ON CONSTRUCTING THE FRACTIONAL REPLICATES OF THE 2^m DESIGNS WITH BLOCKS[1]

By M. S. Patel[2]

Research Triangle Institute

0. Introduction and summary. The problem of constructing fractional replicates of the s^m designs, where s is a prime power is not new in literature. There are several papers which deal with this problem. However, so far as the subject matter of this paper is concerned, the contributions made by Banerjee [2], Rao [8], Dykstra [6] and very recently by Addelman [1] are of special interest. This is yet another attempt in the same direction. The basic concept is the same as in [8], where Rao gives a method of obtaining block designs for the fractional replicates of the s^m designs so as to estimate the main effects and the two-factor interactions orthogonally assuming all other interactions to be absent. With the same assumptions, a method of construction is given in this paper which gives in many cases block designs for the fractional replicates of the 2^m designs with lesser number of treatment combinations than that of the corresponding fractional designs given earlier. This is achieved by allowing the estimates to be correlated. The scheme allows the estimates of treatment-effects and block effects to be mutually orthogonal. An additional example is given at the end to indicate the possibility of improving the construction.

1. Notations and preliminaries. Let the m factors of a 2^m design be denoted by $A_1, A_2, \cdots, A_m$ and a treatment combination in which these factors appear at levels $x_1, x_2, \cdots, x_m$ by

$$a_1^{x_1}a_2^{x_2}\cdots a_m^{x_m} \quad \text{or} \quad (x_1, x_2, \cdots, x_m) \tag{1}$$

where $x_i = 0, 1; i = 1, 2, \cdots, m$. In what follows, (1) will also be referred to as a treatment or an assembly. In accordance with the standard convention, we shall denote a treatment or the mean response to a treatment by the same symbol. Thus, if $y(x_1, x_2, \cdots, x_m)$ denotes an observed response of the treatment (1), then

$$E[y(x_1, x_2, \cdots, x_m)] = (x_1, x_2, \cdots, x_m) \tag{2}$$

where E stands for "expectation".

Any effect in this experiment will be denoted by

$$A_1^{\lambda_1}A_2^{\lambda_2}\cdots A_m^{\lambda_m} \qquad (\lambda_i = 0, 1; i = 1, 2, \cdots, m), \tag{3}$$

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[2] Now at Purdue University.

and will be defined by the equation

$$(4)\quad \begin{pmatrix}1\\a_1\end{pmatrix} \times \begin{pmatrix}1\\a_2\end{pmatrix} \times \cdots \times \begin{pmatrix}1\\a_m\end{pmatrix} = H^{(m)} \cdot \begin{pmatrix}I\\A_1\end{pmatrix} \times \begin{pmatrix}I\\A_2\end{pmatrix} \times \cdots \times \begin{pmatrix}I\\A_m\end{pmatrix}$$

as given by Bose and Connor [3], where

$$(5)\qquad H = \begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}, \qquad H^{(m)} = H \times H \times \cdots \times H, m \text{ times.}$$

The symbol " $\times$ " stands for the symbolic Kronecker Product or Direct Product, and I is the average of the mean responses of all treatments with the convention

$$(6)\qquad I \cdot I = I, \qquad I \cdot A = A = A \cdot I, \qquad A^0 = I.$$

If $\lambda_1 = 1$ and the rest of the λ's are zero, then (3) represents the main effect A_1. If $\lambda_1 = \lambda_2 = 1$ and the rest of the λ's are zero, then (3) represents the two-factor interaction A_1A_2. Any other main effect or a two-factor interaction is similarly represented. Assuming three and higher factor interactions to be absent, (4) reduces to

$$(7)\quad (x_1, x_2, \cdots, x_m) = I + \sum_{i=1}^{m} c(x_i)A_i + \sum_{\substack{i<i' \\ i,i'=1,2,\cdots,m}} c(x_i)c(x_{i'})A_iA_{i'}$$

where $c(x_i) = -1$ when $x_i = 0$ and $c(x_i) = 1$ when $x_i = 1$; $i = 1, 2, \cdots, m$.

The effects defined above are known as the Product Effects. With the help of finite geometries, Bose and Kishen [4] defined the same effects in another way, known as the Geometric Effects. Considering only the main effects and the two-factor interactions, Connor [5] has shown that in a 2^m design, the main effects have the same meaning in both definitions and the two-factor interactions differ in sign only.

