ON A GENERALIZATION OF THE FINITE ARCSINE LAW[1]

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1. Introduction. Throughout this paper $\{X_k\}$ will denote a sequence of independent, identically distributed random variables with continuous and symmetric distributions.

Among the neatest and most startling results concerning the behavior of the partial sums $S_n = X_1 + \cdots + X_n (S_0 = 0)$ are those which are *distribution free*, i.e., those which do not depend on the distribution of X_1. For example, if we define

$$(1) \quad \begin{aligned} &N_n : \text{the number of positive sums among } S_1, S_2, \cdots, S_n. \\ &L_n : \text{the smallest } k(=0, 1, \cdots, n) \text{ for which } S_k = \max_{0 \leqq j \leqq n} S_j, \end{aligned}$$

then Sparre Andersen [3, 4] showed that N_n and L_n have a common distribution which does not depend on the distribution of X_1:

$$(2) \qquad P\{N_n = m\} = P\{L_n = m\} = \binom{2m}{m}\binom{2n-2m}{n-m}(1/2^{2n}), \quad 0 \leqq m \leqq n.$$

We give here another distribution free result which generalizes (2) and which includes in particular information about the *joint* distribution of N_n and L_n. It is disappointingly easy to construct examples (even for $n = 3$) to show that the total joint distribution of N_n and L_n is *not* distribution free. Yet, for the special case $L_n = n$ we can find explicitly the distribution of N_n, namely

$$(3) \qquad P\{N_n = m, L_n = n\} = (1/2n)\binom{2n-2m}{n-m}(1/2^{2n-2m}), \qquad 1 \leqq m \leqq n.$$

Our method consists of finding a pair of "differential equations" for the generating functions of quantities like those appearing in (3). These equations are then solved and the generating functions inverted.

Before we can state our main result we must introduce more notation. Let $R_{n0} \geqq R_{n1} \geqq \cdots \geqq R_{nn}$ be an ordering of the partial sums $S_0, S_1, \cdots, S_n$. Since the distribution of X_1 is continuous, the probability that two S_k's are equal is zero. This means that with probability one there is a unique index m such that $R_{nk} = S_m$. We say $L_{nk} = m$ in case $R_{nk} = S_m$, and we note that L_{nk} is well defined with probability one. Darling [2] found the distribution of L_{nk} in terms of products of binomial coefficients, but he gave no results for joint distributions.

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Our main theorem, which gives information concerning the joint distribution of the L_{nk}'s, is as follows.

THEOREM 1. *For all* $0 \leqq m, k \leqq n$, $(n \geqq 1)$,

$$(4)\quad P\{L_{nm}=0, L_{nk}=n\} = \begin{cases} (1/2n)\binom{2m}{m}\binom{2n-2k}{n-k}(1/2^{2n+2m-2k}), & m<k, \\ 0, & m=k, \\ (1/2n)\binom{2k}{k}\binom{2n-2m}{n-m}(1/2^{2n+2k-2m}), & m>k. \end{cases}$$

We note that $L_{nm} = 0$ is equivalent to $N_n = m$. Moreover if $L_{nk} = n$, then there are exactly k partial sums greater than S_n. Thus, Theorem 1 gives the joint distribution of the number of partial sums *less* than $S_0 = 0$ and the number of partial sums *greater* than S_n. This latter way of stating Theorem 1 has the advantage of being more symmetric with respect to the "time" scale n. If $\bar{N}_n = n - N_n$, the substitution $\hat{X}_k = X_{n+1-k}$ (reversing the time scale) takes the set $\{\bar{N}_n = m, L_{nk} = n\}$ into another one of the same type, namely $\{\hat{\bar{N}}_n = k, \hat{L}_{nm} = n\}$.

