THE DISTRIBUTION OF THE PRODUCT OF TWO CENTRAL OR NON-CENTRAL CHI-SQUARE VARIATES[1]

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1. Summary. The operational method of Mellin transforms is employed here to derive some interesting distribution functions. The distribution of the product of two non-central chi-square variates is obtained and some special cases exhibited. Finally, an application of the derived distributions to a problem in products of complex numbers is discussed.

2. Introduction. In many problems which arise in the physical sciences the use of complex variable notation greatly facilitates the description of some form of rotational motion. This is especially true in the mathematical theory of spin-stabilized rockets. For statistical analysis purposes it is sometimes desirable to find an expression for the probability that the magnitude of the product of two complex numbers (whose real and imaginary parts are independent random variables) is contained in a circle of some specified radius.

Such a problem stimulated the work described here. The solution of the original problem indicated the solution to a more general problem which is given here with the thought that this also may be of interest. The basic problem was to find the probability density function of a random variable $w = y_1 y_2$ where y_1 is a Rayleigh variate (with scale parameter equal to one), y_2 is a non-central Rayleigh variate (with scale parameter equal to one) and y_1 is independent of y_2. Since the square of a Rayleigh variate is a chi-square variate this problem was solved by obtaining the distribution of the product of a chi-square variate and a non-central chi-square variate, each with two degrees of freedom. The solution of this problem was generalized to include the distribution of the product of two non-central chi-square variates with arbitrary degrees of freedom.

The distributions derived here were obtained by the use of Mellin transforms. The operational advantages of Mellin transforms in problems of this type have been described by Epstein [4], and additional information on Mellin transforms can be found in [7] and [8].

3. The distribution of the product of two non-central chi-square variates. Let us suppose that y_1 and y_2 are two independent random variables distributed according to the non-central χ^2 density function [6] with non-centrality parameters Δ_1 and Δ_2 and degrees of freedom k_1 and k_2 respectively. Thus the density function of y_j is

$$(1) \qquad f_j(y_j) = \tfrac{1}{2}(y_j/\Delta_j^2)^{\frac{1}{4}(k_j-2)} e^{-\frac{1}{2}(y_j+\Delta_j^2)} I_{\frac{1}{2}(k_j-2)}(\Delta_j y_j^{\frac{1}{2}}), \qquad j = 1, 2,\ y_j > 0,$$

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where $I_n(x)$ is the modified Bessel function of the first kind defined by

$$(2) \qquad I_n(x) = \sum_{m=0}^{\infty} \{x^{2m+n}/[2^{2m+n} m! \Gamma(m + n + 1)]\}.$$

The non-centrality parameter Δ_j is defined by $\Delta_j^2 = \sum d_i^2$ when

$$y_j = \sum_{i=1}^{n} (x_i - d_i)^2$$

and the x_i are independent normal variates with zero means and unit standard deviations. Now we want to find the probability density function of the variate $w = y_1 y_2$. This problem will be solved by the use of Mellin transforms.

A statistical application of Mellin transforms was considered by Epstein [4]. The Mellin transform of a density function, $f(x)$, of a positive random variable is defined by

$$(3) \qquad M\{f(x)\} = \int_0^\infty x^{s-1} f(x)\, dx = g(s).$$

If the Mellin transform is known, the density function is obtained by the inversion integral,

$$(4) \qquad f(x) = 1/(2\pi i) \int_{a-i\infty}^{a+i\infty} x^{-s} g(s)\, ds.$$

Therefore the Mellin transform of the density function of y_j is

$$(5) \qquad M\{f_j(y_j)\} = \tfrac{1}{2} e^{-\frac{1}{2}\Delta_j^2} \Delta_j^{-\frac{1}{2}(k_j-2)} \int_0^\infty y_j^{s+\frac{1}{4}(k_j-6)} e^{-\frac{1}{2}y_j} I_{\frac{1}{2}(k_j-2)}(\Delta_j y_j^{\frac{1}{2}})\, dy_j .$$

The integral exists and is a continuous function of y_j since, for small values of y_j with $s + (\frac{1}{2}k_j) > 1$, the integrand behaves like $y_j^a (a > -1)$; for large values of y_j, the integrand behaves like $y_j^b e^{-(\frac{1}{2}y_j) + \Delta_j y_j^{\frac{1}{2}}}$, $[b = s + \frac{1}{4}(k_j - 7)]$.