2. Orthogonal arrays and fractional replicates. A one-to-one correspondence can be made between the assemblies of a 2^m design and the points of a finite Euclidian geometry of dimension m denoted by EG $(m, 2)$. An $m - r$ flat F_{m-r} in this geometry is defined by a set of r linearly independent equations

$$(8)\quad L_\alpha = a_{\alpha 1}x_1 + a_{\alpha 2}x_2 + \cdots + a_{\alpha m}x_m = d_\alpha (d_\alpha = 0, 1; \alpha = 1, 2, \cdots, r).$$

There are 2^{m-r} points on F_{m-r}. We shall say that an assembly of a 2^m design lies on the $m - r$ flat F_{m-r} if the corresponding point of EG $(m, 2)$ lies on it. The necessary and sufficient condition that the 2^{m-r} points of F_{m-r} written as column vectors, constitute an orthogonal array of strength $d + 1$ with 2^{m-r} assemblies, m constraints, 2 levels and index $\lambda = 2^{m-r-d-1}$ denoted by $(2^{m-r}, m, 2, d + 1)$ is that every linear form $\sum_{\alpha=1}^{r} \lambda_\alpha L_\alpha$ has at least $d + 2$ non-zero coefficients for $(\lambda_1, \lambda_2, \cdots, \lambda_r) \neq (0, 0, \cdots, 0)$.

The above result when the number of levels is s, where s is a prime power, was first proved by Rao [8]. The linear forms L_α $(\alpha = 1, 2, \cdots, r)$ will be called the generating forms and will be said to generate the orthogonal array in the finite

Euclidian geometry. In this sequel, orthogonal arrays will be referred to as arrays. The array as defined above gives a $1/2^r$ fraction of a 2^m design. To every linear form L_α, let there correspond an interaction of a 2^m design. Then Kempthorne [7] has shown that the estimates of effects corresponding to any other linear form $L \neq \lambda_1 L_1 + \lambda_2 L_2 + \cdot + \lambda_r L_r$ from the fraction defined by (8) are aliased with the estimates of the effects corresponding to

$$L + (\lambda_1 L_1 + \lambda_2 L_2 + \cdot + \lambda_r L_r) \tag{9}$$

$$(\lambda_1, \lambda_2, \cdots, \lambda_r) \neq (0, 0, \cdots, 0); \qquad \lambda_\alpha = 0, 1; \qquad \alpha = 1, 2, \cdots, r.$$

Thus what is actually estimated is not the effect corresponding to L but a linear function of the effects corresponding to L and the linear forms (9). This grouping of the effects can be determined from the identity relationship

$$I = G_1 = G_2 = \cdots = G_r = G_1G_2 = \cdots = G_{r-1}G_r = \cdots = G_1G_2 \cdots G_r, \tag{10}$$

where G's are the interactions corresponding to the linear forms in (8).

3. Construction of the fractional replicates with corresponding block designs. A few lemmas and theorems on which depends the construction of the fractions are given below. The actual construction follows from Theorems 3 and 4.

THEOREM 1. *Let $L_\alpha = a_{\alpha 1}x_1 + a_{\alpha 2}x_2 + \cdots + a_{\alpha m}x_m = 0$ $(\alpha = 1, 2, \cdots, r)$ be the largest possible number of linearly independent equations in* GF(2) *whose solutions constitute an array of strength* 2 *in* EG$(m, 2)$. *Let $U_\alpha = (a_{\alpha 1}, a_{\alpha 2}, \cdots, a_{\alpha m})$, and $w(U_\alpha)$ = the number of non-zero coordinates of U_α be defined as the weight of vector U_α. If ξ_r is the vector space generated by the U_α*'s, *then in ξ_r the number of vectors of weight* 3 *whose* ith *coordinate is unity is $\leq r$ $(i = 1, 2, \cdots, m)$.*

The proof of this theorem depends on the following lemma.

