n in a pair of lines. Hence there exists one and only one line passing through P and intersecting l_1. This is true for every point of l_2 and the number of points of l_2 is $(s+1)$. Hence the lemma follows.

THEOREM 2. *Let **B** be the class of points of a nondegenerate quadric Q_n which does not contain planes and **V** be the class of generators of Q_n. Then $D(\mathbf{B}, \mathbf{V})$ is a PBIB design with two associate classes with the following parameters:*

$$v = N(1, n), \quad r = (s+1), \quad k = N(0, n-2), \quad b = N(0, n), \quad \lambda_1 = 1, \lambda_2 = 0,$$

$$n_1 = (N(0, n-2)-1)\,(s+1), \quad p_{11}^1 = N(0, n-2)-2, \quad \textit{and} \quad p_{11}^2 = (s+1).$$

PROOF. It is sufficient to check the conditions of Lemma 2. It follows easily that v = number of generators of $Q_n = N(1, n)$, r = number of points of Q_n intersecting a generator $= s+1$, k = number of generators of Q_n intersecting a point $= N(0, n-2)$ and b = number of points of $Q_n = N(0, n)$. The association scheme is defined as follows. Two generators l_1 and l_2 of Q_n are first associates if they intersect and are second associates if they do not intersect. Since two generators can intersect in at most one point, we have $\lambda_1 = 1$ and $\lambda_2 = 0$. Let l_1 be a given generator. The number of generators which are first associates of l_1 is equal to the number of generators which intersect l_1. Through every point of l_1 there passes $N(0, n-2)$ generators of which l_1 is one. Hence it follows that $n_1 = (N(0, n-2)-1)\,(s+1)$. Consider two generators l_1 and l_2 which are first associates. By definition l_1 and l_2 intersect each other. The number of generators which are first associates of both l_1 and l_2 is equal to the number of generators which intersect both l_1 and l_2. By Lemma 6, this number is independent of the particular pair of intersecting generators l_1 and l_2 and is equal to $N(0, n-2)-2$. Hence $p_{11}^1 = N(0, n-2) - 2$. Similarly using Lemma 7, we can show that $p_{11}^2 = s+1$. The completes the proof of the theorem.

Specializing Q_n in Theorem 2, we shall get a number of series of PBIB designs with two associate classes. Taking $n = 5$ and Q_5 an elliptic quadric, we get the series with

$$v = (s^3+1)\,(s^2+1), \quad r = s+1, \quad k = s^2+1, \quad b = (s^3+1)\,(s+1), \quad \lambda_1 = 1, \quad \lambda_2 = 0,$$

$$n_1 = s^2(s+1), \quad p_{11}^1 = s^2 - 1, \quad \text{and} \quad p_{11}^2 = s+1.$$

Taking $n = 3$ and Q_3 a hyperbolic nondegenerate quadric in $PG(3, s)$, we get the series with

$$v = 2(s+1), \quad r = (s+1), \quad k = 2, \quad b = (s+1)^2, \quad \lambda_1 = 1, \quad \lambda_2 = 0, \quad n_1 = s+1,$$
$$p_{11}^1 = 0 \quad \text{and} \quad p_{11}^2 = s+1.$$

This series is also given by Clatworthy [7]. Taking $n = 4$ and Q_4 a nondegenerate quadric we get the series with

$$v = b = [(s^4-1)/(s-1)], \quad r = k = s+1, \quad \lambda_1 = 1, \quad \lambda_2 = 0, \quad n_1 = s(s+1),$$
$$p_{11}^1 = s-1, \quad \text{and} \quad p_{11}^2 = s+1.$$

5. PBIB designs from the configuration of generators on generators.

THEOREM 3. *Let* **B** *be the class of generators of a nondegenerate quadric* Q_n *which contains lines but does not contain planes. If a generator is considered as non-intersecting with itself,* $D(\mathbf{B}, \mathbf{V})$ *is a* PBIB *design with two associate classes with the following parameters*:

$$v = b = N(1, n), \quad r = k = (N(0, n-2) - 1)(s+1), \quad \lambda_1 = N(0, n-2) - 2, \quad \lambda_2 = s+1,$$
$$n_1 = (N(0, n-2) - 1)(s+1), \quad p_{11}^1 = N(0, n-2) - 2, \quad \textit{and} \quad p_{11}^2 = s+1.$$

