LIMITING DISTRIBUTION OF THE MAXIMUM TERM IN SEQUENCES OF DEPENDENT RANDOM VARIABLES[1]

By Simeon M. Berman

Columbia University

Introduction. This paper contains an investigation of the limiting distribution of the maximum term in sequences of random variables subject to certain kinds of dependence.

The limiting distribution of the maximum term in a sequence of independent random variables with a common distribution was completely analyzed in a series of works by many writers; this research culminated in the comprehensive paper of Gnedenko [5]. The assumption of a common distribution was dropped by Juncosa [7]. Watson proved that under mild restrictions, the limiting distribution of the maximum in an m-dependent stationary sequence is the same as that for independent random variables with a common distribution [11]. A complete bibliography is contained in the book by Gumbel [6].

The first section of this paper contains a brief review of the classical case of independent and identically distributed random variables as given in Gnedenko's paper [5].

The present work generalizes the classical theory.

1. In the second section, the maximum term in a sequence of exchangeable random variables is considered. The limiting distribution is a mixture of the distributions obtained in the case of independent random variables. This is analogous to the findings of Blum, Chernoff, Rosenblatt, and Teicher [2] and Buhlmann [3], who discovered that the limiting distributions of the *sums* of exchangeable random variables are mixtures of normal distributions.

2. The second generalization is the case where the number of random variables considered in the determination of the maximum is itself a random variable N_n, depending on a nonnegative, integer-valued parameter n. If the sequence $\{N_n\}$ is distributed independently of the observed random variables, and if $N_n \xrightarrow{\text{pr}} \infty$ as $n \to \infty$, then the limiting distribution of the maximum is a mixture of the kind described in the previous paragraph. This is similar to the result of Robbins that the limiting distribution of the sum of a random number of random variables is a mixture of normal distributions [10].

A theorem is also stated for the case where the sequence $\{N_n\}$ may depend on the observed random sequence; an analogous theorem for sums has also been given by Anscombe [1].

Any result obtained for the maximum term is also valid, with appropriate modifications, for the minimum term.

Received April 19, 1961, revised January 12, 1962.

[1] Work done under a grant from the National Science Foundation.

This study treats only the distribution of the maximum term in sequences of dependent random variables with the same marginal distributions. For this reason, the distribution of the maximum partial sum in a sequence of random variables, about which there is a large body of research, is not considered.

1. Review of the classical case. Throughout this paper, the distribution function of a random variable X will be denoted by $P\{X \leqq x\} = F(x)$. The term "distribution function" will be abbreviated as "d.f." A d.f. is said to be proper if

$$\lim_{x\to\infty} F(x) = 1, \qquad \lim_{x\to-\infty} F(x) = 0;$$

otherwise, it is improper. A d.f. will be assumed proper unless the contrary is stated. The expected value of a random variable X will be denoted by EX.

Let $X_1, X_2, \cdots, X_n, \cdots$ be a sequence of independent random variables with the common d.f. $F(x)$. For each integer $n \geqq 1$, let $Z_n = \max(X_1, \cdots, X_n)$. The d.f. of Z_n is $P\{Z_n \leqq x\} = P\{X_1 \leqq x, \cdots, X_n \leqq x\} = F^n(x)$.

The limiting d.f. of Z_n belongs to exactly one of three types; specifically, if there exists a d.f. $\Phi(x)$ and sequences $\{a_n\}$ and $\{b_n\}$ where $a_n > 0$ for $n \geqq 1$, such that

$$\lim_{n\to\infty} P\{a_n^{-1}(Z_n - b_n) \leqq x\} = \Phi(x)$$

at each continuity point x of $\Phi(x)$, then $\Phi(x)$ belongs to one of the following types.

$$\begin{aligned}
\Phi_{(1)}(x) &= 0 && \text{for } x \leqq 0 \\
&= \exp[-x^{-\alpha}] && \text{for } x > 0;\ \alpha > 0 \\
\Phi_{(2)}(x) &= \exp[-(-x)^{\alpha}] && \text{for } x < 0;\ \alpha > 0 \qquad (1.1)\\
&= 1 && \text{for } x \geqq 0 \\
\Phi_{(3)}(x) &= \exp[-e^{-x}] && -\infty < x < \infty
\end{aligned}$$

These are often called the extreme value d.f.'s.

Some useful relationships between the marginal d.f. $F(x)$ of the sequence $\{X_n\}$ and the limiting d.f. $\Phi(x)$ of Z_n are now mentioned. If x is a point such that $0 < \Phi(x) < 1$, then

$$\begin{aligned}
&0 < F(a_n x + b_n) < 1, \qquad \text{for all large } n, \\
&\lim_{n\to\infty} F(a_n x + b_n) = 1.
\end{aligned} \tag{1.2}$$

From (1.2) it follows that

$$\lim_{n\to\infty} -\frac{\log F(a_n x + b_n)}{1 - F(a_n x + b_n)} = 1; \tag{1.3}$$

$$\lim_{n\to\infty} n(1 - F(a_n x + b_n)) = -\log \Phi(x). \tag{1.4}$$

Gnedenko found the conditions for $F(x)$ which are necessary and sufficient for the convergence of the d.f. of Z_n to each of the three types; in other words, he characterized the domains of attraction. The normalizing sequences $\{a_n\}$ and $\{b_n\}$ are computed from the d.f. $F(x)$ according to the domain of attraction to which it belongs.