LEMMA 1. *If $U_{\alpha_1}, U_{\alpha_2}, \cdots, U_{\alpha_k}$ are the vectors of weight* 3 *in ξ_r whose* ith *coordinate is unity, then they are all linearly independent $(i = 1, 2, \cdots, m)$.*

PROOF. If not, there exist constants $b_1, b_2, \cdots, b_k$ not all zero such that

$$b_1U_{\alpha_1} + b_2U_{\alpha_2} + \cdots + b_kU_{\alpha_k} = 0.$$

But this is impossible since no two of the vectors $U_{\alpha_1}, U_{\alpha_2}, \cdots, U_{\alpha_k}$ can have unity as coordinate at the same place except the ith. This can be seen easily from the well-known result

$$w(V_1 + V_2) = w(V_1) + w(V_2) - 2w(V_1V_2), \tag{11}$$

where $V_1 = (d_{11}, d_{12}, \cdots, d_{1m})$, $V_2 = (d_{21}, d_{22}, \cdots, d_{2m})$, and $V_1V_2 = (d_{11}d_{21}, d_{12}d_{22}, \cdots, d_{1m}d_{2m})$. Now suppose two of the vectors, say U_{α_1} and U_{α_2}, have unities at the ith and i'th places $(i \neq i' = 1, 2, \cdots, m)$ and the third unity occurs at different places, then

$$w(U_{\alpha_1} + U_{\alpha_2}) = 3 + 3 - (2 \times 2) = 6 - 4 = 2,$$

which implies that the vector $U_{\alpha_1} + U_{\alpha_2}$ does not belong to ξ_r, a contradiction. Hence the lemma.

PROOF OF THEOREM 1. The proof now follows immediately. Since the k vectors of weight 3 in ξ_r are linearly independent, they form a sub-set of the basis of the vector space generated by the U_α's ($\alpha = 1, 2, \cdots, r$) which implies that k is $\leqq r$.

COROLLARY 1. *In* ξ_r, *the number of vectors of weight* 4 *whose* ith *and* i'th *coordinates are both unity is* $\leqq r$ ($i \neq i' = 1, 2, \cdots, m$). The proof is similar to that given in Lemma 1 for vectors of weight 3.

Consider again the r linearly independent forms $L_\alpha (\alpha = 1, 2, \cdots, r)$ in GF(2) as given in Theorem 1. Let S denote the set of treatments $(x_1, x_2, \cdots, x_m)$ which satisfies the equations $L_\alpha = e_\alpha$ ($\alpha = 1, 2, \cdots, r$), where the e_α's are elements of GF(2) and all operations are in GF(2). For any linear form L, the corresponding treatment comparison will be conveniently denoted by $T(L)$. For example if $L = x_1 + x_2$, $T(L)$ will mean the interaction A_1A_2. The estimate of $T(L)$ from the fraction of the 2^m experiment containing only the treatments of S is given by

$$\widehat{T(L)} = (2^{m-r})^{-1} [(\{L = 1\} \cap S) - (\{L = 0\} \cap S)] \tag{12}$$

where $\{L = e\}$; $e = 0, 1$, means the set of points (treatments) satisfying the equation within the curly bracket in GF(2); $\{L = e\} \cap S$ means the set of treatments common to the two sets $\{L = e\}$ and S; $(\{L = e\} \cap S)$ means the sum of the responses of treatments indicated.

THEOREM 2. $E[\widehat{T(L)}] = \sum (-1)^w c(\lambda' e) A_1^{d_1} A_2^{d_2} \cdots A_m^{d_m}$, *where the summation is over all the* 2^r *vectors* $\lambda' = (\lambda_1, \lambda_2, \cdots, \lambda_r)$, d_i *is the coefficient of* x_i ($i = 1, 2, \cdots, m$) *in the linear form* $L + (\lambda_1 L_1 + \lambda_2 L_2 + \cdots + \lambda_r L_r)$, w *is the weight of the same linear form or of the corresponding coefficient-vector* $(d_1, d_2, \cdots, d_m)$, $\lambda' e = \lambda_1 e_1 + \lambda_2 e_1 + \cdots + \lambda_r e_r$, *and* $c(\lambda' e) = -1$ or 1 *according as* $\lambda' e = 0$ *or* 1 *as defined in* (7).

