DISTRIBUTION-FREE TOLERANCE INTERVALS FOR CONTINUOUS SYMMETRICAL POPULATIONS

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1. Summary. For continuous symmetrical distributions and specified tolerance interval characteristics, distribution-free tolerance intervals are presented that require substantially smaller sample sizes than those required for arbitrary continuous populations. In the case of one-sided tolerance intervals, about twice as many sample values are required for the general situation as for the symmetrical situation. For two-sided intervals, the additional sample values required for the general situation vary from about 35 percent to about 85 percent in the cases considered, depending on the specified tolerance interval characteristics. These intervals furnish at least rough protection against all possible violations of the symmetry assumption. In addition, some specialized one-sided and two-sided tolerance intervals are developed for life-test situations involving continuous symmetrical populations. Results for continuous symmetrical populations with known central median values are also presented; the life-testing versions of these intervals are especially useful.

2. Introduction. Although tolerance intervals that are distribution-free for any continuous population have validity advantages, they require large sample sizes. These large sample sizes can often be reduced if additional knowledge is available about the continuous population sampled. This paper discusses some of the things that can be done when the population is known to be symmetrical.

Let $x(1) \leqq \cdots \leqq x(n)$ be the order statistics of a random sample of size n. The cumulative distribution function (cdf) of the population sampled is denoted by F; it is assumed to be continuous and, for some number φ, to satisfy $F(x) = 1 - F(2\varphi - x)$ for all x. That is, the population sampled is continuous and symmetrical. Clearly φ is the population mean when the mean exists; φ is always the central median of the population (center of the interval of medians when the median is not unique).

Consider two functions, L_1 and L_2, of the sample values, one of which might be specified as infinite, such that the interval $[L_1, L_2]$ has a probability β or more of containing at least 100γ percent of the population. That is, for β and γ constants,

$$P\{F(L_2) - F(L_1) \geqq \gamma\} \geqq \beta, \tag{1}$$

for all F in some specified class of cdfs. An interval $[L_1, L_2]$ with these characteristics is called a tolerance interval, and the functions L_1, L_2 are called tolerance limits. The expressions for $[L_1, L_2]$ developed in this paper are such that (1) holds for every continuous symmetrical cdf F. If $L_1 = -\infty$, or if $L_2 = \infty$, the

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tolerance interval is said to be one-sided. If neither L_1 nor L_2 is required to be a constant, the tolerance interval is said to be two-sided. Ordinarily, the values used for β and γ are near unity.

Two cases are considered. In one case, all the values of the sample are available. Here substitutes are presented for the Wilks one-sided and two-sided tolerance intervals (see, e.g., [3]). Since the Wilks tolerance intervals are valid for arbitrary continuous populations (including the class of symmetrical populations), they are unable to exploit the additional information that the population is symmetrical. The other case is of a life-testing nature where only $x(1), \cdots, x(r), r < n$, are used. This is the situation, for example, when only the r smallest order statistics of a sample of size n are observed.

The Wilks one-sided and two-sided tolerance intervals considered are those based on one or both of $x(1)$ and $x(n)$. The substitutes of this paper could involve up to four different order statistics but, for specified β and γ, require the smallest sample sizes when only $x(1)$ and $x(n)$ are used. If the symmetry assumption could be violated in every possible manner, the required sample sizes for the substitute intervals are at most equal to those for the corresponding Wilks intervals (see Section 5). If, however, the population sampled is known to be symmetrical, substantially fewer sample values are required for the substitute intervals. Specifically, for one-sided intervals and β, γ values in the ranges of principal interest, only about half as many sample values are required when a substitute interval based on $x(1)$ and $x(n)$ is used. For two-sided intervals, substantial reductions in required sample sizes can also be obtained by using substitute intervals (see Table 1).

The substitute intervals developed furnish some protection against an erroneous assumption of symmetry. This protection occurs because a substitute interval $[L_1, L_2]$ always contains the corresponding Wilks interval. Thus the β, γ properties for a substitute interval having a given sample size are always at least as desirable as those for the corresponding Wilks interval based on this sample size. Of course, for given γ, the value of β is reduced compared to that

TABLE 1

Minimum Sample Size Comparison of Wilks and Substitute Two-Sided Tolerance Intervals Using Only $x(1)$ and $x(n)$

γ	β					
	.9		.95		.99	
	Wilks	Sub.	Wilks	Sub.	Wilks	Sub.
.8	18	11	22	14	31	21
.9	38	22	46	29	64	44
.95	77	45	93	59	130	90
.99	388	230	473	300	662	460
.999	3889	2300	4742	3000	6636	4600

for the symmetry case; likewise, for given β, the value of γ is reduced. Lower limits for the protection furnished against any possible violation of symmetry, including some evaluations of how β is reduced for fixed γ, and how γ is reduced for fixed β, are given in Section 5.