If $F(x)$ is in the domain of attraction of $\Phi_{(1)}(x)$, then $F(x) < 1$ for every finite x, and

$$b_n = 0; \qquad 1 - F(a_n) \sim n^{-1}; \qquad a_n \leqq a_{n+1}. \tag{1.5}$$

If $F(x)$ is in the domain of attraction of $\Phi_{(2)}(x)$, there exists a finite number x_0 such that for any $\epsilon > 0$,

$$\begin{aligned} &F(x_0) = 1, \qquad F(x_0 - \epsilon) < 1; \\ &b_n = x_0, \qquad 1 - F(x_0 - a_n) \sim n^{-1}, \qquad a_{n+1} \leqq a_n. \end{aligned} \tag{1.6}$$

If $F(x)$ is in the domain of attraction of $\Phi_{(3)}(x)$, and if x_0 is the least upper bound of all x such that $F(x) < 1$, then x_0 may be either finite or infinite. In both cases, the relations

$$1 - F(b_n) \sim n^{-1}; \qquad 1 - F(a_n + b_n) \sim (ne)^{-1}, \qquad b_n \leqq b_{n+1} \tag{1.7}$$

are valid.

2. Exchangeable random variables. Let $(\Omega, \mathcal{A}, P)$ be a probability space: Ω is a set of points ω, $\mathcal{A}$ is a Borel field of subsets of Ω, and P is a probability measure on $\mathcal{A}$. A sequence $\{X_n : n \geqq 1\}$ of random variables defined on this space is called exchangeable if the joint d.f. of any number m of these random variables does not depend on their subscripts but only on their number m [8; p. 364]. The joint d.f. of m of these random variables will be denoted by $G_m(x_1, \cdots, x_m)$, for each m. According to the fundamental theorem of de Finetti, $G_m(x_1, \cdots, x_m)$ can be represented as a mixture of joint d.f.'s of m independent and identically distributed random variables [8; p. 365]. Specifically, one can write

$$G_m(x_1, \cdots, x_m) = \int_\Omega G_\omega(x_1) \cdots G_\omega(x_m)\, dP(\omega), \tag{2.1}$$

where for fixed x, $G_\omega(x)$ is a random variable, and for each ω, $G_\omega(x)$ is a d.f. in x.

From the representation (2.1), the d.f. of the maximum of n variables, $Z_n = \max(X_1, \cdots, X_n)$, is obtained:

$$P\{Z_n \leqq x\} = G_n(x, \cdots, x) = \int_\Omega G_\omega^n(x)\, dP(\omega) = EG_\omega^n(x). \tag{2.2}$$

The problem to be presented here is that of finding the limiting d.f. of Z_n, that is, finding sequences $\{a_n\}$ and $\{b_n\}$ and a d.f. $L(x)$ such that $a_n > 0$ and

$$\lim_{n\to\infty} P\{a_n^{-1}(Z_n - b_n) \leqq x\} = \lim_{n\to\infty} EG_\omega^n(a_n x + b_n) = L(x) \tag{2.3}$$

for all points x in the continuity set of $L(x)$. This paper will consider the class of d.f.'s $L(x)$ which can be obtained by using pairs of normalizing sequences which are, in fact, normalizing sequences for the limiting d.f. of the maximum in *some* sequence of independent random variables with *some* common d.f. $F(x)$. In other words, the pairs of sequences $(\{a_n\}, \{b_n\})$ to be used in (2.3) are those for which there exists a d.f. $F(x)$ such that for all x,

$$\lim_{n\to\infty} F^n(a_n x + b_n) = \Phi(x) \tag{2.4}$$

where $\Phi(x)$ is an extreme value d.f. (see (1.1)). The continuity set of $\Phi(x)$ is the entire real line.

The following three theorems characterize the limiting d.f.'s of the maximum in the three cases corresponding to the domains of attraction to which $F(x)$ belongs. The necessary and sufficient conditions for convergence to these limiting d.f.'s are also given in each case.

THEOREM 2.1. *Let* $\{X_n : n \geqq 1\}$ *be a sequence of exchangeable random variables on* $(\Omega, \mathcal{A}, P)$ *such that the joint d.f.'s have the representation* (2.1); *let* $Z_n = \max(X_1, \cdots, X_n)$. *Suppose that there exist a sequence of positive numbers* $\{a_n\}$ *and a d.f.* $F(x)$ *in the domain of attraction of* $\Phi_{(1)}(x)$ *such that* (2.4) *is true for*

$$b_n = 0; \qquad \Phi(x) = \Phi_{(1)}(x).$$

Then,

(a) *There exists a nondegenerate d.f.* $L(x)$ *such that for all* x *in its continuity set*

$$\lim_{n\to\infty} P\{a_n^{-1} Z_n \leqq x\} = \lim_{n\to\infty} EG_\omega^n(a_n x) = L(x) \tag{2.5}$$

if and only if there exists a d.f. $A(y)$ *such that for all* y *in its continuity set.*

$$\lim_{u\to\infty} P\left\{\frac{\log G_\omega(u)}{\log F(u)} \leqq y\right\} = A(y), \tag{2.6}$$

where $A(y)$ *satisfies the conditions*

$$A(\infty) - A(0-) = 1; \qquad A(0+) - A(0-) < 1 \tag{2.7}$$

(b) $L(x)$ *is necessarily of the form*

$$\begin{aligned} L(x) &= 0 && x < 0 \\ &= \int_0^\infty [\Phi_{(1)}(x)]^y \, dA(y) && x \geqq 0. \end{aligned}$$

PROOF. It follows from (1.5) that $a_n \to \infty$; if x is negative, then, from (2.5),

$$\begin{aligned} L(x) &= \lim_{n\to\infty} EG_\omega^n(a_n x) \leqq \lim_{n\to\infty} EG_\omega(a_n x) \\ &= \lim_{n\to\infty} P\{X_1 \leqq a_n x\} = 0. \end{aligned}$$