To evaluate the integral it is necessary to replace $I_{(k_j-2)/2}(\Delta_j y_j^{\frac{1}{2}})$ by its power series expansion and integrate termwise. Term by term integration is justified since the series can be shown to converge uniformly. Therefore, we have that

$$(6) \qquad M\{f_j(y_j)\} = \frac{1}{2} \frac{e^{-\frac{1}{2}\Delta_j^2}}{\Delta^{\frac{1}{2}(k_j-2)}} \sum_{m=0}^{\infty} \frac{(\frac{1}{2}\Delta_j)^{2m+\frac{1}{2}(k_j-2)}}{m!\,\Gamma(m + \frac{1}{2}k_j)} \int_0^\infty y_j^{\frac{1}{2}k_j+s+m-2} e^{-\frac{1}{2}y_j}\, dy_j .$$

The integral in (6) is equal to $2^{\frac{1}{2}k_j+s+m-1}\Gamma[\frac{1}{2}k_j + s + m - 1]$. Hence the Mellin transform of $f_j(y_j)$ is

$$(7) \quad M\{f_j(y_j)\} = e^{-\frac{1}{2}\Delta_j^2} \sum_{m=0}^{\infty} \{\Delta_j^{2m} 2^{s-m-1}/m![\Gamma(m + \tfrac{1}{2}k_j)]\} \Gamma(\tfrac{1}{2}k_j + s + m - 1).$$

If we take the limit as $\Delta_j \rightarrow 0$ in equation (7), the result is

$$\text{Lim}_{\Delta_j \rightarrow 0} M\{f(y_j)\} = 2^{s-1}\Gamma(\tfrac{1}{2}k_j + s - 1)/\Gamma(\tfrac{1}{2}k_j),$$

which is the Mellin transform of the central χ^2 distribution with k_j degrees of freedom.

The Mellin transform of the density function of the product of two independent random variables is clearly the product of the Mellin transforms of the density functions of the individual variables; therefore, the Mellin transform of the density function of $w = y_1y_2$ is

$$
\begin{aligned}
M\{h(w)\} &= M\{f_1(y_1)\}M\{f_2(y_2)\} \\
&= e^{-\frac{1}{2}(\Delta_1^2+\Delta_2^2)}\left[\sum_{m=0}^{\infty}\frac{\Delta_1^{2m}2^{s-m-1}\Gamma(\frac{1}{2}k_1+s+m-1)}{m!\,\Gamma(m+\frac{1}{2}k_1)}\right] \\
&\quad\cdot\left[\sum_{n=0}^{\infty}\frac{\Delta_2^{2n}2^{s-n-1}\Gamma(\frac{1}{2}k_2+s+n-1)}{n!\,\Gamma(n+\frac{1}{2}k_2)}\right] = e^{-\frac{1}{2}(\Delta_1^2+\Delta_2^2)}\sum_{m=0}^{\infty}\sum_{j=0}^{m} \\
&\quad\cdot\frac{\Delta_1^{2j}\Delta_2^{2(m-j)}2^{2s-m-2}\Gamma(\frac{1}{2}k_1+s+j-1)\Gamma(\frac{1}{2}k_2+s+m-j-1)}{j!\,(m-j)!\Gamma(j+\frac{1}{2}k_1)\Gamma(m-j+\frac{1}{2}k_2)}.
\end{aligned}
\tag{8}
$$

This theorem is stated in reference [4]. Equation (8) defines all of the moments about the origin of w. To obtain the expected value of w, set $s = 2$, etc.