One-sided intervals for life-testing situations are given for the symmetrical case. These tolerance intervals are based on $x(1)$ and $x(r)$; they have the advantage that the experiment can be discontinued as soon as the r smallest order statistics are determined (the first r items fail). Here r cannot be too small and is expressed as a function of n, β, and another quantity whose value can be selected by the experimenter for cases where r is at his disposal. Although the actual data may not be from a symmetrical population, transformations (based on past experience, technical considerations, etc.) are sometimes available for changing the data so that the symmetry assumption is acceptable. In some cases, such as for the "wear out" failures considered in [2], it may be permissible to assume that the actual data are from a symmetrical population.

In some cases the value of φ is known (by hypothesis, from past experience, etc.). The required sample sizes for the substitute intervals and the corresponding tolerance intervals based on φ (known) are about the same. However, use of φ does yield intervals which have more precisely determined properties, in the sense that the inequality $\geqq \beta$ in (1) is an approximate equality. Also, satisfactory one-sided and two-sided intervals can be obtained for life-testing situations when $r = 1$ (i.e., only one item is failed). One possible use of results with φ known is for tests with null hypothesis $F = F_0$, where F_0 is completely specified, continuous, and symmetrical. When F_0 is not symmetrical, the observations can be transformed so that they are from a symmetrical cdf under the null hypothesis. The intervals corresponding to the substitute intervals furnish the same general level of protection against violation of the symmetry assumption as do the substitute intervals.

The basic idea underlying this paper is the following: For a continuous symmetrical population, the $|x(i) - \varphi|$, considered as an unordered set, form a minimal sufficient statistic. The $|x(i) - \varphi|$ can be considered as observations on the cdf obtained by "folding over" $F(x)$ at φ. The $|x(i) - \varphi|$ then provide tolerance intervals in the usual way. If φ is not known, it can be bounded, on one side or the other (in the confidence sense) with high probability. The resulting bounds, used in conjunction with the results for φ known, yield tolerance intervals for the situation where φ is unknown.

Section 3 is titled Results; it contains a statement of the tolerance intervals presented and their properties. Section 4, Derivations, gives detailed verifications. The final section, Bounds for Symmetry Violations, is concerned with the protection furnished against violations of the symmetry assumption by the substitute intervals and by the corresponding intervals with φ known.

3. Results. Two kinds of one-sided tolerance intervals are presented for the case where only $x(1)$, $x(n)$ are used and φ is unknown. Let

$$X_u = \max[x(n), 2x(u) - x(1)], \qquad Y_v = \min[x(1), 2x(v) - x(n)].$$

The one-sided intervals considered, which are applicable for $\gamma > \frac{1}{2}$, are $[-\infty, X_n]$ and $[Y_1, \infty]$. In both cases, an upper bound for the minimum n such that (1) holds is the smallest integer n_1 such that

$$1 - (2\gamma - 1)^{n_1} - (\tfrac{1}{2})^{n_1} \geqq \beta.$$

The value for n_1 satisfies

$$\log(1 - \beta)/\log(2\gamma - 1) \leqq n_1 \leqq 1 + \log[1 - \beta - (\tfrac{1}{2})^{\log(1-\beta)/\log(2\gamma-1)}]/\log(2\gamma - 1)$$

and approximately equals $\log(1 - \beta)/\log(2\gamma - 1)$ when β and γ are not too small (say, $\beta \geqq .85$ and $\gamma \geqq .9$).

The two-sided interval considered when only $x(1)$, $x(n)$ are used and φ is unknown is $[Y_1, X_n]$. An upper bound for the minimum n such that (1) holds is the smallest integer n_2 such that

$$1 - \gamma^{n_2} - (\tfrac{1}{2})^{n_2 - 1} \geqq \beta.$$

The value for n_2 satisfies

$$\log(1 - \beta)/\log\gamma \leqq n_2 \leqq 1 + \log[1 - \beta - 2(\tfrac{1}{2})^{\log(1-\beta)/\log\gamma}]/\log\gamma$$

and approximately equals $\log(1 - \beta)/\log\gamma$ when β and γ are not too small (say, $\beta \geqq .8$ and $\gamma \geqq .85$).

Other types of one-sided and two-sided substitute intervals can be obtained from the theorems of Section 4. These other intervals, however, require larger minimum sample sizes and also use $x(1)$ and $x(n)$.