It will now be shown that (2.6) and (2.7) imply (2.5) for $x > 0$. If (2.6) holds, then by the extended Helly Bray Lemma [8; p. 181], for every $s > 0$,

$$\lim_{u\to\infty} E\left\{\exp\left[-s\frac{\log G_\omega(u)}{\log F(u)}\right]\right\} = \int_0^\infty e^{-sy} \, dA(y).$$

The left side is the limit of a sequence of monotone functions and the right side is a continuous function; hence, the convergence is uniform in s on each closed and bounded interval. Since $x > 0, \Phi_{(1)}(x)$ is not 0 or 1, and from (1.2) it follows that for all sufficiently large n,

$$0 < -\log F(a_n x + b_n) < \infty .$$

It follows from these facts that if one uses the conventions, $\log 0 = -\infty$; $e^{-\infty} = 0$; $\infty \cdot A = \infty$; $\infty/A = \infty$, that

$$\lim_{n\to\infty} EG_\omega^n(a_n x) = \lim_{n\to\infty} E\left\{\exp\left[\log F^n(a_n x)\cdot\frac{\log G_\omega(a_n x)}{\log F(a_n x)}\right]\right\}$$
$$= \int_0^\infty \exp\,[y \log \Phi_{(1)}(x)]\, dA(y) = \int_0^\infty [\Phi_{(1)}(x)]^y\, dA(y).$$

The condition (2.6) implies the convergence in (2.5) to a function $L(x)$ of the form given in (b).

Such a function is a nondegenerate d.f. if and only if (2.7) holds; in fact

$$\lim_{x\to\infty} L(x) = \lim_{x\to\infty}\int_0^\infty [\Phi_{(1)}(x)]^y\, dA(y) = A(\infty) - A(0-)$$

and

$$L(0+) = \lim_{x\to 0+}\int_0^\infty [\Phi_{(1)}(x)]^y\, dA(y) = A(0+) - A(0-).$$

This completes the proof of the sufficiency of (2.6) and (2.7) for the convergence in (2.5) to a d.f. of the type in (b).

The necessity will now be verified. The sequence $\{a_n\}$ may be replaced by the sequence $\{a_{nk}k^{-1/\alpha}\}$ without any change in the limiting d.f. [5]. The sequence of random variables $\{G_\omega^n(a_n)\}$ has a limiting d.f. In fact, for each positive integer k, it follows that $\lim_{n\to\infty} E[G_\omega^n(a_n)]^k = \lim_{n\to\infty} EG_\omega^{nk}(a_{nk}k^{-1/\alpha}) = L(k^{-1/\alpha})$. The sequence of real numbers $\{L(k^{-1/\alpha}) : k \geqq 1\}$ is bounded by 0 and 1; hence, it is the moment sequence of a unique d.f. By the moment convergence theorem [8; p. 185], the d.f. of $G_\omega^n(a_n)$ converges to the d.f. with the given moment sequence. Since $0 \leqq G_\omega^n(a_n) \leqq 1$, it follows that $0 \leqq -n \log G_\omega(a_n) \leqq \infty$, and that the d.f. of the latter random variable converges at all points of continuity to some d.f. $A(y)$, which is not necessarily a proper d.f.

Let $\{u_n\}$ be a sequence of real numbers such that $u_n \to \infty$. Since $F(x)$ is in the domain of attraction of $\Phi_{(1)}(x)$, $0 < F(u_n) < 1$ for all sufficiently large n, so that

$$[\log G_\omega(u_n)]/[\log F(u_n)]$$

is a well-defined, extended real valued nonnegative random variable. By the weak compactness theorem, one can extract a subsequence

$$[\log G_\omega(u_{n_k})]/(\log F(u_{n_k})]$$

whose d.f.'s converge to a monotone function at all continuity points.

Since $\{u_{n_k}\}$ is a subsequence of $\{u_n\}$, the former also tends to infinity; the sequence $\{a_n\}$ also tends to infinity; furthermore, it follows from (1.5) that the $\{a_n\}$ increase monotonically. For every k sufficiently large, there is an index m depending on k such that $a_m \leqq u_{n_k} \leqq a_{m+1}$; hence, by the monotonicity of $G_\omega(x)$ and $F(x)$,

$$\frac{\log G_\omega(a_{m+1})}{\log F(a_m)} \leqq \frac{\log G_\omega(u_{n_k})}{\log F(u_{n_k})} \leqq \frac{\log G_\omega(a_m)}{\log F(a_{m+1})}.$$

Since, from (1.3) and (1.5),

$$-\log F(a_m) \sim 1 - F(a_m) \sim m^{-1},$$

it follows that the left and right ends of the inequality above are asymptotic to $-m \log G_\omega(a_m)$. From this it follows that

$$[\log G_\omega(u_{n_k})]/[\log F(u_{n_k})]$$

has the same limiting d.f. as $-m \log G_\omega(a_m)$. But it was shown above that this d.f. is $A(y)$. The sequence $\{u_n\}$ and the subsequence $\{u_{n_k}\}$ were chosen independently of $\{a_n\}$; since

$$[\log G_\omega(u_{n_k})]/[\log F(u_{n_k})]$$

is an arbitrary subsequence which has a limiting d.f., every such subsequence has the limiting d.f. $A(y)$; therefore, the sequence itself has the limiting d.f. $A(y)$. Furthermore $\{u_n\}$ is an arbitrary sequence of numbers tending to infinity; therefore, for all y in the continuity set of $A(y)$

$$\lim_{u\to\infty} P\{([\log G_\omega(u)]/[\log F(u)]) \leqq y\} = A(y).$$

This completes the proof of (2.6); $L(x)$ is necessarily of the form (b), by the proof of the sufficiency of (2.6) given above. The fact that $A(y)$ satisfies (2.7) follows from the assertion confirmed before that $L(x)$ is a nondegenerate d.f. if and only if (2.7) is true.