Using Theorem 1, we can find a generalization of the arcsine law for infinitely divisible stochastic processes. In fact, let $\{x(t), 0 \leqq t \leqq T\}$ denote a separable, infinitely divisible process with continuous and symmetric distributions and denote by V_τ the amount of "time" in $[0, T]$ that $x(t)$ is greater than $x(\tau)$. Then, for $0 \leqq \alpha, \beta \leqq T$

$$(5)\quad P\{V_0 < \alpha, V_T < \beta\} = \begin{cases} \frac{2}{\pi}\arcsin\left(\frac{\beta}{T}\right)^{\frac{1}{2}} - \frac{2}{\pi T}[\beta(T-\alpha)]^{\frac{1}{2}}, & \beta \leqq \alpha, \\ \frac{2}{\pi}\arcsin\left(\frac{\alpha}{T}\right)^{\frac{1}{2}} - \frac{2}{\pi T}[\alpha(T-\beta)]^{\frac{1}{2}}, & \beta \geqq \alpha. \end{cases}$$

2. Basic formula. Let $N_n(x)$ denote the number of partial sums among $S_0, S_1, \cdots, S_n$ that are greater than x and let $\bar{N}_n(x)$ denote the number less than x. Then, there is the following basic formula.

FORMULA I. *For* $x \geqq 0$ *and* $n \geqq 1$

$$(6)\quad \begin{aligned} &P\begin{Bmatrix} N_n(x) \leqq K \\ \bar{N}_n(0) \leqq M \end{Bmatrix} - P\begin{Bmatrix} N_n(0) \leqq K \\ \bar{N}_n(0) \leqq M \end{Bmatrix} \\ &\quad = \sum_{k=0}^{K}\sum_{m=0}^{M}\sum_{v=1}^{n}\int_0^x P\begin{Bmatrix} N_{n-v}(y) \leqq M-m \\ \bar{N}_{n-v}(0) = K-k \end{Bmatrix} d_y P\begin{Bmatrix} S_v \leqq y \\ L_{vk} = v \\ \bar{N}_v(0) = m \end{Bmatrix}, \end{aligned}$$

where we interpret as zero all terms on the right of (6) *which involve* $L_{vk} = v$ *for* $k > v$ *or* $\bar{N}_v(0) = m$ *for* $m > v$.

PROOF. By symmetry we can rewrite the integrals on the right in (6) in the form

$$(7)\quad \int_0^x P\begin{Bmatrix} N_{n-v}(0) = K-k \\ \bar{N}_{n-v}(-y) \leqq M-m \end{Bmatrix} d_y P\begin{Bmatrix} S_v \leqq y \\ L_{vk} = v \\ \bar{N}_v(0) = m \end{Bmatrix}.$$

For fixed v, let $\tilde{N}_{n-v}(x)$ and $\tilde{\bar{N}}_{n-v}(x)$ be defined for the variables $\tilde{X}_j = X_{j+v}$, $j = 1, 2, \cdots, n-v$, according to the previously given definitions. Because the variables $\{X_k\}$ are identically distributed, (7) can also be written

$$(8)\qquad \int_0^x P\begin{Bmatrix}\tilde{N}_{n-v}(0) = K-k \\ \tilde{\bar{N}}_{n-v}(-y) \leqq M-m\end{Bmatrix} d_y P\begin{Bmatrix}S_v \leqq y \\ L_{vk} = v \\ \bar{N}_v(0) = m\end{Bmatrix}.$$

Using the independence property of the X_k's and the fact that the integrand in (8) is a continuous function of y, we can evaluate the integral in (8), getting

$$(9)\qquad P\begin{Bmatrix}0 < S_v \leqq x, & \tilde{N}_{n-v}(0) = K-k \\ L_{vk} = v \quad , & \tilde{\bar{N}}_{n-v}(-S_v) \leqq M-m \\ \bar{N}_v(0) = m & \end{Bmatrix}.$$