For life-testing situations, the experiment can be terminated when $x(1)$, $\cdots$, $x(r)$ are determined. Here r depends on n, β, and p, where p satisfies

$$(\tfrac{1}{2})^n/(1 - \beta) \leqq p < 1. \tag{2}$$

When p is given, r is the smallest integer $R (1 \leqq R \leqq n)$ such that

$$P\left\{F[x(R)] \geqq \frac{1}{2}\right\} = \left(\frac{1}{2}\right)^n \sum_{s=0}^{R-1} \binom{n}{s} \geqq 1 - (1 - \beta)p.$$

When r is given, β must be such that (2) holds for p. For n sufficiently large,

$$r \doteq n/2 + \tfrac{1}{2}n^{\frac{1}{2}} K_{(1-\beta)p}, \tag{3}$$

where $K_{(1-\beta)p}$ is the deviate that is exceeded with probability $(1 - \beta)p$ for the standardized normal distribution.

The one-sided interval considered for φ unknown is $[-\infty, 2x(r) - x(1)]$ in the life-testing case. An upper bound for the minimum n such that (1) holds is the smallest integer at least equal to $\log[(1 - p)(1 - \beta)]/\log\gamma$. The two-sided interval considered is $[x(1), 2x(r) - x(1)]$. An upper bound for the minimum n such that (1) holds is the smallest integer at least equal to $\log[(1 - p)(1 - \beta)]/\log(\frac{1}{2} + \gamma/2)$. For both types of intervals, choice of a suitable value for p depends on many considerations; when the time and cost situation is known, however, selection of an optimum value for p should often be possible.

For one-sided Wilks intervals, the minimum sample size for (1) to hold is the smallest integer, denoted by N_1, at least equal to $\log(1 - \beta)/\log\gamma$. Thus, for β and γ not too small,

$$N_1/n_1 \doteq \log(2\gamma - 1)/\log\gamma \doteq 2, \qquad N_1/n_2 \doteq \log\gamma/\log\gamma = 1.$$

For two-sided Wilks intervals, the minimum sample size required for (1) to hold is the smallest integer N_2 such that

$$N_2 \geqq \log(1 - \beta)/\log\gamma \ + \log[1 + (1/\gamma - 1)N_2]/\log(1/\gamma).$$

Thus, for β and γ not too small,

$$N_2 \doteq n_2 + \log[1 + (1/\gamma - 1)N_2]/\log(1/\gamma).$$

The values listed in Table 1 were furnished by Professor Z. W. Birnbaum and were determined by use of the graph in [1].

For φ known and $x(1)$, $x(n)$ available, the one-sided intervals considered (applicable for $\gamma > \frac{1}{2}$) are $[-\infty, X]$ and $[Y, \infty]$, where

$$X = \max[x(n), 2\varphi - x(1)], \qquad Y = \min[x(1), 2\varphi - x(n)].$$

In both cases, an upper bound for the minimum n such that (1) holds is the smallest integer at least equal to $\log(1 - \beta)/\log(2\gamma - 1)$. Thus, for β and γ not too small, the minimum required sample size approximately equals n_1. The two-sided interval considered is $[Y, X]$ and the minimum n such that (1) holds is the smallest integer at least equal to $\log(1 - \beta)/\log\gamma$. Thus, for β and γ not too small, the minimum required n approximately equals n_2.

For life-testing situations where φ is known and the population is symmetrical, only $x(1)$ is needed for obtaining one-sided and two-sided intervals. That is, the experiment can be terminated when the first item fails ($r = 1$). For given n, as indicated by (3), this can result in substantial saving of time or cost, or both, compared to the situation for φ unknown; moreover, the minimum n required for (1) to hold is less (sometimes much less). The one-sided intervals available through the use of $x(1)$ are $[-\infty, 2\varphi - x(1)]$ and $[x(1), \infty]$, with minimum required sample size N_1 (the second is a Wilks interval). The minimum required sample size for the corresponding results with φ unknown exceeds N_1 by approximately $\log(1 - p)/\log\gamma$ for β and γ not too small. The two-sided interval is $[x(1), 2\varphi - x(1)]$ and the minimum n such that (1) holds is the smallest integer at least equal to $\log(1 - \beta)/\log(\frac{1}{2} + \gamma/2)$. The excess number of sample values required for the corresponding interval with φ unknown is approximately $\log(1 - p)/\log(\frac{1}{2} + \gamma/2)$ for β and γ not too small. Thus, in all cases, the excess number of sample values required can be appreciable for γ near unity and p not near zero.