PROOF. Two generators are defined to be first associates if they intersect each other. If two generators do not intersect, they are defined to be second associates. With the association scheme so defined, the proof of the theorem follows easily from Lemma 6 and 7. Taking Q_n, a nondegenerate quadric in $PG(4, s)$, we get the following:

$$v = b = s^3 + s^2 + s + 1, \quad r = k = s(s+1), \quad \lambda_1 = s-1, \quad \lambda_2 = s+1,$$
$$n_1 = s(s+1), \quad p_{11}^1 = s-1, \quad \text{and} \quad p_{11}^2 = s+1.$$

Taking $n = 5$ and Q_5 an elliptic nondegenerate quadric in $PG(5, s)$, we get the series with

$$v = b = (s^3+1)(s^2+1) \; r = k = s^2(s+1), \quad \lambda_1 = s^2 - 1, \quad \lambda_2 = s+1$$
$$n_1 = s^2(s+1), \quad p_{11}^1 = s^2 - 1, \quad \text{and} \quad p_{11}^2 = s+1.$$

Taking $n = 3$ and Q_3 a hyperbolic nongenerate quadric, we get the series with

$$v = b = 2(s+1) \; r = k = s+1, \quad \lambda_1 = 0, \quad \lambda_2 = s+1, \quad n_1 = s+1$$
$$p_{11}^1 = 0, \quad \text{and} \quad p_{11}^2 = s+1.$$

Now we shall give an example to illustrate the actual method of constructing the blocks of the design. Consider a design of the series given in Theorem 1. We take $n = 4$ and Q_4 a non-degenerate quadric and $s = 2$. The parameters are $v = b = 15$, $r = k = 3$, $\lambda_1 = 1$, $\lambda_2 = 0$, $n_1 = 6$, $p_{11}^1 = 1$ and $p_{11}^2 = 3$. The design is obtained by taking generators of Q_4 as blocks and points of Q_4 as treatments. The equation of Q_4 can be taken as

$$x_0^2 + x_1x_2 + x_3x_4 = 0.$$

The 15 points of Q_4 are

$$\begin{aligned} P_1 &= (00001), & P_9 &= (11101),\\ P_2 &= (00010), & P_{10} &= (00110),\\ P_3 &= (10011), & P_{11} &= (01010),\\ P_4 &= (00100), & P_{12} &= (11110),\\ P_5 &= (01000), & P_{13} &= (10111),\\ P_6 &= (11100), & P_{14} &= (11011),\\ P_7 &= (00101), & P_{15} &= (01111).\\ P_8 &= (01001), \end{aligned}$$

To write down the blocks systematically we can proceed as follows. Consider treatment 1. The blocks which contain treatment 1 correspond to the generators containing the point P_1. To find out the generators passing through P_1, we find out $Q_4 \cap \tau(P_1)$ where $\tau(P_1)$ is the tangent space at P_1. For any point P of $Q_4 \cap \tau(P_1)$, PP_1 is a generator. In this way we can exhaust all the blocks containing treatment 1. Next we proceed to treatment 2 and by a similar procedure can find out the blocks containing treatment 2 which are not already included. We continue in this manner until all the blocks are obtained.

In our example the 15 generators are P_1P_4, P_1P_5, P_1P_6, P_2P_4, P_2P_5, P_2P_6, P_3P_4, P_3P_5, P_3P_6, P_7P_{11}, P_7P_{12}, P_8P_{10}, P_8P_{12}, P_9P_{10}, and P_9P_{11}. So the blocks of the designs are

$$\begin{array}{lll} (1, 4, 7), & (1, 5, 8), & (1, 6, 9),\\ (2, 4, 10), & (2, 5, 11), & (2, 6, 12),\\ (3, 4, 13), & (3, 5, 14), & (3, 6, 15),\\ (7, 11, 15), & (7, 12, 14), & (8, 10, 15),\\ (8, 12, 13), & (9, 10, 14), & (9, 11, 13). \end{array}$$

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