Now, the conditions which define the set whose probability appears in (9) can be rewritten in a more convenient form. In the first place, $\tilde{S}_j = S_{j+v} - S_v > 0$ if and only if $S_{j+v} > S_v (j = 1, 2, \cdots, n-v)$. Thus, $\tilde{N}_{n-v}(0) = K-k$ means that exactly $K-k$ of the partial sums $S_{v+1}, \cdots, S_n$ are greater than S_v. But, if $L_{vk} = v$ and $\tilde{N}_{n-v}(0) = K-k$, then there are exactly K partial sums among $S_0, S_1, \cdots, S_n$ greater than S_v, i.e. $L_{nK} = v$. In the second place, $\tilde{S}_j = S_{j+v} - S_v < -S_v$ if and only if $S_{j+v} < 0, j = 1, 2, \cdots, n-v$. Hence, $\tilde{\bar{N}}_{n-v}(-S_v) \leqq M-m$ means that less than or equal to $M-m$ of the partial sums $S_{v+1}, \cdots, S_n$ are less than zero. In view of the condition $\bar{N}_v(0) = m$, there will be less than or equal to M negative sums in all among $S_0, S_1, \cdots, S_n$, i.e., $\bar{N}_n(0) \leqq M$. From (9) and the previous argument, we see that the right side of (6) can be written

$$(10)\qquad \begin{aligned} \sum_{k=0}^{K}\sum_{m=0}^{M}\sum_{v=1}^{n} P\begin{Bmatrix}0 < S_v \leqq x, & L_{nK} = v \\ L_{vk} = v \quad , & \bar{N}_n(0) \leqq M \\ \bar{N}_v(0) = m & \end{Bmatrix} &= \sum_{v=1}^{n} P\begin{Bmatrix}0 < S_v \leqq x \\ L_{nK} = v \\ \bar{N}_n(0) \leqq M\end{Bmatrix} \\ = \sum_{v=1}^{n} P\begin{Bmatrix}0 < R_{nK} \leqq x \\ L_{nK} = v \\ \bar{N}_n(0) \leqq M\end{Bmatrix} &= P\begin{Bmatrix}R_{nK} \leqq x \\ \bar{N}_n(0) \leqq M\end{Bmatrix} - P\begin{Bmatrix}R_{nK} \leqq 0 \\ \bar{N}_n(0) \leqq M\end{Bmatrix}. \end{aligned}$$

We finish the proof with the observation that $R_{nK} \leqq x$ is equivalent to $N_n(x) \leqq K$.

3. Generating functions and a pair of equations. We introduce two generating functions which play an important role in the evaluation of the probabilities of Theorem 1. Let

$$(11)\qquad U_{m,k}^{(n)}(x) = P\begin{Bmatrix}N_n(x) \leqq k \\ \bar{N}_n(0) \leqq m\end{Bmatrix}, \qquad \alpha_{m,k}^{(n)}(x) = P\begin{Bmatrix}0 < S_n \leqq x \\ L_{nk} = n \\ \bar{N}_n(0) = m\end{Bmatrix}.$$

We also introduce the generating functions of these quantities

$$(12)\qquad \begin{aligned} U(x) &\equiv U(\lambda, s, t; x) = \sum_{n,k,m=0} U_{m,k}^{(n)}(x) s^m t^k \lambda^n, \\ \alpha(x) &\equiv \alpha(\lambda, s, t; x) = \sum_{n,k,m=0} \alpha_{m,k}^{(n)}(x) s^m t^k \lambda^n. \end{aligned}$$

We will first show that $\alpha(x)$ is left unchanged under the interchange of s and t, i.e., $\alpha(\lambda, s, t; x) = \alpha(\lambda, t, s; x)$. To do this, one considers the substitution $\hat{X}_v = X_{n+1-v}$ in the set whose probability is $\alpha_{m,k}^{(n)}(x)$. Of course, $S_n = \hat{S}_n$. On the other hand, $S_j < 0$ is equivalent to $\hat{S}_{n-j} > \hat{S}_n$, and $S_j > S_n$ is equivalent to $\hat{S}_{n-j} < 0$. Thus, under this substitution

$$\left\{\begin{matrix} 0 < S_n \leqq x \\ L_{nk} = n \\ \bar{N}_n(0) = m \end{matrix}\right\} = \left\{\begin{matrix} 0 < \hat{S}_n \leqq x \\ \hat{L}_{nm} = n \\ \hat{\bar{N}}_n(0) = k \end{matrix}\right\}. \tag{13}$$

In other words, using the identical distribution property of the X_k's and (13),

$$\alpha_{m,k}^{(n)}(x) = \alpha_{k,m}^{(n)}(x), \tag{14}$$

which shows that $\alpha(x)$ is left unchanged if s and t are interchanged.