THEOREM 2.2. *Let $\{X_n : n \geqq 1\}$ be a sequence of exchangeable random variables on $(\Omega, \mathcal{A}, P)$ such that the joint d.f.'s have the representation* (2.1); *let $Z_n = \max(X_1, \cdots, X_n)$. Suppose that there exist a sequence of positive numbers $\{a_n\}$, a real number x_0, and a d.f. $F(x)$ in the domain of attraction of $\Phi_{(2)}(x)$ such that* (2.4) *is true for $b_n = x_0$; $\Phi(x) = \Phi_{(2)}(x)$. Then,*

(a) *there exists a nondegenerate d.f. $L(x)$ such that for all x in its continuity set,*

$$(2.8) \qquad \lim_{n\to\infty} P\{a_n^{-1}(Z_n - x_0) \leqq x\} = \lim_{n\to\infty} EG_\omega^n(a_n x + x_0) = L(x)$$

if and only if there exists a monotone nondecreasing function $A(y)$ such that for all y in its continuity set

$$(2.9) \qquad \lim_{u\to x_0-} P\{([\log G_\omega(u)/\log F(u)]) \leqq y\} = A(y)$$

where $A(y)$ satisfies the conditions

$$(2.10) \qquad A(0+) - A(0-) = 0; \qquad 0 < A(\infty) - A(0-) \leqq 1;$$

and furthermore,

$$P\{X_1 \leqq x_0\} = EG_\omega(x_0 +) = 1. \tag{2.11}$$

(b) $L(x)$ *is necessarily of the form*

$$L(x) = \int_0^\infty [\Phi_{(2)}(x)]^y \, dA(y) \qquad x < 0$$
$$= 1 \qquad x \geqq 0.$$

PROOF. The proof of the sufficiency of (2.9), (2.10), and (2.11) in the case where $x < 0$ is analogous to the proof of the case $x > 0$ in Theorem 2.1. It is necessary only to replace $a_n x$ by $a_n x + x_0$.

If $x \geqq 0$, then $\lim_{n\to\infty} EG_\omega^n(a_n x + x_0) = 1$. In fact, since $1 - G_\omega(x_0 +) \geqq 0$, and from (2.11),

$$E(1 - G_\omega(x_0 +)) = 1 - P\{X_1 \leqq x_0\} = 0,$$

it is clear that $G_\omega(x_0 +) = 1$; hence,

$$\lim_{n\to\infty} EG_\omega^n(a_n x + x_0) \geqq \lim_{n\to\infty} EG_\omega^n(x_0 +) = 1.$$

The proof of (b) is similar to that of (b) in Theorem 2.1.

The necessity of (2.9) and (2.10) follows as in the latter theorem: $\{a_n\}$ is replaceable by $\{a_{nk}k^{1/\alpha}\}$, and the sequence of random variables $\{G_\omega(a_n - x_0)\}$ is used in place of $\{G_\omega(a_n)\}$. It remains only to show the necessity of (2.11). Suppose that (2.8) holds for some d.f. $L(x)$; let

$$D = \{\omega : G_\omega(x_0 +) < 1\}; \qquad D' = \{\omega : G_\omega(x_0 +) = 1\};$$

then,

$$L(x) = \lim_{n\to\infty} \left(\int_D + \int_{D'} \right) G_\omega^n(a_n x + x_0) \, dP.$$

It follows from (1.2) that for $x \geqq 0$, $a_n x + x_0 \to x_0 +$; hence, for ω in D,

$$\lim_{n\to\infty} G_\omega^n(a_n x + x_0) = 0,$$

and the bounded convergence theorem shows that

$$\lim_{n\to\infty} \int_D G_\omega^n(a_n x + x_0) \, dP = 0, \qquad x \geqq 0.$$

The integral over D' clearly converges to $P(D')$ so that for $x \geqq 0$, $L(x) = P(D')$; since $L(x)$ is assumed to be a proper d.f., $P(D') = \lim_{x\to\infty} L(x) = 1$, so that (2.11) must hold.

THEOREM 2.3. *Let* $\{X_n\}$ *be a sequence of exchangeable random variables on* $(\Omega, \mathcal{A}, P)$ *such that the joint d.f.'s have the representation* (2.1); *let* $Z_n = \max(X_1, \cdots, X_n)$. *Suppose that there exist sequences* $\{a_n\}$ *and* $\{b_n\}$ *and a d.f.*

$F(x)$ in the domain of attraction of $\Phi_{(3)}(x)$ such that (2.4) is valid for

$$\Phi(x) = \Phi_{(3)}(x).$$

Let x_0 be the least upper bound of all real x such that $F(x) < 1$; x_0 may be finite or $+\infty$. Then,

(a) *there exists a nondegenerate d.f. $L(x)$ such that for all x in its continuity set*

$$(2.12)\qquad \lim_{n\to\infty} P\{a_n^{-1}(Z_n - b_n) \leqq x\} = \lim_{n\to\infty} EG_\omega^n(a_n x + b_n) = L(x)$$

if and only if there exists a d.f. $A(y)$ such that for all y in its continuity set

$$(2.13)\qquad \lim_{u\to x_0-} P\{[\log G_\omega(u)/\log F(u)] \leqq y\} = A(y),$$

where $A(y)$ satisfies the conditions

$$(2.14)\qquad A(0+) - A(0-) = 0; \qquad A(\infty) - A(0-) = 1.$$

(b) *$L(x)$ is necessarily of the form*

$$L(x) = \int_0^\infty [\Phi_{(3)}(x)]^y \, dA(y) \qquad -\infty < x < \infty.$$

PROOF. The sufficiency of (2.13) for the convergence in (2.12) has the same proof as the case $x > 0$ in the proof of Theorem 2.1; it is necessary only to substitute $a_n x + b_n$ for $a_n x$.