PROOF. The expectation of the observed response $y(x_1, x_2, \cdots, x_m)$ of the treatment $(x_1, x_2, \cdots, x_m)$ is given by

$$E[y(x_1, x_2, \cdots, x_m)] = \prod_{i=1}^{m} [I + c(x_i) A_i] \tag{13}$$

which follows from (4). With the help of this expectation equation, we shall determine the coefficient of $A_1^{d_1} A_2^{d_2} \cdots A_m^{d_m}$ in the expectation of $\widehat{T(L)}$ for any arbitrary linear form $d_1x_1 + d_2x_2 + \cdots + d_mx_m$. First we notice that if the linear form $d_1x_1 + d_2x_2 + \cdots + d_mx_m$ is not of the form

$$L + (\lambda_1 L_1 + \lambda_2 L_2 + \cdots + \lambda_r L_r),$$

then in the coefficient of $A_1^{d_1} A_2^{d_2} \cdots A_m^{d_m}$, there will be 2^{m-r-1} plus signs and 2^{m-r-1} minus signs and hence the required coefficient is zero. Next consider a linear form

$$d_1x_1 + d_2x_2 + \cdots + d_mx_m = L + (\lambda_1 L_1 + \lambda_2 L_2 + \cdots + \lambda_r L_r).$$

Case (1). $\lambda' e = 0$. The weight of the linear form is w. For the sake of definiteness, suppose $d_1 = d_2 = \cdots = d_w = 1$. Then for any treatment $(x_1, x_2, \cdots, x_m)$ belonging to $\{L = 1\} \cap S$, we shall have $x_1 + x_2 + \cdots + x_w = 1$. So among $x_1, x_2, \cdots, x_w$ an odd number would be equal to 1. Therefore from (13) it follows that the contribution to the coefficient of $A_1^{d_1}A_2^{d_2} \cdots A_m^{d_m} = A_1^{d_1}A_2^{d_2} \cdots A_w^{d_w}$ from the response $y(x_1, x_2, \cdots, x_m)$ of any treatment $(x_1, x_2, \cdots, x_m)$ belonging to $\{L = 1\} \cap S$ in $\widehat{T(L)}$ would be

$$(14) \qquad (2^{m-r})^{-1}(-1)^{w-(2q-1)} = -(2^{m-r})^{-1}(-1)^w = (2^{m-r})^{-1}c(\lambda' e)(-1)^w,$$

where q is a positive integer.

Similarly for any treatment $(x_1, x_2, \cdots, x_m)$ belonging to $\{L = 0\} \cap S$, we have $x_1 + x_2 + \cdots + x_w = 0$. Hence an even number of x's would be 1. This means that the response of any treatment belonging to $\{L = 0\} \cap S$ would contribute

$$(15) \qquad -(2^{m-r})^{-1}(-1)^{w-2q} = (2^{m-r})^{-1}c(\lambda' e)(-1)^w$$

to the coefficient of $A_1^{d_1}A_2^{d_2} \cdots A_w^{d_w}$ in the expectation of $\widehat{T(L)}$. Finally remembering that there are 2^{m-r} treatments in S, we obtain from (14) and (15) $c(\lambda' e)(-1)^w$ as the required coefficient of $A_1^{d_1}A_2^{d_2} \cdots A_w^{d_w}$ in the expression for the expectation of $\widehat{T(L)}$.

Case (2). $\lambda' e = 1$. The coefficient of $A_1^{d_1}A_2^{d_2} \cdots A_w^{d_w}$ in this case can be derived by arguments similar to those in case (1). This completes the proof of Theorem 2.

In what follows, the set of treatments, S, and the corresponding array in the Euclidian geometry will be denoted by the same symbol.

THEOREM 3. *Let* $L_\alpha = a_{\alpha 1}x_1 + a_{\alpha 2}x_2 + \cdots + a_{\alpha m}x_m$ $(\alpha = 1, 2, \cdots, r)$ *be the generating forms for a class of arrays, each of strength* 2 *in* EG$(m, 2)$. *There are* 2^r *arrays in the class. Let* $r + 1$ *of these arrays be given by the equations*

$$L(r \times 1)J(1 \times \overline{r+1}) = [0(r \times 1), \delta(r \times r)]$$

where $L(r \times 1) = (L_1, L_2, \cdots, L_r)'$, $J(1 \times \overline{r+1})$ *is a row vector of* 1's, $0(r \times 1)$ *is a column vector of null elements,* $\delta(r \times r)$ *is a non-singular matrix of* 0's *and* 1's *in* GF(2). *Then, the fractional replicate of the* 2^m *design consisting of the assemblies belonging to the* $(r + 1)$ *arrays defined above estimates the main effects and the two-factor interactions, all other interactions assumed to be absent.*