Formula I can now be rewritten in terms of the notation introduced in (11), i.e.,

$$\begin{aligned} U_{M,K}^{(n)}(x) &- U_{M,K}^{(n)}(0) \\ &= \sum_{k=0}^{K} \sum_{m=0}^{M} \sum_{v=1}^{n} \int_0^x [U_{K-k,M-m}^{(n-v)}(y) - U_{K-k-1,M-m}^{(n-v)}(y)] \, d_y \, \alpha_{m,k}^{(v)}(y). \end{aligned} \tag{15}$$

Relation (15) is equivalent to an equation involving generating functions. In fact, using the notation $V(x) \equiv V(\lambda, s, t; x) = U(\lambda, t, s; x)$, i.e., interchanging s and t in $U(x)$, one has

$$U(x) - U(0) = (1 - t) \int_0^x V(y) \, d_y \, \alpha(y). \tag{16}$$

Interchanging s and t in (16) gives a second equation involving $U(x)$, $V(x)$ and $\alpha(x)$:

$$V(x) - V(0) = (1 - s) \int_0^x U(y) \, d_y \, \alpha(y). \tag{17}$$

Thus, we have a pair of equations, (16) and (17), from which we will eventually determine $\alpha(\infty)$. Now, $U_{M,K}^{(N)}(x)$ is *uniquely* determined by (15) in terms of $U_{m,k}^{(n)}(0)$ and $\alpha_{m,k}^{(n)}(x)$, $m \leqq M$, $k \leqq K$, $n \leqq N$. Thus, there is a *unique* solution to (16) and (17) expressing $U(x)$ and $V(x)$ in terms of $U(0)$, $V(0)$ and $\alpha(x)$. To find this unique solution, let us first assume that the distributions are absolutely continuous. Differentiating (16) and (17), one gets

$$\begin{aligned} U'(x) &= (1 - t)\alpha'(x)V(x) \\ V'(x) &= (1 - s)\alpha'(x)U(x) \end{aligned}, \quad U(0) \text{ and } V(0) \textit{ given.} \tag{18}$$

But, (18) can be solved explicitly. If $a = (1 - t)$ and $b = (1 - s)$, then

$$\begin{aligned} U(x) &= U(0) \cosh (ab)^{\frac{1}{2}}\alpha(x) + V(0)(a/b)^{\frac{1}{2}} \sinh(ab)^{\frac{1}{2}}\alpha(x) \\ V(x) &= V(0) \cosh (ab)^{\frac{1}{2}}\alpha(x) + U(0)(b/a)^{\frac{1}{2}} \sinh (ab)^{\frac{1}{2}}\alpha(x). \end{aligned} \tag{19}$$

Of course, the solution given in (19) is also the unique solution to (16) and (17) in general, as shown by direct substitution, and is completely equivalent to (15), i.e., to Formula I.

The method demonstrated here of forming a pair of differential equations through which the generating functions of two sequences of probabilities are related was also used by the author in [1, Sect. 5] where the special case $m = n$ of (3) was given.