The necessity of (2.13) follows as in the proof of Theorem 2.1 : $\{b_n\}$ is replaceable by $\{b_{nk} - a_{nk} \log k\}$ and the sequence of random variables $\{G_\omega(b_n)\}$ takes the place of $\{G_\omega(a_n)\}$.

Theorems 2.1, 2.2 and 2.3 characterize the limiting d.f.'s and give necessary and sufficient conditions for convergence; yet the conditions (2.6), (2.9) and (2.13) are not in a form that is usable in applications, since the d.f. of $G_\omega(x)$ may be unknown. For this reason, an equivalent, but more convenient condition will be given. In the following, x_0 is defined as in Theorem 2.3.

LEMMA 2.1. *The limiting d.f. of*

$$\log G_\omega(u)/\log F(u) \qquad u \to x_0-$$

is the same as that of

$$[1 - G_\omega(u)]/[1 - F(u)] \qquad u \to x_0-.$$

PROOF. This follows from the asymptotic expansion

$$-\log G_\omega(u) \sim 1 - G_\omega(u) \qquad u \to x_0-$$

over the set of ω where $G_\omega(x_0-)$ has the value 1; from a similar expansion for $\log F(u)$; from the fact that the random variable under consideration in the lemma tends to zero on the set of ω where $G_\omega(x_0-)$ is less than 1; and from the well-known theorem of Cramér [4; p. 254].

The random variable $[1 - G_\omega(u)]/[1 - F(u)]$ is a bounded random variable for each $u < x_0$ so that its d.f. is uniquely determined by its moments. The mth moment is

$$E\left[\frac{1 - G_\omega(u)}{1 - F(u)}\right]^m = \left[\frac{1}{1 - F(u)}\right]^m \sum_{i=0}^{m} \binom{m}{i} (-1)^i E[G_\omega(u)]^{m-i}$$
$$= \left[\frac{1}{1 - F(u)}\right]^m$$
$$\cdot \left\{1 + \sum_{i=1}^{m} \binom{m}{i} (-1)^i P\{X_1 \leqq u, \cdots, X_i \leqq u\right\}.$$

The formula for the probability of the intersection of m events in terms of the intersections of the complements, together with the property of exchangeability of the variables X_n, yields the identity:

$$1 + \sum_{i=1}^{m} \binom{m}{i} (-1)^i P\{X_i \leqq u, \cdots, X_i \leqq u\} = P\{X_1 > u, \cdots, X_m > u\};$$

it follows that

$$E\left\{\frac{1 - G_\omega(u)}{1 - F(u)}\right\}^m = \frac{P\{X_1 > u, \cdots, X_m > u\}}{[1 - F(u)]^m}. \tag{2.15}$$

Let $A_u(y)$ be the unique d.f. with the moments given by (2.15), that is, the d.f. of $[1 - G_\omega(u)]/[1 - F(u)]$.

THEOREM 2.4. *Let $A_u(y)$ be the d.f. with the moments given by* (2.15); *then the condition*

$$\lim_{u\to x_0-} A_u(y) = A(y) \tag{2.16}$$

at all continuity points is equivalent to (2.6), (2.9), *and* (2.13).

PROOF. This is a direct result of Lemma 2.1 and (2.15).

COROLLARY 2.1. *If $A(y)$ is uniquely determined by a moment sequence $\{v_m : m \geqq 1\}$, then* (2.16) *holds if for each m,*

$$\lim_{u\to x_0-} \{P\{X_1 > u, \cdots, X_m > u\}/[1 - F(u)]^m\} = v_m.$$

PROOF. This follows from the moment convergence theorem [8; p. 185].

COROLLARY 2.2. *If there exists a constant $c > 0$ such that*

$$\begin{aligned} &\lim_{u\to x_0-} \{P\{X_1 > u\}/1 - F(u)\} = c \quad \text{and} \\ &\lim_{u\to x_0-} \{P\{X_1 > u, X_2 > u\}/[1 - F(u)]^2\} = c^2, \end{aligned} \tag{2.17}$$

then the limiting d.f. $L(x)$ in Theorems 2.1, 2.2, *and* 2.3 *is of the same type as $\Phi_{(i)}(x)$, $i = 1, 2, 3$.*

PROOF. It follows from (2.15) and (2.17) that

$$\lim_{u\to x_0-} E\{[1 - G_\omega(u)]/[1 - F(u)]\} = c,$$
$$\lim_{u\to x_0-} E\{[1 - G_\omega(u)]/[1 - F(u)]\}^2 = c^2.$$

Therefore, $[1 - G_\omega(u)]/[1 - F(u)]$ converges in mean square to c as $u \to x_0-$. Mean square convergence implies the convergence of the d.f.; hence, (2.16) holds with $A(y)$ the d.f. which is degenerate at c. It follows from Theorems 2.1, 2.2 and 2.3 that for each x,

$$\lim_{n\to\infty} P\{a_n^{-1}(Z_n - b_n) \leqq x\} = \Phi^c_{(i)}(x).$$

It is evident for the three types of $\Phi_{(i)}(x)$ given in (1.1) that $\Phi^c_{(i)}(x)$ is of the same type as $\Phi_{(i)}(x)$, $i = 1, 2, 3$.