PROOF. Let us denote the $r + 1$ arrays by $S_0, S_1, S_2, \cdots, S_r$ and the fractional replicate by S. Then $S = \bigcup_{u=0}^{r} S_u$ and the fraction consists of the $(r + 1)2^{m-r}$ assemblies of the 2^m design. The array S_0 corresponds to $LJ(1 \times 1) = 0(r \times 1)$ and the array S_u $(u = 1, 2, \cdots, r)$ corresponds to $LJ(1 \times 1) = \delta_u$ where δ_u is the uth column of $\delta(r \times r)$. Since each is an array of strength 2, the fraction S obviously estimates all the main effects. Next consider the two-factor interaction $T(L)$. There may be three cases:

Case (1). $T(L)$ is not aliased with any main effect or a two-factor interaction In this case $T(L)$ is estimable.

Case (2). $T(L)$ is aliased with a main effect A_{i_1}. Suppose $T(L) = A_{i_2}A_{i_3}$. Then there exists $(\lambda_{11}, \lambda_{12}, \cdots, \lambda_{1r})$ such that $L + \lambda_{11}L_1 + \lambda_{12}L_2 + \cdots +$

$\lambda_{1r}L_r = x_{i_1}$. Therefore

$$\lambda_{11}L_1 + \lambda_{12}L_2 + \cdots + \lambda_{1r}L_r = x_{i_1} + x_{i_2} + x_{i_3}. \tag{16}$$

Suppose the interactions $A_{i_4}A_{i_5}$, $A_{i_6}A_{i_7}$, $\cdots$, $A_{i_{2k}}A_{i_{2k+1}}$ are also aliased with A_{i_1}. Then, from Lemma 1 and Theorem 1, it follows that all the indices i_2, i_3, $\cdots$, i_{2k+1} are distinct and $k \leqq r$. Suppose

$$\begin{aligned} \lambda_{21}L_1 + \lambda_{22}L_2 + \cdots + \lambda_{2r}L_r &= x_{i_1} + x_{i_4} + x_{i_5}, \\ \lambda_{31}L_1 + \lambda_{32}L_2 + \cdots + \lambda_{3r}L_r &= x_{i_1} + x_{i_6} + x_{i_7}, \\ \vdots \qquad \vdots \qquad\qquad \vdots \qquad & \qquad \vdots \\ \lambda_{k1}L_1 + \lambda_{k2}L_2 + \cdots + \lambda_{kr}L_r &= x_{i_1} + x_{i_{2k}} + x_{i_{2k+1}}. \end{aligned} \tag{17}$$

Then, from Lemma 1, the linear forms on the right-hand side in (16) and (17) are mutually independent. So the matrix $\lambda(k \times r) = ((\lambda_{j\alpha}))$; $j = 1, 2, \cdots k$; $\alpha = 1, 2, \cdots, r$ is of rank k. Premultiplying the equations defining the arrays by $\lambda(k \times r)$, we get

$$\begin{aligned} \lambda(k \times r)L(r \times 1)J(1 \times \overline{r+1}) &= [0(k \times 1), \lambda(k \times r)\delta(r \times r)] \\ &= [0(k \times 1), \nu(k \times r)], \text{ say,} \end{aligned} \tag{18}$$

where all operations are in GF(2). The matrix $\nu(k \times r) = ((\nu_{ju}))$ where $\nu_{ju} = 0$ or 1; $j = 1, 2, \cdots k$; $u = 1, 2, \cdots r$ is of rank k. Without loss of generality, we shall assume that the $(k \times k)$ principal minor

$$\nu_P = ((\nu_{ju})); \qquad j = 1, 2, \cdots, k; u = 1, 2, \cdots, k \tag{19}$$

is non-singular. The linear forms given by $\lambda(k \times r)L(r \times 1)$ are the same as those appearing on the right-hand side in (16) and (17).