4. Derivations. Throughout, the data are a random sample of size n from an arbitrary continuous population that is symmetrical about φ. The following lemma is useful in performing the verifications.

LEMMA. *If δ is any relation on the sample space of the* $x(1), \cdots, x(n)$, *then* $P\{\delta[x(1), \cdots, x(n)]\}$ *equals* $P\{\delta[2\varphi - x(n), \cdots, 2\varphi - x(1)]\}$. This lemma follows immediately from the continuity and symmetry about φ.

The first theorem furnishes the basis for the tolerance intervals with φ known and $x(1), x(n)$ available.

THEOREM 1. $P[F(X) \geqq \gamma] = P\{F[x(n)] \geqq 2\gamma - 1\} = 1 - (2\gamma - 1)^n$, *for* $\gamma > \frac{1}{2}$.

PROOF. $P[F(x) \geqq \gamma] = P\{F[x(n)] \geqq \gamma, F[x(1)] \leqq \gamma\}$ by the lemma. However, by the theorem for statistically equivalent blocks, the result follows. That is, the probability that n independent uniform observations are observed outside the interval $(1 - \gamma, \gamma)$ is the same as that they are observed outside $(0, 2\gamma - 1)$.

The replacements of $1 - F(X)$ by $F(X)$ and of $F(X) - F(Y)$ by $2F(X) - 1$, combined with the lemma and Theorem 1, furnish the lower one-sided and the two-sided intervals. Here it is to be noted that Y equals $2\varphi - X$.

For unknown φ, an order statistic $x(u)$, which exceeds φ with high probability (to be on the safe side), replaces φ in X, yielding X_u. Likewise $x(v)$, which is below φ with high probability, replaces φ in Y, yielding Y_v.

THEOREM 2.

$$P[F(X_u) \geqq \gamma] \geqq P[F(X) \geqq \gamma] - P[x(u) \leqq \varphi],$$

$$P[1 - F(Y_v) \geqq \gamma] \geqq P[F(Y) \leqq 1 - \gamma] - P[x(v) \geqq \varphi],$$

$$P[F(X_u) - F(Y_v) \geqq \gamma] \geqq P[F(X) - F(Y) \geqq \gamma] - P[x(u) \leqq \varphi] - P[x(v) \geqq \varphi].$$

PROOF. The proof is immediate in all cases since if A, B, C, D are events such that $A \supset BCD$, then

$$P(A) \geqq P(B) - P(\bar{C}) - P(\bar{D})$$

whether or not $\bar{D}$ is empty; here the overbar stands for complementation.

The lower bounds furnished by Theorem 2 are easily calculated from

$$P[x(n + 1 - i) \leqq \varphi] = P[x(i) \geqq \varphi] = \left(\frac{1}{2}\right)^n \sum_{j=0}^{i-1} \binom{n}{j},$$

and the above results for φ known.

In life-testing, only $x(1), \cdots, x(r)$ are available for some $r < n$. When φ is known, $r = 1$ can be used. When φ is unknown, however, r must be large enough for p to be less then unity. The following theorem, which is proved by the same methods as are used for Theorems 1 and 2, verifies the results for life-testing situations.

THEOREM 3.

$$P\{1 - F[x(1)] \geqq \gamma\} = P\{F[2\varphi - x(1)] \geqq \gamma\} = 1 - \gamma^n,$$

$$P\{F[2x(r) - x(1)] \geqq \gamma\} \geqq P\{1 - F[x(1) \geqq \gamma\} - P[x(r) \leqq \varphi],$$

$$P\{F[2x(r) - x(1)] - F[x(1)] \geqq \gamma\} \geqq P\{2F[x(1)] \leqq 1 - \gamma\} - P[x(r) \leqq \varphi].$$

5. Bounds for symmetry violations. The algebraic relations

$$X_u \geqq x(n), \qquad X \geqq x(n), \qquad Y_v \leqq x(1), \qquad Y \leqq x(1)$$

imply that a Wilks interval $[L_1, L_2]$ is contained in both its substitute interval $[L_1', L_2']$ and the corresponding interval for φ known $[L_1'', L_2'']$. Let $\beta(n)$, $\gamma(n)$ be any pair of values such that

$$P\{F(L_2) - F(L_1) \geqq \gamma(n)\} = \beta(n)$$

for given n. The relations between intervals imply that

$$P\{F(L_2') - F(L_1') \geqq \gamma(n)\} \geqq \beta(n), \qquad P\{F(L_2'') - F(L_1'') \geqq \gamma\} \geqq \beta(n).$$

Let β', γ' be a pair of values such that (1) holds for $[L_1', L_2']$ when the minimum required sample size equals n and F is symmetrical; likewise for β'', γ'' and $[L_1'', L_2'']$ with φ known. Then the allowable values of $\beta(n)$, $\gamma(n)$ such that $\beta(n) \leqq \beta'$ and $\gamma(n) \leqq \gamma'$ furnish limits for the various possible reductions in the true values of β, γ for $[L_1', L_2']$ due to violation of the symmetry assumption; likewise for $[L_1'', L_2'']$.