The class of d.f.'s $L(x)$ obtained in Theorems 2.1, 2.2, and 2.3 will now be analyzed. For each d.f. $A(y)$, there are three types of d.f.'s $L(x)$, corresponding to $\Phi_{(i)}(x)$, $i = 1, 2, 3$, given in (1.1).

$$\begin{aligned} L_{(1)}(x) &= 0 && \text{for } x < 0 \\ &= \int_0^\infty \Phi_{(1)}(xy^{-1/\alpha})\, dA(y) && x \geqq 0 \\ L_{(2)}(x) &= \int_0^\infty \Phi_{(2)}(xy^{1/\alpha})\, dA(y) && x < 0 \\ &= 1 && x \geqq 0. \\ L_{(3)}(x) &= \int_0^\infty \Phi_{(3)}(x - \log y)\, dA(y) && -\infty < x < \infty. \end{aligned}$$

These d.f.'s have the following significance: if Y is a random variable with the d.f. $A(y)$ and X_i, $i = 1, 2, 3$ a random variable distributed independently of Y, with the d.f. $\Phi_{(i)}(x)$, then

$L_{(1)}(x)$ is the d.f. of the product $X_{(1)} \cdot Y^{1/\alpha}$;
$L_{(2)}(x)$ is the d.f. of the quotient $X_{(2)}/Y^{1/\alpha}$;
$L_{(3)}(x)$ is the d.f. of the sum $X_{(3)} + \log Y$.

The normal d.f. is not representable in any of these forms for any choice of $A(y)$. $L_{(1)}(x)$ and $L_{(2)}(x)$ cannot be normal d.f.'s because

$$L_{(1)}(x) = 0 \quad \text{for } x < 0; \qquad L_{(2)}(x) = 1 \quad \text{for } x \geqq 0.$$

But even $L_{(3)}(x)$ cannot be a normal d.f.; it is the d.f. of the sum $X_{(3)} + \log Y$, and according to the normal decomposition theorem [8; p. 271], the sum of two independent random variables has a normal d.f. if and only if both have a normal d.f.

Example. Let

$$\begin{aligned} A(y) &= 0, && x \leqq 0; \\ &= 1 - e^{-x} && x > 0. \end{aligned}$$

Then,

$$\begin{aligned} L_{(1)}(x) &= 0 && x \leqq 0 \\ &= (1 + x^{-\alpha})^{-1} && x > 0. \\ L_{(2)}(x) &= (1 + (-x)^{\alpha})^{-1} && x < 0 \\ &= 1 && x \geqq 0 \\ L_{(3)}(x) &= (1 + e^{-x})^{-1} && -\infty < x < \infty. \end{aligned}$$

$L_{(3)}(x)$ is the "logistic" d.f.; it is the convolution of the d.f.'s of the random variables T and $-T'$, where T and T' have the d.f. $\Phi_{(3)}(x)$.

3. Random Numbers of Random Variables.

Let $\{X_n : n \geqq 1\}$ be a sequence of independent random variables with the common d.f. $F(x)$, which is in the domain of attraction of $\Phi(x)$, one of the extreme value d.f.'s in (1.1).

Let $\{N_n : n \geqq 1\}$ be a sequence of nonnegative, integer-valued random variables distributed independently of the sequence $\{X_n\}$. Let N_n have the d.f. given by $P\{N_n = k\} = p_n(k)$, $k \geqq 0$, where for fixed n, $p_n(k) \geqq 0$; $\sum_{k=0}^{\infty} p_n(k) = 1$. A sequence of random variables W_n is defined as

$$\begin{aligned} W_n &= -\infty && \text{if } N_n = 0 \\ &= \max(X_1, \cdots, X_{N_n}) && \text{if } N_n > 0. \end{aligned}$$

Then the d.f. of W_n is

$$P\{W_n \leqq x\} = \sum_{k=0}^{\infty} p_n(k) F^k(x).$$

This d.f. is not necessarily proper:

$$\lim_{x \to -\infty} P\{W_n \leqq x\} = p_n(0) \geqq 0.$$

Suppose that $N_n \to \infty$ in probability as $n \to \infty$. The following theorem characterizes the limiting d.f. of $a_n^{-1}(W_n - b_n)$ and gives necessary and sufficient conditions for convergence.

THEOREM 3.1. *There exists a d.f. $L(x)$ such that for all x in its continuity set,*

$$\text{(3.1)} \quad \lim_{n \to \infty} P\{a_n^{-1}(W_n - b_n) \leqq x\} = \lim_{n \to \infty} \sum_{k=0}^{\infty} p_n(k) F^k(a_n x + b_n), \\ = L(x)$$

if and only if there exists a d.f. $A(y)$ such that for all y in its continuity set

$$\text{(3.2)} \quad \lim_{n \to \infty} P\{n^{-1} N_n \leqq y\} = A(y),$$

where $A(y)$ satisfies the following condition: If $F(x)$ is in the domain of attraction *of* $\Phi_{(i)}(x)$, $i = 1, 2, 3$, $A(y)$ *satisfies* (2.7), (2.10), *and* (2.14), *respectively; furthermore,* $L(x)$ *is of the form indicated in part* b *of Theorems* 2.1, 2.2, 2.3, *respectively.*

PROOF. If $F(x)$ is in the domain of attraction of $\Phi_{(1)}(x)$, then $b_n = 0$, and if $x < 0$, then

$$\lim_{n\to\infty} P\{a_n^{-1}W_n \leqq x\} = \lim_{n\to\infty} p_n(0) + \lim_{n\to\infty} \sum_{k=1}^{\infty} p_n(k)F^k(a_n x).$$

Since $N_n \to \infty$ in probability, the first term on the right is 0; since $a_n \to \infty$,

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} p_n(k)F^k(a_n x) \leqq \lim_{n\to\infty} F(a_n x) \sum_{k=1}^{\infty} p_n(k) = 0;$$

therefore, the entire expression is 0.