Suppose T_u denotes the estimate of A_{i_1} based on the fraction S_u. The weight of the linear form representing a two-factor interaction is 2. Hence, we get from (18) using Theorem 2

$$E\begin{bmatrix} \hat{T}_0 \\ \hat{T}_1 \\ \vdots \\ \hat{T}_k \end{bmatrix} = \frac{1}{k}\left[\begin{array}{c|c} 1 & -J \\ \hline J' & ((c(\nu_{ju}))) \end{array}\right]_{\substack{\\ 1 \qquad\quad k}} \cdot \begin{bmatrix} A_{i_1} \\ A_{i_2}A_{i_3} \\ \vdots \\ A_{i_{2k}}A_{i_{2k+1}} \end{bmatrix}. \tag{20}$$

The matrix in the equation (20) is non-singular since its determinant is equal to $2^k \times \det((\nu_{ju}))$, i.e., $2^k \times \det \nu_P$ which is $\neq 0$. Therefore $A_{i_2}A_{i_3}$ is estimable.

Case (3). $A_{i_2}A_{i_3}$ is not aliased with any main effect but is aliased with a two-factor interaction. Using the argument similar to that in case (2), we can easily establish the estimability of $A_{i_2}A_{i_3}$ with the help of Corollary 1 and Theorem 2.

As a special case, we may take $\delta(r \times r) = I(r \times r)$, i.e., a diagonal matrix with unit elements and the theorem holds.

THEOREM 4. *If the assemblies belonging to each of the* $(r + 1)$ *arrays as given in Theorem* 3 *are assigned a block, then in the resulting fractional replicate of the* 2^m

design in $(r + 1)$ blocks, all main effects and the two-factor interactions are estimable with their estimates correlated in sets but orthogonal to the r block contrasts.

PROOF. Let (S_u) denote the sum of the responses of the assemblies in the array S_u $(u = 0, 1, 2, \cdots, r)$. These will then be the $(r + 1)$ block totals in some order. The r linearly independent contrasts between the $(r + 1)$ block totals represent linear functions of interactions corresponding to the linear forms

$$\lambda_1 L_1 + \lambda_2 L_2 + \cdots + \lambda_r L_r \tag{21}$$
$$(\lambda_1, \lambda_2, \cdots, \lambda_r) \neq (0, 0, \cdots, 0); \qquad \lambda_\alpha = 0, 1; \qquad \alpha = 1, 2, \cdots, r$$

each of weight $\geqq 3$ (by the condition of the theorem) and the contrasts between the block effects. This implies that the interactions corresponding to the linear forms in (21) are mixed up (or confounded) with the contrasts between the block effects.

Next the r linear forms L_α $(\alpha = 1, 2, \cdots, r)$ partition the effects of the factorial experiment in alias sets, the estimates of any two effects belonging to different alias sets being orthogonal. From this it follows that the estimates of the main effects and the two-factor interactions of the 2^m design are orthogonal to the estimates of the interactions corresponding to the linear forms in (21) since they belong to different alias sets, which in turn implies that they are orthogonal to the r block contrasts. That they are estimable follows from Theorem 3.

4. Examples. The usefulness of Theorems 3 and 4 is indicated by the examples given in this section.

EXAMPLE 1. 1/2 fraction of a 2^6 design in 4 blocks of 8 assemblies each. The fraction S consists of the arrays S_0, S_1, S_2, S_3 given by

$$\begin{bmatrix} x_1 + x_2 + x_3 \\ x_1 + x_4 + x_5 \\ x_2 + x_4 + x_6 \end{bmatrix} [1, 1, 1, 1] = [0(3 \times 1), I(3 \times 3)].$$

Assigning S_0, S_1, S_2, S_3 to different blocks, we get the required block design. The identity relationship for the fractions defined above is

$$I = A_1A_2A_3 = A_1A_4A_5 = A_2A_4A_6 = A_2A_3A_4A_5 = A_1A_3A_4A_6$$
$$= A_1A_2A_5A_6 = A_3A_5A_6 .$$

Hence, the sets of aliased effects are (1) $\{A_1, A_2A_3, A_4A_5\}$, $\{A_2, A_1A_3, A_4A_6\}$, $\{A_4, A_1A_5, A_2A_6\}$, (2) $\{A_3, A_1A_2, A_5A_6\}$, $\{A_5, A_1A_4, A_3A_6\}$, $\{A_6, A_2A_4, A_3A_5\}$. The effects A_1A_6, A_2A_5, A_3A_4 are estimated orthogonally. Following the model (7), the normal equations which give estimates of the effects are