The method of determining the possible reductions in the true values β, γ consists in expressing n as a function of β', γ' for $[L_1', L_2']$ and as a function of β'', γ'' for $[L_1'', L_2'']$. That is, the value used for n in $\beta(n)$, $\gamma(n)$ is determined in this manner. Given n, the possible values of the pairs $\beta(n)$, $\gamma(n)$ are approximately determined by equating $\log[1 - \beta(n)]/\log\gamma(n)$ to n when the intervals are one-sided. The equation

$$(4) \quad \log[1 - \beta(n)]/\log\gamma(n) + \log\{1 + [1/\gamma(n) - 1]n\}/\log[1/\gamma(n)] = n$$

approximately determines the possible values for $\beta(n)$, $\gamma(n)$ when the intervals are two-sided. Section 3 contains methods for evaluating the minimum required sample sizes for the substitute intervals and the corresponding intervals for φ known.

Among the various possible pairs of values for $\beta(n)$, $\gamma(n)$, the pair for which $\beta(n) = \beta'$ and the pair for which $\gamma(n) = \gamma'$ are sometimes of interest for $[L_1', L_2']$; likewise for β'', γ'' and $[L_1'', L_2'']$. Although not considered further here, another pair of potential interest is that for which $\beta(n)/\beta' = \gamma(n)/\gamma'$ (or $\beta(n)/\beta'' = \gamma(n)/\gamma''$).

Let us consider evaluation of $\gamma(n)$ for the situation where $\beta(n) = \beta'$ and β', γ', $\gamma(n)$ are not too small. For one-sided intervals, $\gamma(n) \doteq 2\gamma' - 1$. Thus, for $\beta' \geqq .85$ (say) and $\gamma' = .99$, the value of $\gamma(n) \doteq .98$. For two-sided intervals, expression (4) depends on β' and is not explicitly solvable. For $\beta' = .9, .95, .99$ and $\gamma = .9, .95, .99, .999$, Table 1 furnishes approximate values for $\gamma(n)$ by use of one-dimensional interpolation of the results for Wilks intervals. For example, let $\beta' = .99$ and $\gamma' = .95$; then $\gamma(n) \doteq .915$. These results indicate that, for $[L_1', L_2']$, violation of the symmetry condition does not have a huge effect on γ for $\beta = \beta'$, especially when β', $\gamma' \geqq .95$; also that the effect is much larger for the two-sided than for the one-sided intervals.

Next consider evaluation of $\beta(n)$ for $\gamma(n) = \gamma'$ and β', $\beta(n)$, γ' not too small. For one-sided intervals $\beta(n) \doteq 1 - (1 - \beta')^{\frac{1}{2}}$. Thus, for $\gamma' \geqq .9$ (say) and $\beta' = .99$, the value of $\beta(n) \doteq .9$. For two-sided intervals, (4) depends on γ' and is not explicitly solvable. For $\beta = .95, .99$ and $\gamma' = .9, .95, .99, .999$, Table 1 furnishes approximate values for $\beta(n)$ by one-dimensional interpolation of the results for Wilks intervals. For example, let $\beta' = .99$ and $\gamma' = .95$; then $\beta(n) \doteq .943$. These results indicate that, for $[L_1', L_2']$, violation of the symmetry assumption does not have a huge effect on β for $\gamma = \gamma'$, especially for $\beta', \gamma' \geqq .95$; also that the effect is much larger for the one-sided than for the two-sided intervals.

The minimum required sample size for $[L_1'', L_2'']$ is approximately the same as that for $[L_1', L_2']$ when $\beta' = \beta''$, $\gamma' = \gamma''$ and these quantities are not too small. Thus, for both $[L_1', L_2']$ and $[L_1'', L_2'']$, violation of the symmetry assumption does not result in a huge reduction in the possible values for $\beta(n)$ and $\gamma(n)$, especially if $\beta', \gamma' \geqq .95$ (or $\beta'', \gamma'' \geqq .95$). That is, other possible pairs $\beta(n)$, $\gamma(n)$ would have values intermediate (one value reduced and the other increased) to those for the two extreme pairs that are explicitly considered.

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