If $F(x)$ is in the domain of $\Phi_{(2)}(x)$ and if $x > 0$, then $F(a_n x + x_0) \geqq F(x_0) = 1$ and

$$\lim_{n\to\infty} P\{a_n^{-1}(W_n - x_0) \leqq x\} = \lim_{n\to\infty} \sum_{k=0}^{\infty} p_n(k)F^k(a_n x + x_0)$$
$$\geqq \lim_{n\to\infty} \sum_{k=0}^{\infty} p_n(k)F^k(x_0) = \sum_{k=0}^{\infty} p_n(k) = 1.$$

If

(a) $F(x)$ is in the domain of attraction of $\Phi_{(1)}(x)$ and $x > 0$; or
(b) $F(x)$ is in the domain of attraction of $\Phi_{(2)}(x)$ and $x < 0$; or
(c) $F(x)$ is in the domain of attraction of $\Phi_{(3)}(x)$ and $-\infty < x < \infty$; then

$$0 < \Phi_{(i)}(x) < 1 \qquad i = 1, 2, 3, \text{ respectively},$$

and, by (1.2), $\log F(a_n x + b_n)$ is finite and not zero for all sufficiently large n, and

$$P\{a_n^{-1}(W_n - b_n) \leqq x\} = \sum_{k=0}^{\infty} p_n(k)F^k(a_n x + b_n)$$
$$= E\{\exp[n \log F(a_n x + b_n)(N_n/n)]\}.$$

The proof of the sufficiency of (3.2) is identical with the corresponding proof of the sufficiency in Theorem 2.3, after the replacement of $\{\log G_\omega(a_n x + b_n)/\log F(a_n x + b_n)\}$ by $\{n^{-1}N_n\}$.

It remains only to prove the necessity of (3.2). The sequence of random variables $\{n^{-1}N_n\}$ has a subsequence $\{n_k^{-1}N_{n_k}\}$ whose d.f.'s converge to some monotone function $A(y)$ at all continuity points; for every $s > 0$, it follows that

$$\lim_{k\to\infty} E\{\exp[-sn_k^{-1}N_{n_k}]\} = \int_0^\infty e^{-sy}\, dA(y)$$

uniformly in s on each closed and bounded interval, as in Theorem 2.1. If x is a number such that $0 < \Phi(x) < 1$, it follows from (3.1) that

$$L(x) = \lim_{k\to\infty} E\,[\exp\,(\log F^{n_k}(a_{n_k}x + b_{n_k})n_k^{-1}N_{n_k})]$$
$$= \int_0^\infty \exp\,[y \log \Phi(x)]\, dA(y).$$

Since $L(\Phi^{-1}(e^{-s}))$ is the Laplace transform of $A(y)$ and $-\log \Phi(x)$ assumes every positive value as x goes through its range of values, it follows from the uniqueness of the Laplace transform that every convergent subsequence of d.f.'s of $\{n^{-1}N_n\}$ has the same limit $A(y)$; hence (3.2) is valid. The conditions given on $A(y)$ in each of the three cases are proved as in the preceding theorems. The proof is now complete.

In Theorem 3.1 it was assumed that $\{N_n\}$ and $\{X_n\}$ were distributed independently of each other. The case in which this is not assumed is more difficult, and there is only a partial analogy to the theory of the limiting distribution of the sum of a random number of random variables. It was shown by Rényi [9] that the limiting distribution for the partial sums is normal under the condition that $n^{-1}N_n$ converges in probability to a discrete positive random variable. It is clear from Theorem 3.1 that the usual extreme value distribution is not, in general, the limiting distribution of the maximum of a random number of random variables in the case where $n^{-1}N_n$ converges in probability to a discrete positive random variable.

In the case where $\{N_n\}$ and $\{X_n\}$ are not assumed to be independent the writer was able to prove only the following analog of a less general theorem of Anscombe [1], who showed that the ordinary central limit theorem holds for the partial sums when $n^{-1}N_n$ converges in probability to a positive constant.

THEOREM 3.2. *If $\{N_n\}$ and $\{X_n\}$ are not necessarily independent of each other, and if there exists a positive number c such that*

$$(3.3) \qquad n^{-1}N_n \to c \qquad \text{in probability,}$$

then, for every x,

$$\lim_{n\to\infty} P\{a_n^{-1}(W_n - b_n) \leqq x\} = \Phi^c(x).$$

Since $\Phi^c(x)$ is of the same type as $\Phi(x)$ (see (1.1)), W_n has a limiting d.f. of the same type as $\Phi(x)$.

PROOF: If (3.3) is true, then for every $\delta > 0$,

$$(3.4) \quad \lim_{n\to\infty} P\{N_n \leqq n(c - \delta)\} = \lim_{n\to\infty} P\{N_n \geqq n(c + \delta)\} = 0.$$

The event $\{W_n \leqq a_nx + b_n\}$ can be decomposed into the union of the mutually exclusive events,

$$\bigcup_{k=0}^{\infty} \{k \text{ random variables are observed and none are greater than } a_nx + b_n\}$$
$$= \bigcup_{k=0}^{\infty} \{N_n = k;\, Z_k = \max\,(X_1, \cdots, X_k) \leqq a_nx + b_n\}.$$

Then,

$$(3.5)\qquad \begin{aligned} P\{W_n \leqq a_n x + b_n\} &= \sum_{k=0}^{\infty} P\{N_n = k, Z_k \leqq a_n x + b_n\} \\ &= \sum_{k=0}^{[n(c-\delta)]} + \sum_{k=[n(c-\delta)]+1}^{[n(c+\delta)]} + \sum_{k=[n(c+\delta)]+1}^{\infty}, \end{aligned}$$

where $[t]$ stands for the largest integer less than or equal to t.