$$\begin{bmatrix} 32 & -16 & -16 \\ & 32 & 0 \\ \text{Sym.} & & 32 \end{bmatrix} \begin{bmatrix} \widehat{A_1} \\ \widehat{A_2A_3} \\ \widehat{A_4A_5} \end{bmatrix} = \begin{bmatrix} Y(A_1) \\ Y(A_2A_3) \\ Y(A_4A_5) \end{bmatrix},$$

and

$$\begin{bmatrix} 32 & -16 & -16 \\ & 32 & 0 \\ \text{Sym.} & & 32 \end{bmatrix} \begin{bmatrix} \widehat{A_3} \\ \widehat{A_1A_2} \\ \widehat{A_5A_6} \end{bmatrix} = \begin{bmatrix} Y(A_3) \\ Y(A_1A_2) \\ Y(A_5A_6) \end{bmatrix},$$

where $\wedge$ above the effect means the estimate of that effect, and $Y(A_1^{\lambda_1}A_2^{\lambda_2}\cdots A_m^{\lambda_m})$ denotes the linear form of the observed responses on the right-hand side of the normal equations corresponding to the interaction $A_1^{\lambda_1}A_2^{\lambda_2}\cdots A_m^{\lambda_m}$. The first equation holds for any set of effects in (1) and the second equation, for any set of effects in (2). After inverting the matrices, we get

$$\begin{bmatrix} \widehat{A_1} \\ \widehat{A_2A_3} \\ \widehat{A_4A_5} \end{bmatrix} = \frac{1}{64} \begin{bmatrix} 4 & 2 & 2 \\ & 3 & 1 \\ \text{Sym.} & & 3 \end{bmatrix} \begin{bmatrix} Y(A_1) \\ Y(A_2A_3) \\ Y(A_4A_5) \end{bmatrix},$$

and

$$\begin{bmatrix} \widehat{A_3} \\ \widehat{A_1A_2} \\ \widehat{A_5A_6} \end{bmatrix} = \frac{1}{64} \begin{bmatrix} 4 & 2 & -2 \\ & 3 & -1 \\ \text{Sym.} & & 3 \end{bmatrix} \begin{bmatrix} Y(A_3) \\ Y(A_1A_2) \\ Y(A_5A_6) \end{bmatrix}.$$

The grand average I is estimated by $G/32$ where G is the total response of all assemblies. In what follows, the term "correlated effects" will be used for "aliased effects" since the effects are estimable. The inverse matrix provides the estimates and also the variance-covariance matrix of the corresponding estimates. For each set of effects in every group, there is just one inverse matrix.

The model (7) may be modified to allow for block effects assumed to be fixed and unknown. The contrasts between block effects being orthogonal to the treatment effects by Theorem 4, the error sum of squares S_e is calculated from the formula

$$S_e = \sum_{t=1}^{32} y_t^2 - \sum_{u=0}^{3} (S_u)^2/8 - \left[\sum_{i=1}^{6} \widehat{A_i}\, Y(A_i) + \sum_{i<i'} \widehat{A_i A_{i'}} Y(A_i A_{i'})\right],$$

$$(i, i' = 1, 2, \cdots, 6)$$

where y_t is the observed response of the tth assembly and (S_u) is as defined in Theorem 4. With this value of S_e, the "t" test for testing the significance of any main effect or interaction can be carried out as usual.

Example 2. 5/16th fraction of a 2^7 design in 5 blocks of 8 assemblies each. The generating forms are $L_1 = x_1 + x_4 + x_5$, $L_2 = x_1 + x_3 + x_7$, $L_3 = x_1 + x_2 + x_6$, $L_4 = x_2 + x_3 + x_4$. The fraction and its corresponding block design are obtained as in Theorems 3 and 4.

The sets of correlated effects are (1) $\{A_1, A_4A_5, A_2A_6, A_3A_7\}$, (2) $\{A_2, A_3A_4, A_1A_6, A_5A_7\}$, $\{A_3, A_2A_4, A_1A_7, A_5A_6\}$, $\{A_4, A_2A_3, A_1A_5, A_6A_7\}$, (3) $\{A_5, A_1A_4, A_3A_6, A_2A_7\}$, $\{A_6, A_1A_2, A_3A_5, A_4A_7\}$, $\{A_7, A_1A_3, A_4A_6, A_2A_5\}$. Their estimates can be worked out using model (7).