4. Generating function of $\alpha(\infty)$ and inversion. To find $\alpha(\infty)$ we let x become infinite in (19). This leaves only the problem of determining $U(\infty)$ and $U(0)$. However, these power series are easily computed from their definitions and from the known result of Andersen [3]

$$\sum_{n=0}^{\infty} \sum_{v=0}^{\infty} \lambda^n s^v P\{N_n = v\} = (1 - \lambda)^{-\frac{1}{2}}(1 - \lambda s)^{-\frac{1}{2}}. \tag{20}$$

On the one hand

$$\begin{aligned} U(\infty) &= \sum_{m,k,n=0}^{\infty} \lambda^n s^m t^k P \begin{Bmatrix} N_n(\infty) \leqq k \\ \bar{N}_n(0) \leqq m \end{Bmatrix} \\ &= \sum_{m,k,n=0}^{\infty} \lambda^n s^m t^k P\{\bar{N}_n(0) \leqq m\} \\ &= \sum_{m,k,n=0}^{\infty} \lambda^n s^m t^k P\{N_n \leqq m\} \\ &= (1 - t)^{-1}(1 - s)^{-1} \sum_{n=0}^{\infty} \sum_{v=0}^{n} \lambda^n s^v P\{N_n = v\} \\ &= (1 - t)^{-1}(1 - s)^{-1}(1 - \lambda)^{-\frac{1}{2}}(1 - \lambda s)^{-\frac{1}{2}}. \end{aligned} \tag{21}$$

On the other hand

$$\begin{aligned} U(0) &= \sum_{m,k,n=0}^{\infty} \lambda^n s^m t^k P \begin{Bmatrix} N_n(0) \leqq k \\ \bar{N}_n(0) \leqq m \end{Bmatrix} \\ &= \sum_{m,k,n=0}^{\infty} \lambda^n s^m t^k P\{n - m \leqq N_n \leqq k\} \\ &= \sum_{m,k,n=0}^{\infty} \lambda^n s^m t^k \sum_{v=n-m}^{k} P\{N_n = v\} \\ &= (1 - t)^{-1} \cdot (1 - s)^{-1} \sum_{n=0}^{\infty} \sum_{v=0}^{n} \lambda^n s^{n-v} t^v P\{N_n = v\} \\ &= (1 - t)^{-1} \cdot (1 - s)^{-1} \cdot (1 - \lambda s)^{-\frac{1}{2}}(1 - \lambda t)^{-\frac{1}{2}}. \end{aligned} \tag{22}$$

If we substitute these expressions into (19) with $x = \infty$, remembering that $V(\infty)$ and $V(0)$ are formed from $U(\infty)$ and $U(0)$ by interchanging s and t, and perform an obvious simplification, we find

$$(23)\quad \begin{aligned}[(1-\lambda t)/(1-\lambda)]^{\frac{1}{2}} &= \cosh (ab)^{\frac{1}{2}}\alpha(\infty) + (a/b)^{\frac{1}{2}}\sinh (ab)^{\frac{1}{2}}\alpha(\infty)\\ [(1-\lambda s)/(1-\lambda)]^{\frac{1}{2}} &= \cosh (ab)^{\frac{1}{2}}\alpha(\infty) + (b/a)^{\frac{1}{2}}\sinh (ab)^{\frac{1}{2}}\alpha(\infty).\end{aligned}$$

Solving (23) for $\cosh (ab)^{\frac{1}{2}}\alpha(\infty)$ and $\sinh (ab)^{\frac{1}{2}}\alpha(\infty)$ and then adding one finds:

$$(24)\qquad e^{(ab)^{\frac{1}{2}}\alpha(\infty)} = \frac{b^{\frac{1}{2}}(1-\lambda t)^{\frac{1}{2}} + a^{\frac{1}{2}}(1-\lambda s)^{\frac{1}{2}}}{(b+a)^{\frac{1}{2}}(1-\lambda)^{\frac{1}{2}}}.$$

In summary, we have found the generating function of the quantities $\alpha_{m,k}^{(n)}(\infty)$, i.e.,

$$(25)\quad \alpha(\infty) = [(1-s)(1-t)]^{-\frac{1}{2}} \log\left[\frac{(1-s)^{\frac{1}{2}}(1-\lambda t)^{\frac{1}{2}} + (1-t)^{\frac{1}{2}}(1-\lambda s)^{\frac{1}{2}}}{[(1-s)^{\frac{1}{2}} + (1-t)^{\frac{1}{2}}](1-\lambda)^{\frac{1}{2}}}\right].$$