In order to find the density function of w we need to obtain the inverse Mellin transform of each term of the series (8). This means that we want to find the inverse Mellin transform of

$$M_j = 2^{2s-m-2}\Gamma(\tfrac{1}{2}k_1 + s + j - 1)\Gamma(\tfrac{1}{2}k_2 + s + m - j - 1).$$

This can be expressed as

$$M^{-1}\{M_j\} = (1/2\pi i)\int_{a-i\infty}^{a+i\infty} w^{-s}2^{2s-m-2}\Gamma(\tfrac{1}{2}k_1+s+j-1)\Gamma(\tfrac{1}{2}k_2+s+m-j-1)\,ds.$$

Making a change of variables let $s'/2 = s + \frac{1}{4}k_1 + \frac{1}{4}k_2 + \frac{1}{2}m - 1$, and obtain

$$
\begin{aligned}
M^{-1}\{M_j\} &= 2^{1-\frac{1}{2}k_1-\frac{1}{2}k_2-2m}\, w^{\frac{1}{4}k_1+\frac{1}{4}k_2+\frac{1}{2}m-1}\,(1/2\pi i)\int_{a'-i\infty}^{a'+i\infty} 2^{s'-2}\,\Gamma \\
&\quad\cdot[\tfrac{1}{2}s' + \tfrac{1}{4}(k_1 - k_2 + 4_j - 2m)]\Gamma[\tfrac{1}{2}s' - \tfrac{1}{4}(k_1 - k_2 + 4_j - 2m)]\,(w^{\frac{1}{2}})^{-s'}\,ds'.
\end{aligned}
$$

The inversion integral is now of such a form that one can use the tables of Mellin transforms in reference [5]. Formula 26, page 331, Volume I, of [5] gives

$$M^{-1}\{a^{-s}2^{s-2}\Gamma(\tfrac{1}{2}s - \tfrac{1}{2}\nu)\Gamma(\tfrac{1}{2}s + \tfrac{1}{2}\nu)\} = K_\nu(ax) \tag{9}$$

where $a > 0$ and $Rl\, s > |Rl\, \nu|$. In formula (9), $K_\nu(ax)$ is the modified Bessel function of the second kind defined by

$$K_\nu(ax) = \frac{\pi[I_{-\nu}(ax) - I_\nu(ax)]}{2\sin\nu\pi}$$

and tabulated in [2]. Therefore we have

$$M^{-1}\{M_j\} = \frac{w^{\frac{1}{4}k_1+\frac{1}{4}k_2+\frac{1}{2}m-1}\,K_{\frac{1}{2}k_1-m-\frac{1}{2}k_2+2j}(w^{\frac{1}{2}})}{2^{2m+\frac{1}{2}k_1+\frac{1}{2}k_2-1}}. \tag{10}$$

Hence the density function of the variate w is

$$(11)\quad h(w) = e^{-\frac{1}{2}(\Delta_1^2+\Delta_2^2)} \sum_{m=0}^{\infty} \sum_{j=0}^{m} \cdot \frac{\Delta_1^{2j}\Delta_2^{2(m-j)}2^{1-\frac{1}{2}k_1-\frac{1}{2}k_2-2m+\frac{1}{4}k_1+\frac{1}{4}k_2+\frac{1}{2}m-1}}{j!(m-j)!\,\Gamma(j+\frac{1}{2}k_1)\Gamma(m-j+\frac{1}{2}k_2)} K_{\frac{1}{2}k_1-\frac{1}{2}k_2-m+2j}(w^{\frac{1}{2}}).$$

Thus we have obtained an expression for the distribution of the product of two non-central chi-square variates.

The distribution of the product of two central chi-square variates can be obtained from equation (11) by simply setting $\Delta_1 = \Delta_2 = 0$. That is, if y_1 is a chi-square variate with k_1 degrees of freedom and y_2 (independent of y_1) is a chi-square variate with k_2 degrees of freedom then the density function of $w = y_1 y_2$ is

$$(12)\quad h(w) = \frac{w^{\frac{1}{4}k_1+\frac{1}{4}k_2-1} K_{\frac{1}{2}k_1-\frac{1}{2}k_2}(w^{\frac{1}{2}})}{2^{\frac{1}{2}k_1+\frac{1}{2}k_2-1}\Gamma(\frac{1}{2}k_1)\Gamma(\frac{1}{2}k_2)}.$$