It follows from (3.4) that the first and third sums in (3.5) tend to zero. The limit of the middle sum in (3.5) is now computed. Let E_n be the event $E_n = \{[n(c - \delta)] < N_n < [n(c + \delta)] + 1\}$, and $\bar{E}_n$ the complement of this event. For a fixed n, the events

$$\{Z_k \leqq a_n x + b_n\} = \bigcap_{i=1}^{k} \{X_i \leqq a_n x + b_n\}, \qquad k = 1, 2, \cdots$$

form a monotone nonincreasing sequence; hence

$$(3.6)\qquad \begin{aligned} &\sum_{k=[n(c-\delta)]+1}^{[n(c+\delta)]} P\{N_n = k; Z_{[n(c+\delta)]} \leqq a_n x + b_n\} \\ \leqq &\sum_{k=[n(c-\delta)]+1}^{[n(c+\delta)]} P\{N_n = k; Z_k \leqq a_n x + b_n\} \\ \leqq &\sum_{k=[n(c-\delta)]+1}^{[n(c+\delta)]} P\{N_n = k; Z_{[n(c-\delta)]+1} \leqq a_n x + b_n\}. \end{aligned}$$

The last expression in (3.6) is no greater than

$$P\{\{E_n\};\ \ Z_{[n(c-\delta)]+1} \leqq a_n x + b_n\} \leqq F^{[n(c-\delta)]+1}(a_n x + b_n).$$

The first expression in (3.6) greater than or equal to $F^{[n(c+\delta)]}(a_n x + b_n) - P\{\bar{E}_n\}$ since

$$\begin{aligned} P\{\{E_n\};\ \ Z_{[n(c+\delta)]} \leqq a_n x + b_n\} \\ = F^{[n(c+\delta)]}(a_n x + b_n) - P\{Z_{[n(c+\delta)]} \leqq a_n x + b_n\ ;\ \ \{\bar{E}_n\}\}. \end{aligned}$$

Since, from (3.4), $\lim_{n\to\infty} P\{\bar{E}_n\} = 0$, it follows that

$$\begin{aligned} \lim_{n\to\infty} \{F^{[n(c+\delta)]}(a_n x + b_n) - P\{\bar{E}_n\}\} &= \lim_{n\to\infty} [F^n(a_n x + b_n)]^{c+\delta} \\ &= [\Phi(x)]^{c+\delta}, \end{aligned}$$

and

$$\lim_{n\to\infty} F^{[n(c-\delta)]+1}(a_n x + b_n) = \lim_{n\to\infty} [F^n(a_n x + b_n)]^{c-\delta} = [\Phi(x)]^{c-\delta};$$

hence,

$$[\Phi(x)]^{c+\delta} \leqq \liminf_{n\to\infty} \sum_{k=[n(c-\delta)]+1}^{[n(c+\delta)]} \leqq \limsup_{n\to\infty} \sum_{k=[n(c-\delta)]+1}^{[n(c+\delta)]} \leqq [\Phi(x)]^{c-\delta}.$$

Since δ is arbitrarily small, the limit of the middle sum in (3.5) exists and is equal to $\Phi^c(x)$.

4. Acknowledgments. This paper contains the first part of the author's Ph.D. dissertation written at Columbia University. The author is indebted to Professors Gisiro Maruyama and Ronald Pyke for guidance during the writing of the dissertation, and to Professor E. J. Gumbel who introduced the author to the problems of extreme values, obtained the support necessary for this work, and expressed unceasing interest in it.

REFERENCES

[1] Anscombe, F. J. (1952). Large sample theory of sequential estimation. *Proc. Cambridge Philos. Soc.* **48** 600–607.

[2] Blum, J., Chernoff, H., Rosenblatt, M., and Teicher, H. (1958). Central limit theorems for interchangeable processes. *Canad. J. Math.* **10** 222–229.

[3] Bühlmann, Hans. (1960). Austauchbare stochastische Variabeln und ihre Grenzwertsaetze. *Univ. Calif. Pubs. Stat.* **3** 1–36.

[4] Cramér, Harald. (1946). *Mathematical Methods of Statistics.* Princeton Univ. Press.

[5] Gnedenko, B. V. (1943). Sur la distribution limite du terme maximum d'une série aléatoire. *Ann. Math.* **44** 423–453.

[6] Gumbel, E. J. (1958). *Statistics of Extremes,* Columbia Univ. Press.

[7] Juncosa, M. L. (1949). On the distribution of the minimum in a sequence of mutually independent random variables, *Duke Math. J.* **16** 609–618.

[8] Loève, Michel. *Probability Theory.* 2nd Ed., Van Nostrand, Princeton.

[9] Rényi, A. (1960). On the central limit theorem for the sum of a random number of independent random variables. *Acta Math. Acad. Sci. Hungar.* **11** 97–102.

[10] Robbins, Herbert. (1948). The asymptotic distribution of the sum of a random number of random variables. *Bull. Amer. Math. Soc.* **54** 1151–1161.

[11] Watson, G. S. (1954). Extreme values in samples from m-dependent stationary stochastic processes. *Ann. Math. Statist.* **25** 798–800.