A simple observation will enable us to invert the generating function in (25). Let us write

$$(26)\quad \begin{aligned} P &= [(1-s)(1-t)]^{-\frac{1}{2}} \log\left[\frac{(1-s)^{\frac{1}{2}}(1-\lambda t)^{\frac{1}{2}} + (1-t)^{\frac{1}{2}}(1-\lambda s)^{\frac{1}{2}}}{(1-s)^{\frac{1}{2}} + (1-t)^{\frac{1}{2}}}\right]\\ Q &= -\tfrac{1}{2}\cdot[(1-s)(1-t)]^{-\frac{1}{2}} \log (1-\lambda),\end{aligned}$$

so that $\alpha(\infty) = P + Q$. Now, all non-zero terms $P_{nmk}\lambda^n s^m t^k$ in the expansion of P must have $m + k \geqq n$. This follows simply because λ appears in P only together with s or t. On the other hand $\alpha_{m,k}^{(n)}(\infty) = 0$ if $m + k \geqq n$; for clearly, if $L_{nk} = n$ and $S_n > 0$, then there are at least $k + 1$ positive sums. Thus, $\alpha_{m,k}^{(n)}(\infty)$ is non-zero only if $m + k < n$. The *only* contribution to the non-zero terms of $\alpha(\infty)$ comes from Q. Thus, we find

$$(27)\quad \alpha_{m,k}^{(n)}(\infty) = P\begin{Bmatrix} 0 < S_n < \infty \\ L_{nk} = n \\ \bar{N}_n(0) = m \end{Bmatrix} = \frac{1}{2n}\binom{2m}{m}\binom{2k}{k}\frac{1}{2^{2m+2k}}, \qquad m + k < n.$$

Replacing m by $n - m$ in (27) yields

$$(28)\quad P\begin{Bmatrix} 0 < S_n < \infty \\ L_{nk} = n \\ N_n = m \end{Bmatrix} = \frac{1}{2n}\binom{2k}{k}\binom{2n-2m}{n-m}\frac{1}{2^{2n+2k-2m}}, \qquad k < m,$$

which is equivalent to the last line of (4) since $S_n > 0$ is equivalent to $k < m$ in (4). The first line of (4) follows by symmetry.

5. Limiting case. To find the distribution indicated in (5) we compute the limit

$$(29)\qquad \lim_{n\to\infty} \sum_{m=0}^{[\alpha n/T]} \sum_{k=0}^{[\beta n/T]} P\{N_n = m, L_{nk} = n\}.$$

It follows that $(\beta < \alpha)$

$$P\{V_0 < \alpha, V_T < \beta\} = \lim_{n\to\infty} (1/2\pi) \sum_{k=1}^{[\beta n/T]} \sum_{m=k}^{[\alpha n/T]} (1/n)[k(n-m)]^{-\frac{1}{2}}$$

$$+ \lim_{n\to\infty} (1/2\pi) \sum_{k=0}^{[\beta n/T]} \sum_{m=1}^{k} (1/n)[m(n-k)]^{-\frac{1}{2}}$$

(30)
$$= 1/(2\pi) \int_0^{\beta/T} \int_x^{\alpha/T} (x(1-y))^{-\frac{1}{2}}\, dx\, dy$$

$$+ (1/(2\pi)) \int_0^{\beta/T} \int_0^{x} (y(1-x))^{-\frac{1}{2}}\, dx\, dy$$

$$= \frac{2}{\pi} \arcsin \left(\frac{\beta}{T}\right)^{\frac{1}{2}} - \frac{2}{\pi}\left(\frac{\beta}{T}\left(1-\frac{\alpha}{T}\right)\right)^{\frac{1}{2}}, \qquad \beta < \alpha.$$

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