It is known that if y_1 and y_2 are independent chi-square variates with degrees of freedom differing by one (i.e., $k_2 = k_1 + 1$) then the variate $2(w^{\frac{1}{2}})$ is also distributed as chi-square with $2k_1$ degrees of freedom. Probably the most familiar application of this result is in connection with the sample generalized variance from a multinormal distribution. This result is proved on page 172 of Anderson [1]. This fact follows from equation (12) by using the relationship

$$K_{\frac{1}{2}}(w^{\frac{1}{2}}) = \tfrac{1}{2}[(2\pi)^{\frac{1}{2}}e^{-w^{\frac{1}{2}}}w^{-\frac{1}{4}}],$$

and the duplication formula for the gamma function which is

$$\Gamma(\tfrac{1}{2}k_1)\Gamma(\tfrac{1}{2}k_1+\tfrac{1}{2}) = \pi^{\frac{1}{2}}2^{1-k_1}\Gamma(k_1).$$

It has also been shown in [3] that the density function of a variate, say z, which is the product of two $N(0, 1)$ variates is $f(z) = (1/\pi)K_0(z)$. This result follows from equation (12) by setting $k_1 = k_2 = 1$ and making the appropriate transformation.

4. Application to the distribution of the magnitude of the product of two complex numbers. As an example of the application of the previous results let us consider the following problem. Let $z_1 = u_1 + iv_1$ be a complex random variable with u_1 and v_1 being $N(0, 1)$ variates. Also let $z_2 = u_2 + iv_2$ be a complex random variable where u_2 is $N(\mu_u, 1)$ and v_2 is $N(\mu_v, 1)$, and u_1, u_2, v_1, v_2 are all independent. We wish to find the density function of the magnitude of the product of z_1 and z_2 or

$$r = |z_1 z_2| = [(u_1^2 + v_1^2)(u_2^2 + v_2^2)]^{\frac{1}{2}}.$$

The density function of $r_1 = |z_1| = (u_1^2 + v_1^2)^{\frac{1}{2}}$ is $f(r_1) = r_1 e^{-\frac{1}{2}r_1^2}$ which is the Rayleigh density function with scale parameter 1. Of course r_1^2 is a chi-square variate with two degrees of freedom. The density function of r_2 is

$$f(r_2) = r_2 e^{-\frac{1}{2}(r_2^2+\Delta_2^2)} I_0(\Delta_2 r_2).$$

It is also true that r_2^2 is a non-central chi-square variate with two degrees of freedom and non-centrality parameter Δ_2. Hence, we can obtain the distribution $r^2 = r_1^2 r_2^2$ from equation (11) by letting $k_1 = k_2 = 2$, $\Delta_1^2 = 0$ and $\Delta_2^2 = \mu_u^2 + \mu_v^2$. A simple transformation of variables then gives the density function of r which is

$$f(r) = e^{-\Delta_2^2/2} \sum_{n=0}^{\infty} \{\Delta_2^{2n} r^{n+1} K_n(r)/[2^{2n}(n!)^2]\}. \tag{13}$$

As a second example let us consider a special case of the first example. Let us assume that u_1, u_2, and v_1 and v_2 are all NID (0, 1) variates. As before, suppose we would like to find the probability density function of $r = |z_1 z_2|$.

The distribution of $w = r^2 = r_1^2 r_2^2$ can be obtained from equation (13) by setting $\Delta_2 = 0$.

The density function of r is $f(r) = rK_0(r)$. To evaluate the probability that r is less than or equal to some value a we have

$$\text{Prob } (r \leqq a) = \int_0^a rK_0(r)\, dr.$$

Reference [5] (page 367, Volume II, formula 22) gives

$$\int_0^a y^{n+1} K_n(y)\, dy = 2^n\, \Gamma(n+1) - a^{n+1} K_{n+1}(a).$$

Thus

$$\text{Prob } (r \leqq a) = 1 - aK_1(a) \tag{14}$$

and the $K_1(a)$ function is tabulated in [2].

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