Example 3. 5/32nd fraction of a 2^9 design in 5 blocks of 16 assemblies each. The generating forms are $L_1 = x_1 + x_2 + x_3$, $L_2 = x_1 + x_4 + x_5$, $L_3 = x_1 + x_6 + x_7$, $L_4 = x_1 + x_8 + x_9$, $L_5 = x_2 + x_4 + x_6$. The fraction and its corresponding block design consist of arrays given by the equations

$$L(5 \times 1)J(1 \times 5) = I(5 \times 5).$$

This follows from the special case of Theorem 3.

The sets of correlated effects are (1) $\{A_1, A_2A_3, A_4A_5, A_6A_7, A_8A_9\}$, (2) $\{A_2, A_1A_3, A_4A_6, A_5A_7\}$, $\{A_4, A_1A_5, A_2A_6, A_3A_7\}$, $\{A_6, A_1A_7, A_2A_4, A_3A_5\}$, (3) $\{A_3, A_1A_2, A_5A_6, A_4A_7\}$, $\{A_5, A_1A_4, A_3A_6, A_2A_7\}$, $\{A_7, A_1A_6, A_3A_4, A_2A_5\}$, (4) $\{A_8, A_1A_9\}$, $\{A_9, A_1A_8\}$, (5) $\{A_4A_8, A_5A_9\}$, $\{A_4A_9, A_5A_8\}$, $\{A_2A_8, A_3A_9\}$, $\{A_2A_9, A_3A_8\}$, $\{A_6A_8, A_7A_9\}$, $\{A_6A_9, A_7A_8\}$.

For any other value of m, a corresponding fraction can be obtained similarly.

5. Remark. It should be noted that the assemblies given by the $r + 1$ arrays in Theorem 3 are sufficient in order to estimate the main effects and the two-factor interactions. However, it is not necessary to have all of them and in fact they may be selected in a different way as shown by the following example.

Example 4. 3/16th fraction of a 2^9 design in 3 blocks of 32 assemblies each. The fraction S consists of the arrays S_1, S_2, S_3 given by

$$\begin{bmatrix} x_1 + x_2 + x_3 + x_8 \\ x_1 + x_4 + x_7 + x_9 \\ x_2 + x_5 + x_6 + x_9 \\ x_2 + x_3 + x_4 + x_5 \end{bmatrix} [1, 1, 1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

The blocks correspond to S_1, S_2, S_3. The sets of correlated effects are (1) $\{A_1A_2, A_3A_8\}$, $\{A_1A_6, A_3A_7\}$, $\{A_1A_9, A_4A_7\}$, $\{A_2A_4, A_3A_5\}$, $\{A_2A_9, A_5A_6\}$, $\{A_5A_7, A_8A_9\}$, (2) $\{A_1A_5, A_4A_8\}$, $\{A_2A_7, A_6A_8\}$, $\{A_3A_9, A_4A_6\}$, (3) $\{A_1A_3, A_2A_8, A_6A_7\}$, $\{A_1A_7, A_3A_6, A_4A_9\}$, $\{A_2A_5, A_3A_4, A_6A_9\}$, $\{A_7A_9, A_5A_8, A_1A_4\}$, $\{A_2A_3, A_1A_8, A_4A_5\}$, $\{A_5A_9, A_2A_6, A_7A_8\}$.

Each set of effects in (1), (2) and (3) is estimated by the matrices

(1) $\frac{1}{256}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$, (2) $\frac{1}{256}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$, and (3) $\frac{1}{128}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$

respectively. All main effects are estimated orthogonally.

The technique of estimation is not essentially different from that given in Theorem 3. Consider for example, the set of aliased effects A_1A_3, A_2A_8, A_6A_7 in (3). The linear forms connected with these effects give the equations

$$\begin{bmatrix} x_1 + x_2 + x_3 + x_8 \\ x_1 + x_3 + x_6 + x_7 \end{bmatrix} [1, 1, 1] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

as associated with the arrays S_2, S_1, S_3. The $\delta(2 \times 2)$ matrix on the right is the unit matrix $I(2 \times 2)$. Hence, these effects are estimable. Similar argument holds for every other set of effects in (1), (2), and (3).

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